Ultracold Fermions

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Ideal Fermi Gases (I)

- Experiments are always in a trap:
Ideal Fermi Gases (II)

- Number of states below a certain energy (for one spin state) is:

\[
N(\varepsilon) = \frac{1}{(\hbar \omega)^3} \int_0^\varepsilon d\varepsilon_x \int_0^{\varepsilon-\varepsilon_x} d\varepsilon_y \int_0^{\varepsilon-\varepsilon_x-\varepsilon_y} d\varepsilon_z = \frac{\varepsilon^3}{6(\hbar \omega)^3}
\]

- This means that the Fermi energy is:

\[
\varepsilon_F = (6N)^{1/3} \hbar \omega
\]
Ideal Fermi Gases (III)

- Differently: For the homogeneous gas: $n = \frac{k^3_F}{6\pi^2}$. So in the trap

$$n(x) = \frac{1}{6\pi^2} \left\{ \frac{2m}{\hbar^2} [\varepsilon_F - V(x)] \right\}^{3/2}$$

- By integrating over space we find again:

$$\varepsilon_F = (6N)^{1/3} \hbar \omega$$

- Note that the size of the cloud is:

$$R = \sqrt{\frac{2\varepsilon_F}{m\omega^2}}$$
Ideal Fermi Gases (IV)

- Comparison between bosons and fermions:
Exercise 1:

- Show that $n = k_F^3 / 6\pi^2$ for a homogeneous gas.

- The density of states $dN(\varepsilon) / d\varepsilon$ can also be calculated semiclassically as:

$$
\frac{dN(\varepsilon)}{d\varepsilon} = \int dx \int \frac{dk}{(2\pi)^3} \delta \left( \frac{\hbar^2 k^2}{2m} + \frac{1}{2} m\omega^2 x^2 - \varepsilon \right)
$$

Show that this gives the same result as before, i.e.,

$$
\frac{dN(\varepsilon)}{d\varepsilon} = \frac{\varepsilon^2}{2(\hbar\omega)^3}
$$
Exercise 2:

- Calculate the Fermi energy $\varepsilon_F$ from $N = \int d\mathbf{x} n(\mathbf{x})$ using

\[
\int_0^1 dy y^2 \left(1 - y^2\right)^{3/2} = \frac{\pi}{32}
\]

- Show also that:

\[
n(0) = \frac{8}{\pi^2} \frac{N}{R^3}
\]

where $R = \sqrt{\frac{2\varepsilon_F}{m\omega^2}}$
Ultracold Fermions (I)

- Collisions are $s$-wave

\[ \hbar k_F r_v \ll \hbar \]

and we thus only have interactions between two different spin states.

- This implies also: \[ n r_v^3 \ll 1 \]
Ultracold Fermions (II)

- Open channel coupled to closed channel by (exchange) interaction
- Closed channel has (bare) bound state close to two-atom continuum threshold.
- Energy difference between two atoms and bare molecular state

\[
\delta (B) = \Delta \mu (B - B_0)
\]

is experimentally tunable!
Ultracold Fermions (III)

- Interaction strength or scattering length:

\[
a_{bg} = \frac{\Delta B}{B - B_0}
\]

- Binding energy:

\[
E_b(B) = -\frac{\hbar^2}{m[a(B)]^2} \propto -(B - B_0)^2
\]
Superfluidity (I)

- Flow without friction. Described by a macroscopic wave function:

\[ \Psi(x) = \sqrt{n_s} e^{i(mv_s / \hbar) \cdot x} \]

or more general \( v_s(x) = \hbar \nabla \mathcal{J}(x) / m \) and \( n_s(x) = |\Psi(x)|^2 \).

- This implies the existence of quantized vortices with

\[ \mathcal{J}(x) = \ell \varphi \]

which is really the trademark of superfluidity.
Superfluidity (II)

- Observed in a rotating Bose-Einstein condensate:

- What about a Fermi gas?
Superfluidity (III)

- Presently much debate over:

\[ P = \frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}} \]
BEC (I)

- In second-quantization language the hamiltonian is

\[
H = \int dx \psi^\dagger(x) \left\{ -\frac{\hbar^2 \nabla^2}{2m} - \mu \right\} \psi(x) + ... \\
... + \frac{1}{2} V_0 \int dx \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x),
\]

- How do we treat Bose-Einstein condensation now?
BEC (II)

- Our most simple variational ground-state wave function for a Bose-Einstein condensed gas is now

\[
|\Psi\rangle \propto \left( \int d\mathbf{x} \phi(\mathbf{x})\psi^\dagger(\mathbf{x}) \right)^N |0\rangle .
\]

- However, for \( N \gg 1 \) we expect that we are also allowed to use the more convenient wave function

\[
|\Psi\rangle \propto \exp\left( \int d\mathbf{x} \phi(\mathbf{x})\psi^\dagger(\mathbf{x}) \right) |0\rangle .
\]
The latter ground-state wave function has the property that

\[ \psi(x) |\Psi\rangle = \phi(x) |\Psi\rangle. \]

This suggests that Bose-Einstein condensation is associated with spontaneous symmetry breaking, i.e.,

\[ \langle \psi(x) \rangle \neq 0. \]

This is the macroscopic wavefunction of superfluidity!
Symmetry Breaking (I)

- It is nice to understand spontaneous symmetry breaking a bit better. At a fixed number we have

\[ |N\rangle \propto \frac{1}{\sqrt{N!}} \left( \int dx \phi(x)\psi^{\dagger}(x) \right)^N |0\rangle. \]

- At fixed phase we have, however,

\[ |\vartheta\rangle \propto \sum_N \frac{\exp(iN\vartheta)}{\sqrt{N!}} |N\rangle. \]
Symmetry Breaking (II)

- This shows that the phase and the number of particles are conjugate variables, i.e.,

$$[N, \mathcal{G}]_\_ = -i.$$ 

- Moreover, the energy obeys due to the definition of the chemical potential

$$E \approx E_0 + \mu \Delta N + \frac{1}{2} \frac{d \mu}{dN} \Delta N^2.$$
Symmetry Breaking (III)

- The thermodynamic potential thus obeys

\[ \Omega \approx \Omega_0 + \frac{1}{2} \frac{d\mu}{dN} \Delta N^2, \]

- which leads to the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} \Psi(\mathcal{G}) = \frac{1}{2} \frac{d\mu}{dN} \left( \frac{1}{i} \frac{\partial}{\partial \mathcal{G}} - N \right)^2 \Psi(\mathcal{G}). \]
Symmetry Breaking (III)

- So the absolute ground state of the gas is the symmetry unbroken state

\[ \Psi(\mathcal{G}) = \frac{1}{\sqrt{2\pi}} \exp(iN\mathcal{G}). \]

- However, if \( N \gg 1 \) it takes a very long time for the gas to ‘diffuse’ to this state and we can safely assume that

\[ \Psi(\mathcal{G}) = \delta(\mathcal{G}). \]
In second-quantization language the hamiltonian is

\[
H = \sum_\sigma \int dx \psi_\sigma^\dagger(x) \left\{ -\frac{\hbar^2 \nabla^2}{2m} - \mu \right\} \psi_\sigma(x) + \ldots
\]

\[
\ldots + V_0 \int dx \psi_{\uparrow}^\dagger(x) \psi_{\downarrow}^\dagger(x) \psi_{\downarrow}(x) \psi_{\uparrow}(x),
\]

Now we have Bose-Einstein condensation of pairs so:

\[
\langle \psi_{\downarrow}(x) \psi_{\uparrow}(x) \rangle \neq 0.
\]
Introducing $\Delta = V_0 \langle \psi_\downarrow(x) \psi_\uparrow(x) \rangle$ the Hamiltonian can be approximated by

$$H \approx \sum_\sigma \int dx \psi_\sigma^\dagger(x) \left\{-\frac{\hbar^2 \nabla^2}{2m} - \mu\right\} \psi_\sigma(x) + \ldots$$

$$\ldots + \int dx \Delta \psi_\uparrow^\dagger(x) \psi_\downarrow^\dagger(x) + \int dx \Delta^* \psi_\downarrow(x) \psi_\uparrow(x),$$

This is thus a mean-field theory!
Exercise 3

- Show that this leads to:

\[
    n = \int \frac{d\mathbf{k}}{(2\pi)^3} \left( 1 - \frac{\varepsilon(\mathbf{k}) - \mu}{\hbar \omega(\mathbf{k})} \right)
\]

and

\[
    - \frac{1}{V_0} = \int \frac{d\mathbf{k}}{(2\pi)^3} \left( \frac{1}{2\hbar \omega(\mathbf{k})} \right)
\]

- Here \( \varepsilon(\mathbf{k}) = \hbar^2 \mathbf{k}^2 / 2m \) and \( \hbar \omega(\mathbf{k}) = \sqrt{(\varepsilon(\mathbf{k}) - \mu)^2 + |\Delta|^2} \)
Renormalization

- To make contact with experiments we must use:

\[
\frac{1}{V_0} = \frac{m}{4\pi a\hbar^2} - \int \frac{d\mathbf{k}}{(2\pi)^3} \left( \frac{1}{2\varepsilon(\mathbf{k})} \right)
\]

- This gives the desired gap equation:

\[
- \frac{m}{4\pi a\hbar^2} = \int \frac{d\mathbf{k}}{(2\pi)^3} \left( \frac{1}{2\hbar\omega(\mathbf{k})} - \frac{1}{2\varepsilon(\mathbf{k})} \right)
\]

so we have two equations for \( \Delta \) and \( \mu \).
Results (I)

- Solution for atomic lithium-6:
Results (II)

- Experiments with atomic lithium-6:
Results (III)

- Experiments with atomic lithium-6:
Results (IV)

- Experiments with atomic lithium-6:
Imbalanced Fermi Mixture (I)

- Atomic Fermi mixtures in an optical trap:

\[ P = 0 \quad \rightarrow \quad P = 1 \]
MIT Science experiment (I)

- Uses rotation to create vortices and measures density profiles after expansion and magnetic field sweep:

MIT Science experiment (II)

Imbalanced Fermi Mixture (II)

- Up to recently only $N_{\uparrow} = N_{\downarrow}$. However, the most recent experiments do not have this condition any longer. This means that $N_{\uparrow} > N_{\downarrow}$. 

![Diagram](image_url)
Rice Science experiment (I)

- Uses *in-situ* imaging:

Rice Science experiment (II)

\[ P = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow} \]

[G.B. Partridge et al., Science 311, 503 (2006).]
Using the BCS ansatz we find for the homogeneous case:

![Phase diagram]

- BCS
- Normal
Zero Temperature Theory (II)

- Local-density theory and BCS ansatz:

![Graph showing density distribution and BCS core radius](image)

- BCS core radius: $R_{BCS} = 407.8 \, \mu m$
- Minority $R_{TF} = 444.2 \, \mu m$
- Majority $R_{TF} = 675.8 \, \mu m$

![Graph showing radial distance and effective chemical potential](image)

- $\mu_{eff}$
- $R_{TF}(\text{min.})$
- $R_{BCS}$
- $R_{TF}(\text{maj.})$
- Radial distance

- $N_1 = 72197$
- $N_2 = 39692$
- Spherical trap

![Diagram of spherical trap with labels A, B, C](image)
Zero Temperature Theory (III)

- Comparison with Rice experiment:

See also F. Chevy, PRL 96, 130401 (2006) and
T.N. De Silva and E.J. Mueller, PRA 73, 051602(R) (2006).]
Nonzero Temperature Theory (I)

- Proposed (universal) phase diagram:

[K.B. Gubbels et al., PRL 97, 210402 (2006).]
Nonzero Temperature Theory (II)

- *In-situ* density profiles are again calculated in local-density approximation:

[K.B. Gubbels *et al.*, PRL 97, 210402 (2006).]
Nonzero Temperature Theory (III)

- Comparison with MIT experiment:

[K.B. Gubbels et al., PRL 97, 210402 (2006).]
Rice PRL experiment (I)

- Uses *in-situ* imaging:

[G.B. Partridge *et al.*, PRL 97, 190407 (2006).]
Rice PRL experiment (II)

- Comparison between theory and experiment:

[G.B. Partridge et al., PRL 97, 190407 (2006).]
Rice PRL experiment (III)

- Comparison between ‘low’ and ‘high’ temperature:

\[ \text{[Y. Shin et al., PRL 97, 030401 (2006).]} \]

\[ \text{[G.B. Partridge et al., PRL 97, 190407 (2006).]} \]
Rice PRL experiment (IV)

- Superfluid core shape:

\[
\gamma \left(\frac{r}{R}\right)^\gamma + \left(\frac{z}{Z}\right)^\gamma = 1
\]
The Quantum Fluids and Solids Group

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