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Abstract

In this thesis we consider the nonminimal inflationary scenario, where the scalar inflaton field is coupled to the Ricci scalar $R$ through a nonminimal coupling $\xi$. We show that in the limit of strong negative coupling chaotic inflation with a quartic potential is successful in this model. Moreover, the quartic self-coupling $\lambda$ must be extraordinary small in the minimal model in order to agree with the observed spectrum of density perturbations, but in the nonminimally coupled model the constraint on $\lambda$ can to a large extent be relaxed if $\xi$ is sufficiently large.

This allows the Higgs boson to be the inflaton and excludes the need to introduce any exotic new particles to explain inflation. We show that we can make a transformation to the Einstein frame where the Higgs boson is not coupled to gravity, but where the potential is modified. The potential reduces to the familiar Higgs potential for small values of the Higgs field, but becomes asymptotically flat for large values of the field, making sure that slow-roll inflation is successful. We introduce the two-Higgs doublet model and show that inflation also works in this model. The extra phase degree of freedom in this model is a source for CP violation and a possible source for baryogenesis.

Special attention is made to the quantization of the nonminimally coupled inflaton in a FLRW universe. In an expanding universe the vacuum choice is not unique and therefore the notion of a particle is not well defined. In quasi de Sitter space there is natural vacuum state, called the Bunch-Davies vacuum that corresponds to zero particles in the infinite past. With the choice of this vacuum state the scalar propagator in quasi de Sitter space can be calculated.

The Higgs boson interacts with the Standard Model fermions and therefore effects on the dynamics of fermions during inflation. It is shown that in the two-Higgs doublet model the additional phase degree of freedom gives rise to an axial vector current for the fermions that violates CP and might be converted into a baryon asymmetry through sphaleron processes. The effect of the Higgs inflaton on the dynamics of massless fermions is derived by calculating the one-loop fermion self-energy. It is shown that the Higgs boson effectively generates a mass for the massless fermions that is proportional to the Hubble parameter $H$. 
Contents

Abstract v

1 Introduction 1

2 Cosmology 3
   2.1 The Cosmological Principle and Einstein’s equations 3
   2.2 An expanding universe 4
   2.3 Einstein equations from an action principle 6

3 Cosmological Inflation 9
   3.1 Evolution of the universe 9
   3.2 Cosmological puzzles 10
   3.3 Inflation as solution of the puzzles 11
   3.4 Inflationary models 13
   3.5 Inflationary dynamics 14
   3.6 Constraints on inflationary models 15

4 Nonminimal inflation 19
   4.1 Real scalar field 20
      4.1.1 Dynamical equations 20
      4.1.2 Analytical solutions of the dynamical equations 22
      4.1.3 Numerical solutions of the field equation 25
   4.2 The conformal transformation 28
   4.3 The Higgs boson as the inflaton 30
   4.4 The two-Higgs doublet model 35
      4.4.1 Dynamical equations 38
      4.4.2 Numerical solutions of the field equations 40
   4.5 Summary 42

5 Quantum field theory in an expanding universe 45
   5.1 Quantum field theory in Minkowski space 46
      5.1.1 Bogoliubov transformation 49
      5.1.2 In and out regions 51
   5.2 Quantization of the nonminimally coupled inflaton field 53
      5.2.1 Adiabatic vacuum 56
      5.2.2 Solutions of the mode functions 57
      5.2.3 Scalar propagator 59
   5.3 Quantization of the two-Higgs doublet model 64
      5.3.1 Propagator for the two-Higgs doublet model 67
   5.4 Summary 69
Chapter 1

Introduction

Cosmology is in many respects the physics of extremes. On the one hand it handles the largest length scales as it tries to describe the dynamics of the whole universe. On these scales gravity is the dominant force and general relativistic effects become very important. Cosmology also deals with the longest time scales. It describes the evolution of the universe during the last 13.7 billion years and tries to make predictions for the future of our universe. In fact most cosmologists are working on events that happened in the very early universe with extreme temperatures and densities. These conditions are so extreme that we will most likely never be able to recreate them. Extreme conditions are also present and can be observed today in our universe in for example supernovae, black holes, quasars and highly energetic cosmic rays.

Many cosmologists actually try to explain the large scale effects from microscopic theories. A main field in this sense is the field of baryogenesis, where the matter-antimatter of the universe is tried to be explained. This matter-antimatter asymmetry is tiny, but is the reason why our universe now only consists of matter and no antimatter. Sakharov[1] realized that this asymmetry can be dynamically generated in particle physics models where particles decay in a tiny \( CP \) violating way. Another field of research that tries to explain a large scale phenomenon from a microscopic effect is the dark energy sector. The cosmological constant functions as a uniform energy density in our universe and is incredibly small. However, our universe is so big that the dark energy now dominates our universe and is in fact the source for the recent acceleration of our universe.

Combining the extremely large with the incredibly small is perhaps most evident when we try to do quantum mechanics in an expanding universe. In a heuristic picture a particle-antiparticle pair can be created out of the vacuum for a very short time through the Heisenberg uncertainty principle. The expansion of the universe however can elongate the existence of such a pair because it increases the spatial distance between the particles. In fact, when the expansion of the universe is rapid enough, the particles can have a lifetime longer than the lifetime of the universe. Although heuristic, this picture shows that particles can be created out of the vacuum though the expansion of the universe. These particles form tiny inhomogeneities in the early universe, and in fact these leave their imprint on the spectrum of the cosmic microwave background radiation, which has been measured with incredible precision by the WMAP mission[2]. The inhomogeneities are actually the origin of the formation of stars and planets.

As mentioned above, a rapid expansion of the universe is necessary for particle creation. This is most evident in the inflationary era, where the universe exponentially grows to a size much larger than the visible universe today. Cosmological inflation, proposed by Alan Guth in 1981 [3], is a beautiful combination of general relativity in an expanding universe...
and particle physics. The Einstein equations that follow from Einstein’s theory of general relativity allow for a solution that leads to an accelerated expansion of our universe. A special type of matter with negative pressure is needed for this particular solution, and Guth realized that this is possible within certain scalar field models. The inflationary hypothesis could solve many of the problems that cosmologists faced at the time, and it lead to a revolution in cosmology. Today there are literally hundreds of inflationary models and new models are being build all the time. Many of these models involve exotic new particles and are in that sense not very appealing.

In this thesis our main focus is on a particular inflationary model, which we will call nonminimal inflation. In this model a scalar field is coupled to gravity which can effectively reduce Newton’s gravitational constant and lead to inflation. The main advantage of this model is that in the limit of strong coupling the only scalar particle in the Standard Model, the Higgs boson, is an appropriate candidate for the inflaton, the scalar field that drives inflation. Since there is no need to introduce any exotic new particles, the beauty of this model is evident.

The outline of this thesis is the following. In chapter 2 we will introduce some basic cosmology. Then in chapter 3 we outline the basics of cosmological inflation. In chapter 4 we will introduce the nonminimal inflationary model and we will see that inflation is successful if the scalar field is nonminimally coupled to gravity. Furthermore we will see that observations of the spectrum of density perturbations constrain the quartic self-coupling of the inflaton field in the minimal coupling case to a tiny value, but the constraint can be relaxed by several orders of magnitude if the nonminimal coupling is sufficiently strong. This allows the Higgs boson to be the inflaton, proposed by Bezrukov and Shaposhnikov in [4]. We also introduce the two-Higgs doublet model in chapter 4, and show that nonminimal inflation works in this model as well. The extra degrees of freedom and couplings in the two-Higgs doublet model allow for CP violation and this might lead to baryogenesis.

Chapter 5 is all about combining quantum field theory and general relativity. Although it is well known that quantum gravity is a non-renormalizable theory, quantum gravitational effects become important only at energies above the reduced Planck scale of \( M_P = 2.4 \times 10^{18} \) GeV where the universe was so small that quantum mechanics and gravity become equally important. At lower energies we do not see these effects and we can ‘just’ do quantum field theory on a curved background. Quantization of the nonminimally coupled inflaton field in an expanding universe is however not so trivial. We will see that there is no unique vacuum choice and that in general particle number is not well defined in an expanding universe, which we also mentioned above in the heuristic picture of particle creation. Fortunately we will see that we can make a natural vacuum choice during inflation, the Bunch-Davies vacuum with zero particles. With this specific vacuum we can uniquely quantize the nonminimally coupled inflaton field and do quantum field theory. We calculate the propagator for the scalar inflaton field and eventually extend our results to calculate the propagator for the two-Higgs doublet model in quasi de Sitter space.

In chapter 6 we will calculate the effect of the Higgs boson as the inflaton on the Standard Model fermions. The Higgs boson interacts with the fermions and gives a mass to the fermions through the Higgs mechanism. As a final application we use the massless fermion propagator and the propagator for the two-Higgs doublet model in quasi de Sitter space to calculate the one-loop self-energy for the massless fermions. This one-loop fermion self-energy effectively generates a mass for the massless fermions.
Chapter 2

Cosmology

2.1 The Cosmological Principle and Einstein’s equations

Most cosmological models are based on the assumption that the universe is the same everywhere. This might seem strange to us, since we clearly see other planets and the sun around us, with empty space everywhere in between. However, on the largest scales, when we average over all the local density fluctuations, the universe is the same everywhere. This statement is formulated more precisely by the cosmological principle, which states that the universe is isotropic and homogeneous on the largest scales. Isotropy means that no matter what direction you look, the universe looks the same. The observation of the cosmic microwave background radiation (CMBR) supports the idea of isotropy. The COBE and WMAP missions found that the deviations in the CMBR from a perfect isotropic universe are of the order of $10^{-5}$ [5, 2].

Homogeneity is the idea that the metric is the same on every point in space. Stated otherwise, when we look at the universe from our planet earth and do the same on a planet 1 billion lightyears away, we should still observe the same universe. Again, the same universe means that the universe looks the same on the largest scales. Since we have been assuming for quite a while that our planet is not the center of the universe, we should also observe an isotropic universe anywhere else.

Thus, cosmologists describe the homogeneous and isotropic universe on the largest scales. Since the universe is electrically neutral on these scales, the infinite ranged electromagnetic force does not play a role. Therefore, the dynamics of the universe are determined by the gravitational interaction, described by the Einstein equations

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu},$$

(2.1)

where $G_{\mu\nu}$ is defined in Eq. (B.10), $G_N$ is Newton's constant and $T_{\mu\nu}$ is the energy momentum or stress-energy tensor. This tensor describes the influence of matter on the dynamics of the universe. For the isotropic and homogeneous universe we can describe the matter and energy content as a perfect fluid, a fluid which is isotropic in its rest frame. For a perfect fluid the energy momentum tensor takes the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu},$$

(2.2)

where $\rho$ and $p$ are the energy density and pressure in the rest frame and $u^\mu$ is the four velocity of the fluid. Choosing our frame to be the rest frame of the fluid where $u^\mu = (1,0,0,0)$, we find that the energy momentum tensor is

$$T_{\mu\nu} = (\rho + p)\delta^\mu_\alpha \delta^\nu_\beta + p g_{\mu\nu}.$$  

(2.3)
2.2 An expanding universe

We are now in a position to write down the explicit Einstein equations (2.1) for our universe. We therefore have to find the specific form of the metric in our universe. Although the universe is both isotropic and homogeneous, it is not static. When we observe the sky we see that distant stars and galaxies are moving away from us! The explanation is that the universe itself is expanding in time. Such an expanding universe is generally described by the Friedmann-Lemaître-Robertson-Walker metric, where the line element in spherical coordinates is

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \tag{2.4} \]

Here \( a(t) \) is the scale factor which describes the expansion of the universe and \( k \) describes the curvature of space. If \( k > 0 \) the universe is positively curved and closed, for \( k < 0 \) the universe is negatively curved and open. If \( k = 0 \), the universe is flat, and this last possibility is supported by distant supernovae observations. In that case the metric simplifies to the flat FLRW metric with the line element

\[ ds^2 = -dt^2 + a^2(t)\mathbf{d}x \cdot \mathbf{d}x. \tag{2.5} \]

In Appendix B.2 we derive the Einstein tensor for the flat FLRW metric in 4 dimensions. Using the result (B.18) and the expression for the energy momentum tensor from Eq. (2.3) we find for the 00 component of the Einstein equations,

\[ 3 \left( \frac{\dot{a}}{a} \right)^2 = 8\pi G_N \rho \tag{2.6} \]

For the \( ij \) components we find

\[ -\left[ 2\frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right] g_{ij} = 8\pi G_N p g_{ij} \tag{2.7} \]

Rewriting this equation by using the 00 equation gives us the following equation

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3} (\rho + 3p). \tag{2.8} \]

Now we introduce the Hubble parameter \( H \)

\[ H = \frac{\dot{a}}{a}, \tag{2.9} \]

that determines the expansion rate of the universe. We also define the quantity \( \epsilon \), which is the time derivative of the Hubble parameter,

\[ \epsilon = -\frac{H}{H^2}. \tag{2.10} \]

When \( \epsilon > 1 \), the expansion rate decreases and the expansion of the universe decelerates, whereas for \( \epsilon < 1 \) the expansion of the universe accelerates. This allows us to write the Einstein equations (2.6) and (2.8) in terms of \( H^2 \) as

\[ H^2 = \frac{8\pi G_N}{3} \rho \tag{2.11} \]

\[ (1-\epsilon)H^2 = -\frac{4\pi G_N}{3} (\rho + 3p). \tag{2.12} \]
Table 2.1: Scaling of the energy density $\rho$, scale factor $a$, Hubble parameter $H$ and the deceleration parameter $\epsilon$ in different era.

<table>
<thead>
<tr>
<th>Era</th>
<th>$\omega$</th>
<th>$\rho$</th>
<th>$a$</th>
<th>$H$</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>General</td>
<td>$\omega \propto a^{-3(1+\omega)}$</td>
<td>$\propto t^{\frac{2}{3(1+\omega)H}}$</td>
<td>$\propto t^\frac{2}{3(1+\omega)}$</td>
<td>$\frac{3}{2}(1+\omega)$</td>
<td></td>
</tr>
<tr>
<td>Matter</td>
<td>0</td>
<td>$\propto a^{-3}$</td>
<td>$\propto t^\frac{2}{3}$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>Radiation</td>
<td>$\frac{1}{3}$</td>
<td>$\propto a^{-4}$</td>
<td>$\propto t^\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>2</td>
</tr>
<tr>
<td>de Sitter inflation</td>
<td>$-1$</td>
<td>constant</td>
<td>$\propto e^{H_0 t}$</td>
<td>$H_0 = \text{constant}$</td>
<td>0</td>
</tr>
</tbody>
</table>

There is a third equation which is due to the fact that the Einstein tensor is covariantly conserved, i.e. $\nabla^\mu G_{\mu\nu} = 0$, where $\nabla^\mu$ is the covariant derivative defined in Eq. (B.2). Therefore we also have that $\nabla^\mu T_{\mu\nu} = 0$, and the zeroth component gives

$$0 = \nabla^\mu T_{\mu0} = g^{\mu\lambda}(\partial_\lambda T_{\mu0} - \Gamma^\alpha_{\mu\lambda} T_{\alpha0} - \Gamma^\alpha_{\lambda0} T_{\mu\alpha} = -\partial_0 \rho - \frac{\dot{a}}{a} g^{ij} g_{ij}(\rho + p) = -\rho - 3\frac{\dot{a}}{a}(\rho + p)$$

An important remark is that the conservation of the stress-energy tensor implies that the Einstein equations are not independent. We have derived three equations (Eqs. (2.11), (2.12) and (2.13)), but only two are independent. We now assume that our perfect fluid obeys the equation of state

$$p = \omega \rho,$$

where $\omega$ is a constant. For a matter dominated universe $\omega = 0$, for a radiation dominated universe $\omega = \frac{1}{3}$ and for an inflationary universe $\omega \approx -1$. The equation of state allows us to solve Eq. (2.13) for $\omega \neq -1$, and we get

$$\rho \propto a^{-3(1+\omega)},$$

which gives for the scale factor when we solve Eq. (2.6)

$$a \propto t^{\frac{2}{3(1+\omega)}}.$$ (2.16)

Now we plug this in Eq. (2.10) to find that $\epsilon$ is a constant,

$$\epsilon = 3 \frac{1}{2}(1 + \omega).$$ (2.17)

We see that for a matter dominated universe $\epsilon = \frac{3}{2}$ and for a radiation dominated universe $\epsilon = 2$, so in both cases the expansion of the universe decelerates. For a general inflationary universe, $\omega \approx -1$ and $\epsilon \ll 1$. For future references we will now summarize the above in Table 2.1

The fact that $\epsilon$ is constant allows us to solve Eq. (2.10) for both $H(t)$ and $a(t)$, and we can write the solutions as

$$a(t) = (1 + cH_0 t)\frac{1}{\epsilon}, \quad H(t) = \frac{H_0}{1 + cH_0 t}.$$ (2.18)
Here, \( H_0 = H(t = 0) \) and we have chosen \( a(t = 0) = a_0 = 1 \). Note that in the limit where \( \epsilon \to 0 \) (de Sitter inflation), the scale factor is \( a(t) = e^{H_0 t} \) and \( H(t) = H_0 \).

Now we switch to conformal time by substituting \( dt = a(\eta)d\eta \) in the metric, such that the line element is given by

\[
 ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x} \cdot d\vec{x}) = a^2(\eta)\eta_{\mu\nu}dx^\mu dx^\nu. \tag{2.19}
\]

Here \( \eta \) is conformal time and \( \eta_{\mu\nu} = (-1, 1, 1, 1) \) is the Minkowski metric. This conformally flat FLRW metric simplifies calculations for the curvature invariants such as the Ricci tensor and Ricci scalar, as is shown in Appendix B.2. We now want to find solutions for the scale factor and Hubble rate for constant \( \epsilon \) in conformal time. We note the useful relations

\[
 \dot{a} = a' a, \quad H(\eta) = a' a^2, \quad \dot{H} = H' a.
\]

This gives us the following differential equation for \( a(\eta) \)

\[
 \epsilon = -\frac{a'' a}{(a')^2} + 2. \tag{2.20}
\]

Since the relation between real and conformal time is \( d\eta = \frac{dt}{a(t)} \), i.e. an integral relation between \( \eta \) and \( t \), we have a freedom in choosing the zero of conformal time and also of the boundary conditions for \( a(\eta) \). This means we can choose some conformal time \( \eta_0 \) where we have the boundary conditions \( a(\eta_0) = 1 \) and \( H(\eta_0) = H_0 \). We can choose this \( \eta_0 \) such that the solutions of Eq. (2.20) are

\[
 a(\eta) = \frac{1}{\left[-(1-\epsilon)H_0 \eta \right]^{\frac{1}{1-\epsilon}}}, \quad H(\eta) = \frac{H_0}{\left[-(1-\epsilon)H_0 \eta \right]^{\frac{1}{1-\epsilon}}}. \tag{2.21}
\]

This choice reduces for \( \epsilon = 0 \) to \( H(\eta) = H_0 \) and \( a(\eta) = -\frac{1}{H_0 \eta} \), which is the conformal scale factor in de Sitter space. Also, the universe is expanding (both \( a \) and \( H \) positive) for either \( \epsilon < 1 \) and \( -\infty < \eta < 0 \), or for \( \epsilon > 1 \) and \( 0 < \eta < \infty \). A useful relation is

\[
 H(\eta)a(\eta) = \frac{1}{(1-\epsilon)\eta}, \tag{2.22}
\]

which gives the simple expression for the scale factor

\[
 a(\eta) = \frac{1}{H\eta(1-\epsilon)}. \tag{2.23}
\]

For \( \epsilon \ll 1 \) we are in the so-called *quasi de Sitter space*, which reduces to de Sitter space in the limit where \( \epsilon \to 0 \).

### 2.3 Einstein equations from an action principle

The Einstein equations (2.1) can also be derived from an action. We therefore introduce the Einstein-Hilbert action,

\[
 S_{EH} = \int d^Dx \sqrt{-g} \frac{R}{16\pi G_N}, \tag{2.24}
\]

where \( g = \det(g_{\mu\nu}) \) and \( R \) is the Ricci scalar defined in Eq. (B.7). We now vary this action with respect to the metric \( g^{\mu\nu} \). We use that

\[
 \delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \tag{2.25}
\]
\[ \delta R = \delta (g^\mu{}^\nu R_{\mu^\nu}) = R_{\mu^\nu} \delta g^{\mu^\nu} + g^{\mu^\nu} \delta R_{\mu^\nu} = (R_{\mu^\nu} - \nabla_\mu \nabla_\nu + g_{\mu^\nu} g^{\rho^\sigma} \nabla_\rho \nabla_\sigma) \delta g^{\mu^\nu}. \] (2.26)

Since no fields are coupled to \( R \) in the Einstein-Hilbert action, the covariant derivatives vanish and we get

\[ \delta S_{EH} = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} (R_{\mu^\nu} - \frac{1}{2} R g_{\mu^\nu}) \delta g^{\mu^\nu}. \] (2.27)

Thus, we find that

\[ \frac{16\pi G_N}{\sqrt{-g}} \frac{\delta S_{EH}}{\delta g^{\mu^\nu}} = R_{\mu^\nu} - \frac{1}{2} R g_{\mu^\nu} = 0. \] (2.28)

We have recovered the Einstein equations in vacuum! Of course we want to find the full non-vacuum Einstein equations, so we need to include matter in our equations. We can do this by adding matter fields to the action and defining a new total action

\[ S = S_{EH} + S_M, \] (2.29)

where

\[ S_M = \int d^D x \sqrt{-g} \mathcal{L}_M, \] (2.30)

is the matter action with \( \mathcal{L} \) the matter Lagrangian density containing the matter fields. When we now again do a variation of this action with respect to the metric, we find that

\[ \frac{16\pi G_N}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu^\nu}} = \left( R_{\mu^\nu} - \frac{1}{2} R g_{\mu^\nu} \right) + 16\pi G_N \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu^\nu}} = 0. \] (2.31)

We now recover the non-vacuum Einstein equations (2.1) if we set

\[ T_{\mu^\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu^\nu}}. \] (2.32)

This is a great result! We have shown that we can derive the full Einstein equations from an action. The question remains: what is the matter action? Well, in principle this is the action containing all the matter field, and in our ordinary, low-energy everyday life it is the Standard Model action. In the early universe, at extreme conditions, the matter action is expected to be very different and our Standard Model is only an effective matter action at low temperature. For now we will not choose a specific matter action, but give a simple example that turns out to be quite useful in the following chapters. For a real scalar field \( \phi(x) \) we have the action

\[ S_M = \int d^D x \sqrt{-g} \mathcal{L}_M = \int d^D x \sqrt{-g} \left( -\frac{1}{2} g^{\mu^\nu} (\partial_\mu \phi)(\partial_\nu \phi) - V(\phi) \right). \] (2.33)

Note the minus sign in front of the kinetic term, which is there because we use the metric sign convention \((-+, +, +, +)\). By the definition of the energy momentum tensor in Eq. (2.32) we find that

\[ T_{\mu^\nu} = (\partial_\mu \phi)(\partial_\nu \phi) + g_{\mu^\nu} \mathcal{L}_M. \] (2.34)
Since the spatial background is homogeneous according to the cosmological principle, our scalar field also has to be spatially homogeneous, i.e $\phi(x) = \phi(t)$. This allows us to identify by using Eq. (2.3)

$$
\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi)
$$

$$
p = \frac{1}{2} \dot{\phi}^2 - V(\phi).
$$

(2.35)

Now we use conservation of the energy momentum tensor, $\nabla^{\mu} T_{\mu\nu} = 0$, which lead us to the energy conservation law Eq. (2.13). Using the expressions for $\rho$ and $p$ for a real homogeneous scalar field, we find

$$
\dot{\phi} + 3H\phi + \frac{dV(\phi)}{d\phi} = 0.
$$

(2.36)

Note that we could just as well have derived the equation of motion for the field $\phi$ from the action (2.33). This would give us

$$
-\Box\phi + \frac{dV(\phi)}{d\phi} = 0,
$$

(2.37)

where

$$
\Box\phi = \frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi.
$$

(2.38)

Using the flat FLRW metric with $g_{\mu\nu} = (-1, a^2, a^2, a^2)$, we recover Eq. (2.36). Since we can derive the two equations (2.11) and (2.12) from the action (2.29) and we can express these two equations in terms of $\rho$ and $\phi$, we find that the equation of motion relates these two equations. Thus we have three equations: two Einstein equations derived by varying the action with respect to the metric $g^{\mu\nu}$ and one equation of motion for $\phi$. In chapter 4 we will actually solve the field equations for a specific matter action, and we will use the fact that only two of these equations are independent. To be precise, we will solve the equation for only $H$ and the field $\phi$ because we will be able to eliminate $H$ from these equations.
Chapter 3

Cosmological Inflation

3.1 Evolution of the universe

In the early days of cosmology the universe was thought to have undergone a certain evolution. The universe starts off with a Big Bang. Since the Big Bang itself is a singularity in space-time, we can not say too much about the Big Bang. The only thing we can say is that general relativity became important only at energies below the Planck scale with $M_P \approx 2.4 \times 10^{18}$ GeV, where $M_P$ is the reduced Planck mass, defined as

$$M_P^2 = \frac{1}{8\pi G_N}. \quad (3.1)$$

The reason is that above these energies the entire universe was so small (of the order of the Planck length $l_p = 1.6 \times 10^{-35}$ m) that quantum effects completely dominate the universe. Therefore above these energies we need a quantum theory of gravity, which so far has not yet been constructed. We call this era with extremely high energies and small length scales the Planck era.

At some point in this very early universe particles are created, which is a process that is not yet understood. For example, in Grand Unified Models heavy particles decay to the lighter quarks, whereas in inflation it is the inflaton that decays. Although we then still do not know when or how these other particles were created, it is safe to say that at high temperatures $T \sim 10^{16}$ GeV the universe was filled with a dense fluid of particles. At these temperatures the particles move at velocities close to the speed of light and are therefore called relativistic. The relativistic fluid of particles obeys an equation of state $p = \frac{1}{3} \rho$. Since these particles essentially behave like radiation, this era is called radiation era.

As the universe expands, it cools down and it undergoes a number of phase transitions. For example at the electroweak phase transition ($T \sim 100$ GeV) the gauge group of the Standard Model is spontaneously broken through the Higgs mechanism and the $W$ and $Z$ bosons acquire a mass whereas the photon remains massless. At the QCD transition ($T \sim 160$ MeV) the quarks are confined into baryons and mesons and for example the protons and neutrons form. Finally, at the scale $T \sim 160$ MeV nucleosynthesis takes place, where the lighter elements are created from protons and neutrons.

At a temperature of about $T \sim 1$ eV the energy density in matter and radiation are roughly equal. Since for radiation $p = \frac{1}{3} \rho$ and for matter $p = 0$, we see from Eq. (2.15) that $\rho_r \propto a^{-4}$ and $\rho_m \propto a^{-3}$. Since the scale factor always increases (the expansion rate of the universe $H$ is positive), we see that after the matter-radiation equality the energy density of matter dominates the total energy density. Therefore, this era is called matter era. Perhaps the most important event during this era was the recombination of protons in electrons.
into neutral hydrogen at a temperature $T \sim 3000$ K. Before this moment the photons were tightly coupled to electrons via Compton scattering. However, as soon as the neutral hydrogen formed the photons decoupled from the electrons and the universe became transparent. These photons from the time of decoupling we can still see today as a constant background radiation, also known as the Cosmic Microwave Background Radiation (CMBR). Since we are essentially looking back into a time where the universe was much smaller than today, the CMBR provides valuable information about the early universe.

### 3.2 Cosmological puzzles

The previous section is in principle a good description of the history of our universe and it has proved to be extremely useful for cosmologists. However, cosmologists could not explain certain puzzles within this standard history of the universe. As observations of our universe became better and better, these problems worsened up to a point where they could not be ignored. Let us explain the puzzles.

**Homogeneity puzzle:** This puzzle concerns one of the two main points of most cosmological models, namely the homogeneity of the universe on the largest scales. From the observation of the CMBR we can derive that the inhomogeneities are of the order of $10^{-4}$ on the Hubble length scale, which is roughly the size of our visible universe. However, gravity is an attractive force, and when stars, planets, galaxies and clusters formed we expect this to create many more inhomogeneities.

**Horizon puzzle:** To state this puzzle, first we have to define some important cosmological distances. The first is the Hubble radius, defined as

$$R_H \equiv \frac{c}{H},$$

where $c$ is the speed of light, which in our convention equals 1. The Hubble radius is a way to see whether particles are causally connected. If particles are separated by distances larger than the Hubble radius, they cannot communicate now. This does not necessarily mean that the particles were also out of causal contact at earlier times. To see this we can actually calculate the distance that light has traveled in a certain period of time by looking at the invariant line element in Eq. (2.5). For light $ds^2 = 0$, so we can integrate this expression to find the comoving distance at time $t$,

$$l_c(t) = \int_0^t \frac{cdt'}{a(t')},$$

where the time $t = 0$ is the time at which the light signal has been sent. This is not yet the physical distance, because physical distances have increased by a factor $a(t)$ due to the expansion of the universe. Thus the physical distance is

$$l_{phys}(t) = a(t) \int_0^t \frac{cdt'}{a(t')},$$

This physical distance is called the particle horizon, since it is the maximum distance from which light could have traveled from particles to an observer. In other words, if two particles are separated by a distance greater than $l_{phys}$, they were never in causal contact. So the situation can occur where particles cannot communicate now because their separation distance is larger than $R_H$, but they were in causal contact before because the distance that separates the particles is smaller than the particle horizon.
If we consider the evolution of the universe with a radiation and matter era, we can calculate both the Hubble radius and particle horizon. We use Table 2.1 and Eqs. (3.2) and (3.4) and find that during radiation era \( R_H = 2t \) and \( l_{\text{phys}} = 2t = R_H \). Thus in a radiation dominated universe the Hubble radius and the particle horizon are equal. During matter era \( R_H = \frac{3}{2}t \) and \( l_{\text{phys}} = 3t = 2R_H \), so the particle horizon is twice as big as the Hubble radius in a matter dominated universe. If we assume that our universe only went through an era of radiation and matter domination, we can safely say that the particle horizon is approximately equal to the Hubble radius and at most twice the Hubble radius. Taking this into account, plus the fact that \( R_H \) is roughly 13 billion lightyears, we find approximately 3000 causally disconnected regions in the CMB map of the universe. To be more precise, these are regions that are not in causal contact now because their separation distance is larger than \( R_H \), and were therefore also never in causal contact because the particle horizon is approximately equal to the Hubble radius. However when we look at the temperature fluctuations of the CMBR, we see that these are of the order of \( \delta T/T \sim 10^{-5} \). The horizon puzzle is: How can these temperature differences be so extremely small for regions that were never in causal contact?

**Flatness puzzle:** Some people do not consider this to be a real problem, but it is an interesting puzzle nonetheless. Our universe is on the largest scales described by the FLRW metric in Eq. (2.4) and observations from the CMBR support a flat universe with \( k = 0 \). But the FLRW metric was derived by assuming the most general metric for an expanding universe, and our universe could just as well have been described by a metric with \( k \neq 0 \) which has the same number of symmetries. How can we explain that our universe is so flat?

**Cosmic relics puzzle:** In the very early universe many cosmic relics could be created at extreme energy scales or phase transitions. Examples of cosmic relics are particles such as gravitons, gravitinos and the lightest supersymmetric particle \( \text{LSP} \). Other examples are topological defects such as domain walls, cosmic strings and the notorious magnetic monopole. The puzzle is that all of these cosmic relics, even the ones that are very long lived, have not been observed.

### 3.3 Inflation as solution of the puzzles

In 1981 Alan Guth[3] proposed a solution to the puzzles in the previous section. He added a new era to the history of the universe known as cosmological inflation. This inflationary era happened prior to the radiation era but below the Planck scale. The idea is that the universe undergoes a period of rapid accelerated expansion in which the universe grows to many orders of magnitude the size of the present universe. We explain now how inflation solves the puzzles:

The **homogeneity puzzle** is solved by noting that the inhomogeneities are formed because of the attractive gravitational interaction. In an inflationary universe the exponential expansion of the universe tends to pull the gravitationally interacting particles apart, such that the inhomogeneities do not form. When inflation ends, the universe is almost perfectly homogeneous. As we will see at the end of the chapter, quantum fluctuations can still cause small inhomogeneities in the homogeneous universe at the end of inflation. Eventually these small inhomogeneities will then form the large scale structures that we see today in our universe.

The **horizon puzzle** is solved in the following way. During (de Sitter) inflation the scale factor increases exponentially in time, i.e. \( a \propto e^{H_0 t} \) (see Table 2.1). We find that the particle
horizon during inflation is
\[ l_{\text{phys}}(t) = \frac{1}{H_0}(e^{H_0 t} - 1). \]  
(3.5)
Thus, the particle horizon increases exponentially with time. The Hubble radius however is constant during inflation (since \( H = \dot{a}/a = H_0 \) is constant) and is equal to \( R_H = 1/H_0 \). So we conclude that during de Sitter inflation the particle horizon grows exponentially with respect to the Hubble radius. If this happens the particle horizon can grow to many orders of magnitude beyond the Hubble radius. So particles that cannot communicate now, because their separation distance is much larger than \( R_H \), were able to communicate in the past because the particle horizon is much larger than the distance between the particles. This solves the horizon puzzle.

The **flatness puzzle** is solved by inflation as well. The FLRW equations can also be derived for the FLRW metric with \( k \neq 0 \) from Eq. (2.4). This changes the first of the FLRW equations to
\[ H^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2}, \]  
(3.6)
which we can rewrite as
\[ 1 = \frac{\rho}{\rho_{\text{crit}}} - \frac{k}{H^2 a^2} \equiv \Omega_{\text{total}} - \Omega_k, \]  
(3.7)
where the critical energy density, the density necessary for a flat universe with \( k = 0 \), is defined as
\[ \rho_{\text{crit}} = \frac{3H_i^2}{8\pi G_N}. \]  
(3.8)
\( H_i \) is the Hubble parameter today and is approximately \( 73 \pm 3 \) km s\(^{-1}\)Mpc\(^{-1}\). Measurements of the energy density of the universe give
\[ \Omega_k = \frac{k}{H^2 a^2} = 0.01 \pm 0.02, \]  
(3.9)
so the energy density in our universe is at this moment very close to the critical density. Now we want to find how the magnitude of the curvature term in Eq. (3.7) at earlier times. By again using Table 2.1, we find that during matter era \( H \alpha \propto t^{-\frac{1}{3}} \) and during radiation era \( H \alpha \propto t^{-\frac{1}{2}} \). Since \( k \) is constant, we conclude that the curvature term \( k^2/(Ha)^2 \) was much smaller in the early universe. This means that if we live in a universe with some \( k \neq 0 \), the energy density in the early universe must have been extremely fine tuned close to the critical density. Although a fine tuning problem is not a problem per se, it is highly unlikely that the energy density of the universe was so extremely close to the critical density such that we now measure a curvature term which is close to zero.

Inflation solves this puzzle by noting that during (de Sitter) inflation the factor \( Ha \propto e^{H_0 t} \).

The curvature term now scales as \( k/(Ha)^2 \propto e^{-2H_0 t} \), so at earlier times this term was much bigger. This also means that the energy density could have been far from the critical density and we do not have a fine tuning problem. A useful quantity to define now is the number of e-folds \( N(t) \),
\[ N(t) = \ln \left( \frac{a(t)}{a_i} \right) = \int_0^t H(t') dt', \]  
(3.10)
where \( a_i = a(t = 0) \). In one e-fold the scale factor has increased by a factor \( e \), which in the case of de Sitter inflation happens when \( t = 1/H_0 \). The flatness puzzle is solved when \( \Omega_k \approx 1 \) at some early time (before inflation), and one can calculate that the number of e-folds necessary for this is \( N(t) \approx 70 \). This solves the flatness puzzle.

Finally, the **cosmic relics puzzle** is solved by inflation. The solution is comparable to the
solution to the homogeneity puzzle. Suppose we have some initial energy density of cosmic relics $\rho_{\text{relics}}$. During inflation, the size of the universe increases by an enormous factor (the scale factor increases by a factor $e^N$, which is huge). Therefore, any preexisting energy density of cosmic relics is diluted to practically nothing and we will never observe cosmic relics such as magnetic monopoles.

3.4 Inflationary models

Inflation is basically an epoch during which the scale factor accelerates, thus the universe undergoes an accelerated expansion. Formally, inflation occurs when

$$\ddot{a} > 0.$$ (3.11)

Since $\frac{\ddot{a}}{a} = H + H^2$, this also means

$$\epsilon = -\frac{\dot{H}}{H^2} < 1.$$ (3.12)

From Eq. (2.8) we see that we need to have

$$\rho + 3p < 0.$$ (3.13)

This means that we need some special field with $p < -\frac{1}{3}\rho$, i.e. negative pressure. This is possible within certain particle physics models. An essential ingredient is a scalar field. As we know from the Standard Model, in scalar field models a phenomenon known as the Higgs mechanism can occur. The potential of a scalar field $\phi$ can be such that there are two minima and the scalar field is initially in the minimum with the lowest value (at $\phi = 0$). However, as time increases and the temperature drops, the second minimum at $\phi \neq 0$ can take a lower value than the initial minimum. We then say that the scalar field is in the false vacuum state and it wants to be in the true vacuum state. A phase transition will take place, in which the scalar field moves from the initial symmetric vacuum phase at $\phi = 0$ to the non-symmetric vacuum phase at $\phi \neq 0$. This is a process known as spontaneous symmetry breaking. Going back to the statement that we need negative pressure for successful inflation, first realize that negative pressure is not that unusual if you think of it as an attractive force. Basically, the scalar field is pulled out of its false vacuum state into the true vacuum, thus the scalar field experiences negative pressure.

Alan Guth used precisely the mechanism of spontaneous symmetry in his original article [3] on "old" inflation. He considers a first order phase transition, which means that there is a potential barrier between the false vacuum and the true vacuum. The idea is that the scalar field wants to move to the true vacuum, but it cannot because of the barrier, and it is trapped in the false vacuum state with some high temperature. The true vacuum will become lower and lower as the temperature drops, until finally at some low temperature the scalar field will finally move to the true vacuum. The latent heat is released and this generates a huge entropy production which solves the flatness and horizon puzzles. The flaw in this original model is that the bubbles of the true vacuum phase that are formed cannot overtake the rapid expansion of the universe. In other words, the universe expands faster that the bubbles grow, so the bubbles never coalesce and the universe never reaches the true vacuum state. Thus, inflation never ends, a problem known as the graceful exit problem.

Linde[6] and Albrecht and Steinhardt[7] proposed a new inflationary model, not surprisingly known as new inflation. In this scenario the universe will undergo a second order phase transition or a crossover, such that no bubbles are formed and the universe will as
Figure 3.1: Chaotic inflationary potential. The field has an initial value of $O(M_P)$ and slowly rolls down the potential well.

a whole move to the true vacuum state. In order for this not to happen almost instantaneously, the potential needs to be very flat. Moreover, the initial conditions of the field must be $\phi_0 = 0, \dot{\phi}_0 \approx 0$ and the field must slowly roll down the potential well. This leads to the so-called slow-roll conditions, which we will discuss in the next section. The flatness of the potential and the initial conditions for the field form the naturalness and fine-tuning problems for the new inflationary scenario.

One of the simplest scenario for inflation is chaotic inflation, proposed by Linde[8]. The idea of chaotic inflation is shown in Fig. 3.1. In this scenario, the universe was initially in a chaotic state where the value of the scalar field could take any value. This happened in the Planck era where quantum effects dominate the universe. In particular, the value of the field could be very large (i.e. $O(M_P)$). Below the Planck era quantum effects become subdominant and the scalar field behaves classically. This means that the field slowly rolls down the potential well and inflation is a success. The chaotic inflationary scenario allows for many forms of the potential, in particular a $m^2\phi^2$ or a $\lambda\phi^4$ potential.

### 3.5 Inflationary dynamics

In this section we will show more explicitly how inflation could happen in scalar field models. Assume again the simple scalar field action from Eq. (2.33) and the energy density and pressure from Eqs. (2.35). The scalar field we will now call the inflaton, the field that drives inflation. Using these expressions we find that for successful inflation we need

$$\dot{\phi}^2 \ll V(\phi). \tag{3.14}$$

In chaotic inflationary models the inflaton field starts at some large initial value and rolls down the potential well. In most of these models the potential energy is much greater than the kinetic energy, i.e. $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$. Also, it turns out that in many models the inflaton slowly rolls down the potential well. Taking the scalar field equation (2.36) into account, this means that $\dot{\phi} \ll 3H\phi$. The equation for $H^2$ and the field equation the become

$$H^2 \approx \frac{V(\phi)}{3M_p^2} \tag{3.15}$$

$$3H\dot{\phi} \approx -V'(\phi). \tag{3.16}$$
In general, we can define the so-called slow-roll parameters $\epsilon$ and $\eta$ by

\[ \epsilon = \frac{1}{2} M_p^2 \left( \frac{V'}{V} \right)^2, \]
\[ \eta = M_p^2 \frac{V''}{V}, \]

where $V' \equiv dV/d\phi$ and the Planck mass was defined in Eq. (3.1). The slow-roll conditions are then

\[ \epsilon \ll 1, \quad \eta \ll 1. \]

The qualitative argument for these slow-roll conditions is the following: in order for inflation to take place the potential energy must dominate the kinetic energy. This means that in a chaotic inflationary scenario (see Fig. 3.1) the scalar field should not roll down the potential well 'too fast'. In order to achieve this, the slope of the potential well should not be too steep. Since the slow-roll parameters in Eq. (3.17) and (3.18) contain derivatives of the potential, they therefore tell us something about the steepness of the slope. The slow-roll conditions (3.19) are such that the slope is not too steep.

Note the two different definitions for $\epsilon$ in Eq. (3.17) and (2.10). One can show that these are equal in the slow-roll approximation. Take the time derivative of Eq. (3.15) to get an equation for $\dot{H}$, eliminate the $\dot{\phi}$ term by substituting Eq. (3.16) and divide by $H^2$. You will recover Eq. (3.17).

In a simple inflationary scenario with a real scalar field in a quartic potential $V(\phi) = \lambda \phi^4$, we can easily see that the slow-roll conditions (3.19) are satisfied when the fields $\phi \sim O(10 M_P)$. In chaotic inflationary scenarios this is a typical initial value for the field $\phi$.

### 3.6 Constraints on inflationary models

So far we have only considered characteristics of general inflationary models. We have however not yet discussed constraints on inflationary models. Although inflation is a hypothetical era that happened in the very early universe, we fortunately can constrain inflationary models. These constraints follow from cosmological perturbation theory, see for example Refs. [9][10][11]. The idea is that in the early universe there are small quantum fluctuations of the scalar field and metric. During inflation some of these fluctuations are frozen in and inflation actually amplifies the vacuum fluctuations. The scalar metric perturbations are responsible for the large scale structure formation in the universe and leave their imprint on the spectrum of the CMBR. This effect can be measured and constrains the inflationary model. Let us sketch in a few steps how this works.

Consider a simple free scalar field theory with field equation

\[ -\Box \phi = \dot{\phi} + 3H\dot{\phi} - \frac{\nabla^2}{a^2} \phi = 0. \]

Suppose now that the spatial derivative term is large compared to the damping term with $H$, so we can neglect the second term. We now see that we have a wave equation with plane wave solutions. If we see the field $\phi$ as a small quantum fluctuation and Fourier transform the field, we see that for sub-Hubble wavelengths with $aH \ll k$ the Fourier-transformed field $\phi_k$ obeys a harmonic oscillator equation and fluctuates.

If the spatial derivative terms are small such that we can neglect the third term, we find that the term $3H\dot{\phi}$ leads to damping of $\phi$ with a characteristic decay rate proportional to $H$. This means that $\dot{\phi} \propto H\phi$, such that we find that $H^2 \gg \frac{k^2}{a^2}$ in this limit. These are
wavelengths that are larger than the Hubble radius and are therefore called super-Hubble wavelengths. The solution of $\phi_k$ in this limit is a constant plus a decaying mode. In other words, on super-Hubble scales with $aH \gg k$ the amplitude of field $\phi_k$ is approximately frozen in.

During inflation the physical length scale increases due to the expansion of the universe, but the Hubble radius remains almost constant. This means that quantum fluctuations with sub-Hubble wavelengths can become super-Hubble during inflation and their amplitudes are frozen in. In the Minkowski vacuum the amplitude of vacuum fluctuations scales as $\frac{k}{l}$, where $l$ is the physical length scale. An expansion of space implies that $l$ increases and that therefore the amplitude decreases. However, for super-Hubble modes the amplitude is frozen in during inflation and therefore the amplitude is effectively amplified with respect to the Minkowski vacuum. The scalar perturbations form gravitational potential wells that are the origin of the formation of large-scale structure. A measure for the amplitude of scalar perturbations then is

$$\langle \phi_k \phi_{k'} \rangle = \frac{1}{k^3} P_s(k) 2\pi^3 \delta^3(k - k'),$$

where $P_s(k)$ is the spectrum of scalar perturbations. It is important that $P_s(k)$ is evaluated at the first Hubble-crossing, which is the time that the quantum fluctuations become super-Hubble, so when $k = aH$. In a thorough calculation it can be shown that

$$P_\phi(k) = \Delta^2 \left( \frac{k}{k_0} \right)^{n_s - 1},$$

where $n_s$ is the called the spectral index and

$$\Delta^2 = \frac{V}{24\pi^2 M_P^4 \epsilon} \bigg|_{k = k_0},$$

is the amplitude of scalar perturbations at some pivot wavelength $k_0$. $V$ is here the inflaton potential evaluated at the first Hubble-crossing and $\epsilon$ is the slow-roll parameter from Eq. (3.17). The power spectrum from Eq. (3.20) can actually be measured from the CMBR. The idea is that the quantum fluctuations create small inhomogeneities in the otherwise homogeneous universe. These small inhomogeneities then form the origin of the formation of large scale structure in the universe. Through complicated but well understood effects these fluctuations leave their imprint on the spectrum of the CMBR. The amplitude of perturbations $\Delta^2$ can then be related to the fractional density perturbations $\frac{\delta \rho}{\rho}$, i.e.

$$\Delta^2 \sim \left( \frac{\delta \rho}{\rho} \right)^2.$$

Furthermore since $\rho \propto T^4$, one finds that $\frac{\delta \rho}{\rho} = 4 \frac{\delta T}{T}$. The fractional temperature perturbations in the CMBR are $\frac{\delta T}{T} = 10^{-5}$, and we then find

$$\Delta^2 = (2.445 \pm 0.096) \times 10^{-9},$$

measured by WMAP [12] at a reference wavelength $k_0 = 0.002\text{Mpc}^{-1}$. Because $\Delta^2$ is related to the inflaton potential, see Eq. (3.20), this gives us constraints on the inflaton potential. For a quartic $\lambda \phi^4$ potential, we find that $\frac{\delta \rho}{\rho} \propto \sqrt{\lambda}$ which gives $\lambda \sim 10^{-12}$. As we will see, in case of the SM Higgs boson the quartic self-coupling is of $\mathcal{O}(10^{-1})$ and is therefore too big to be the inflaton potential.
Figure 3.2: WMAP measurements of the spectral index $n_s$ and the tensor to scalar ratio $r$ at a pivot wavelength $k_0 = 0.002\text{Mpc}^{-1}$. The scale-free Harrison-Zeldovich spectrum is excluded by two standard deviations, as is the $\lambda\phi^4$ inflaton potential.

Now let us consider the other part of the power spectrum (3.20), namely the spectral index $n_s$. If the spectral index $n_s = 1$, the spectrum is said to be scale invariant and this spectrum is called the Harrison-Zeldovich spectrum. However, inflation predicts that the spectral index is not precisely 1, but there are small corrections to $n_s$ in terms of the slow-roll parameters. To be exact,

$$n_s = 1 - 6\epsilon + 2\eta.$$ 

The spectral index has been measured by the WMAP mission and it was found that $n_s = 0.951 \pm 0.016$ [12]. The slow-roll parameters $\epsilon$ and $\eta$ are expressed in terms of derivatives of the inflaton potential, see Eqs. (3.17) and (3.18), and the measurements of the spectral index therefore constrain the form of the inflaton potential.

There are more parameters that constrain the inflaton potential. The power spectrum in Eq. (3.20) corresponds to the spectrum of scalar perturbations, i.e. density perturbations. There are however also tensor perturbations, leading to gravitational waves, and the corresponding spectrum of tensor perturbations can again be calculated. We would then find that the power spectrum of tensor perturbations $P_T \propto k^{n_T}$. $n_T = 0$ for a Harrison-Zeldovich spectrum, but inflation predicts that it is $n_T = -2\epsilon$. Often one calculates the ratio of the tensor to scalar perturbations, and one finds that $r = -8n_T = 16\epsilon$. The tensor to scalar ratio $r$ has also been measured by the WMAP mission, and so far it is found that $r < 0.65$ [12], which means that primordial gravitational waves that leave their imprint on the CMBR have not been detected yet.

In Fig.3.2 we show the 1 and 2 $\sigma$ confidence contours of the spectral index $n_s$ and the tensor to scalar ratio $r$ at a pivot wavelength $k_0 = 0.02\text{Mpc}^{-1}$. These parameters have a specific value for a choice of the inflaton potential, because they are expressed in terms of the slow-roll parameters. We have shown the results for a massive $m^2\phi^2$ potential, a quartic $\lambda\phi^4$ potential and the scale-free Harrison Zeldovich spectrum. Both the scale free Harrison-Zeldovich spectrum and the spectrum for a quartic potential are excluded by over 95% confidence level. As we will see in the next chapter, the Higgs potential is effectively a $\lambda\phi^4$ potential during inflation, and is therefore excluded by the WMAP measurements.

In conclusion, we have found that there are constraints on the inflaton potential coming from measurements of the CMBR. These measurements require the inflaton to be a massive scalar particle with a $\lambda\phi^4$ inflaton potential. The quartic self-coupling should be $\lambda \sim 10^{-12}$. 

In the Standard Model, the only scalar particle is the Higgs boson, which during chaotic inflation has effectively a quartic potential with $\lambda = \mathcal{O}(10^{-1})$. The Standard Model Higgs boson is therefore excluded as the inflaton. In the next chapter we will see that the Higgs boson can still be the inflaton field if we introduce an extra term in our action that couples the scalar field to the Ricci scalar $R$. These models are called nonminimal inflationary models.
Chapter 4

Nonminimal inflation

In this chapter we will consider a certain class of inflationary models, which I will call nonminimal inflationary models. The idea is that we have an additional term in the inflaton action Eq. (2.33). This term is \( \frac{1}{2} \xi R \phi^2 \) and couples the scalar field to the Ricci scalar through a coupling constant \( \xi \). The action for the nonminimally coupled inflaton field is

\[
S_M = \int d^4x \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - V(\phi) - \frac{1}{2} \xi R \phi^2 \right).
\]

The sign of \( \xi \) is chosen such that for \( \xi = \frac{1}{6} \) the action is invariant under a conformal transformation. We will learn more about the conformal transformation in section 4.2. When \( \xi = 0 \), the coupling is said to be minimal. One way to interpret the \( \frac{1}{2} \xi R \phi^2 \) term when \( \xi \neq 0 \) is to see it as an additional mass term for the scalar field \( \phi \), with mass \( m^2 = \xi R \) that could be negative. The second interpretation is that the gravitational constant \( G_N \) is effectively changed and now becomes time dependent.

The nonminimal inflationary model is part of a wider class of models where a field is coupled to gravity, known as Brans-Dicke theories [13]. La and Steinhardt [14] used the Brans-Dicke theory and the interpretation that the gravitational constant changes in time to solve the graceful exit problem in the old inflationary scenario. In their scenario of extended inflation, the time-dependent gravitational constant effectively slows down inflation from exponential to powerlaw inflation. The bubbles formed can overtake the expansion of the universe and the phase transition will be completed.

Futamase and Maeda [15] have investigated the nonminimal inflation scenario for different values of \( \xi \) in chaotic models with potentials \( m^2 \phi^2 \) and \( \lambda \phi^4 \). They find that for successful inflation the nonminimal coupling \( \xi \lesssim 10^{-3} \). The intuitive reason is that for large positive \( \xi \) the effective gravitational constant can become negative in a chaotic inflationary scenario, making the theory unstable. Fakir and Unruh [16] calculated the amplitude of density perturbations in the nonminimal inflation scenario with a potential \( \lambda \phi^4 \). When \( \xi = 0 \), the amplitude of density perturbations \( \delta \rho/\rho \propto \sqrt{\lambda} \) and the self-coupling of the field \( \lambda \) must be tuned to an extremely small value of \( < 10^{-12} \) to agree with observations (see section 3.6). However, when the coupling is non-minimal, Fakir and Unruh find that the amplitude of density perturbations is proportional to \( \sqrt{\lambda/\xi^2} \). To calculate this Fakir and Unruh first write the action in the Einstein frame where the inflaton field is minimally coupled to gravity, which is related to the Jordan frame with nonminimal coupling through a conformal transformation. The proportionality of the amplitude of density perturbations to \( \sqrt{\lambda/\xi^2} \) means that \( \lambda \) can be much bigger as long as the nonminimal coupling \( \xi \) is sufficiently large. Note that the investigation for successful inflation by Futamase and Maeda does not exclude a large negative value of \( \xi \)!

Salopek, Bond and Bardeen [17] have done an extensive
research of density perturbations in nonminimal models. They generalize the single field case with a coupling \( \frac{1}{2} \xi R \phi^2 \) to multiple fields coupled with some function \( f(\phi_i) \) to the Ricci scalar. Komatsu and Futamase [18, 19] have calculated the spectrum of gravitational waves in the nonminimal model with large negative \( \xi \) and their effect on the CMB. This gives a constraint on the initial value of the inflaton field.

Recently, Bezrukov and Shaposhnikov [4] proposed the only scalar particle in the Standard Model, the Higgs boson, as the inflaton. This is possible when the Higgs boson is nonminimally coupled to gravity with a large negative \( \xi \). By making use of the conformal transformation they rewrite the action in the Jordan frame (Eq. (4.1)) to the action in the Einstein frame. In this frame the inflaton potential takes an asymptotically flat form that leads to successful inflation for large values of the field and is the familiar Higgs potential for small field values. In [20, 21, 22, 23, 24] one and two loop radiative corrections to the effective potential have been calculated. It is shown that these do not destroy the flatness of the potential for large field values. The radiative corrections to the effective potential constrain the Higgs mass to lie in the range \( 124 \text{ - } 194 \) GeV and this could be tested at the LHC.

The outline for this chapter will be the following: in section 4.1 we will derive the Friedmann equations and the field equation for the action (4.1). Then we will take the limit of large negative \( \xi \) and we will find that we can solve the equations analytically in the slow-roll approximation. In section 4.1.3 we will show numerical solutions of the field equations and show that inflation is successful. In section 4.2 we will introduce the conformal transformation. Then in section 4.3 we will introduce the Higgs boson as the inflaton as proposed by Bezrukov and Shaposhnikov and we will apply the conformal transformation to make the transformation to the Einstein frame. Finally in section 4.4 we will make a simple supersymmetric extension of the Higgs model, which is the two-Higgs doublet model. We will again derive the Friedmann and field equations for these two doublets and solve the equations numerically. As we will see, one linear combination of the two physical Higgs bosons is a rapidly oscillating mode that quickly decays. The other linear combination serves as the inflaton in the two-Higgs doublet model and leads to successful inflation.

### 4.1 Real scalar field

#### 4.1.1 Dynamical equations

We start with the action (4.1) and we add the Einstein Hilbert action from Eq. (2.24). The total action then becomes

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} M_p^2 R - \frac{1}{2} g^{\mu\nu} \left( \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} \xi \phi^2 \right) - \frac{1}{2} \left( \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} \xi \phi^2 \right) - \frac{1}{2} \left( \partial_{\mu} \phi \partial_{\nu} \phi + g_{\mu\nu} \frac{1}{2} \xi \phi^2 \right) \right],
\]

where \( M_p^2 = (8\pi G_N)^{-1} \). We now derive the Einstein equations by varying this action with respect to the metric. We again use Eqs. (2.25) and (2.26). The difference is that this time there is a scalar field that is coupled to the Ricci scalar, so the covariant derivatives will not vanish. We find

\[
\delta S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \left( \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} \xi \phi^2 \right) - \frac{1}{2} \left( \partial_{\mu} \phi \partial_{\nu} \phi + \frac{1}{2} \xi \phi^2 \right) \right] \delta g^{\mu\nu},
\]
where \( G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} \). If we now use that we want the variation of the action to vanish, we find

\[
(M_p^2 - \xi \phi^2) G_{\mu \nu} - (\partial_\mu \phi)(\partial_\nu \phi) + g_{\mu \nu} \left[ \frac{1}{2} g^{\rho \sigma} (\partial_\rho \phi)(\partial_\sigma \phi) + V(\phi) \right] + \xi (\nabla_\mu \nabla_\nu - g_{\mu \nu} g^{\rho \sigma} \nabla_\rho \nabla_\sigma) \phi^2 = 0. \tag{4.3}
\]

We write the term containing the covariant derivatives explicitly

\[
(\nabla_\mu \nabla_\nu - g_{\mu \nu} g^{\rho \sigma} \nabla_\rho \nabla_\sigma) \phi^2 = 2 \left[ \nabla_\mu (\phi \nabla_\nu \phi) - g_{\mu \nu} g^{\rho \sigma} \nabla_\rho (\phi \nabla_\sigma \phi) \right] = 2 \phi (\nabla_\mu \nabla_\nu - g_{\mu \nu} g^{\rho \sigma} \nabla_\rho \nabla_\sigma) \phi + 2(\nabla_\mu \phi)(\nabla_\nu \phi) - g_{\mu \nu} g^{\rho \sigma} (\nabla_\rho \phi)(\nabla_\sigma \phi)).
\]

Moreover, we use that

\[
g^{\rho \sigma} \nabla_\rho \nabla_\sigma \phi = \Box \phi = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu \nu} \partial_\nu \phi = -\ddot{\phi} - 3H \dot{\phi} + \frac{1}{a^2} \dddot{\phi}. \tag{4.5}
\]

and

\[
\nabla_\mu \nabla_\nu \phi = \partial_\mu \partial_\nu \phi - \Gamma^i_{\mu \nu} \partial_\phi.
\tag{4.6}
\]

When we assume a homogeneous and isotropic universe with \( \phi = \phi(t) \), we find for the 00 component that Eq. (4.4) reduces to

\[
(M_p^2 - \xi \phi^2) 3H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi) + 6\xi H \phi \dot{\phi}
\]

which gives us the constraint equation

\[
H^2 = \frac{1}{3(M_p^2 - \xi \phi^2)} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) + 6\xi H \phi \dot{\phi} \right]. \tag{4.7}
\]

This equation is sometimes called the energy constraint equation, since it contains \( T_{00} \), the 00 component of the stress-energy tensor which is the energy density. If we take the spatial diagonal components of \( G_{\mu \nu} \), we find the equation

\[
(M_p^2 - \xi \phi^2)(-2H - 3H^2) = \frac{1}{2} - 2(\phi)(\phi') + 2\xi \phi \dot{\phi} - 4\xi H \phi \dot{\phi}
\]

Upon inserting Eq. (4.7) into this equation, we find the equation for \( H \)

\[
H = \frac{1}{(M_p^2 - \xi \phi^2)} \left[ -\frac{1}{2} \dot{\phi}^2 + \xi \phi^2 + \xi \phi \dot{\phi} - \xi H \phi \dot{\phi} \right]. \tag{4.8}
\]
Finally, we can also vary the action with respect to the field \( \phi \) to obtain the equation of motion for \( \phi \). The equation of motion for \( \phi \) is

\[
\ddot{\phi} + 3H\dot{\phi} + \xi R + \frac{dV(\phi)}{d\phi} = 0.
\] (4.9)

The Ricci scalar \( R \) can be expressed in terms of \( H^2 \) and \( \dot{H} \) (see appendix, Eq. (B.17)). We now want to express the Ricci scalar in terms of \( \phi \) and \( \phi \) alone. We first substitute Eq. (4.9) into Eq. (4.8) and obtain

\[
H = \frac{1}{(M_P^2 - \xi \phi^2)} \left[ -\frac{1}{2} \phi^2 + \xi \dot{\phi}^2 - 4\xi H \phi \dot{\phi} - 6\xi^2 (H + 2H^2) \phi^2 - \xi \phi \frac{dV(\phi)}{d\phi} \right],
\]

which allows us to write

\[
H = \frac{1}{(M_P^2 - \xi (1 - \xi) \phi^2)} \left[ -\frac{1}{2} \phi^2 + \xi \dot{\phi}^2 - 4\xi H \phi \dot{\phi} - 12\xi^2 H^2 \phi^2 - \xi \phi \frac{dV(\phi)}{d\phi} \right].
\]

Then \( R \) is

\[
R = 6(H + 2H^2)
\]

\[
= \frac{6}{(M_P^2 - \xi (1 - \xi) \phi^2)} \left[ -\frac{1}{2} \phi^2 + \xi \dot{\phi}^2 - 4\xi H \phi \dot{\phi} - 12\xi^2 H^2 \phi^2 - \xi \phi \frac{dV(\phi)}{d\phi} \right] + 2H^2
\]

\[
= \frac{6}{(M_P^2 - \xi (1 - \xi) \phi^2)} \left[ -\frac{1}{2} \phi^2 + \xi \dot{\phi}^2 - 4\xi H \phi \dot{\phi} + 2(M_P^2 - \xi \phi^2) H^2 - \xi \phi \frac{dV(\phi)}{d\phi} \right]
\]

\[
= \frac{1}{(M_P^2 - \xi (1 - \xi) \phi^2)} \left[ -\phi^2 + 6\xi \dot{\phi}^2 + 4V(\phi) - 6\xi \phi \frac{dV(\phi)}{d\phi} \right].
\] (4.10)

Note that for a quartic potential \( V(\phi) = \frac{1}{4} \lambda \phi^4 \) the Ricci scalar vanishes for the special value \( \xi = \frac{1}{6} \). This is the conformal coupling value of \( \xi \) in 4 dimensions. The vanishing of \( R \) suggests that a conformally coupled massless scalar field behaves as in flat space, i.e. it does not feel the expansion of the universe. In fact, one can reduce the field equation for \( \phi \) in Eq. (4.9) for \( R = 0 \) to the field equation for a scalar field in flat space by a conformal rescaling of the scalar field by the scale factor. We will come back to and show this in section 4.2.

Let us continue by inserting Eq. (4.10) into Eq. (4.9). We find that the field \( \phi \) obeys the equation of motion

\[
\ddot{\phi} + 3H\dot{\phi} + \frac{-\xi (1 - \xi) \phi}{(M_P^2 - \xi (1 - \xi) \phi^2)} \dot{\phi}^2 = \frac{1}{(M_P^2 - \xi (1 - \xi) \phi^2)} \left[ -4\xi \phi V(\phi) - (M_P^2 - \xi \phi^2) \frac{dV(\phi)}{d\phi} \right].
\] (4.11)

In the next section we will solve this equation analytically by using the slow-roll approximations and the assumption that \( \xi \ll -1 \).

### 4.1.2 Analytical solutions of the dynamical equations

To be complete I will give again the three equations we obtained from the action (4.2). First the constraint equations (4.7) for \( H^2 \) and the dynamical equation (4.8) for \( H \),

\[
H^2 = \frac{1}{3(M_P^2 - \xi \phi^2)} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) + 6\xi H \phi \dot{\phi} \right]
\]

\[
H = \frac{1}{(M_P^2 - \xi \phi^2)} \left[ -\frac{1}{2} \phi^2 + \xi \dot{\phi}^2 + \xi \phi \dot{\phi} - \xi H \phi \dot{\phi} \right],
\]
and the equation of motion for $\phi$ from Eq. (4.9),

$$\ddot{\phi} + 3H\dot{\phi} + \zeta R\phi + \frac{dV(\phi)}{d\phi} = 0.$$  

In section 2.3 I have shown that only two of these equations are independent because of the Bianchi identity which leads to conservation of the energy-momentum tensor. So let us choose and solve only two of these equations. Since we are mostly interested in the dynamics of the field $\phi$ and this equation of motion contains $H$, we will solve the equations for $\phi$ and $H^2$. First of all we will employ the slow-roll conditions introduced in section 3.5, which in this case translates to

$$\ddot{\phi} \ll H\dot{\phi}$$

$$\dot{\phi} \ll H\phi$$

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi).$$  

(4.12)

In words, the potential energy dominates the kinetic energy and the field slowly rolls down the potential well, such that the acceleration of the field vanishes and the velocity of the field is small. The equation of motion for $\phi$ (4.11) then becomes

$$3H\dot{\phi} = \frac{1}{(M_p^2 - \zeta(1 - 6\zeta)\phi^2)} \left[-4\xi\phi V(\phi) - (M_p^2 - \zeta\phi^2)\frac{dV(\phi)}{d\phi}\right],$$  

(4.13)

while the energy constraint equation simplifies to

$$H^2 = \frac{1}{3(M_p^2 - \zeta\phi^2)} \left[V(\phi) + \frac{2\xi\phi}{(M_p^2 - \zeta(1 - 6\zeta)\phi^2)} \left(-4\xi\phi V(\phi) - (M_p^2 - \zeta\phi^2)\frac{dV(\phi)}{d\phi}\right)\right].$$  

(4.14)

Note that we have not justified the use of the slow-roll conditions. Indeed, not in every model the slow-roll conditions are satisfied. Komatsu and Futamase [18] have shown that for large negative nonminimal coupling and a massive potential term $V(\phi) = \frac{1}{2}m^2\phi^2$, the slow-roll conditions are not satisfied and there will not be successful inflation. However, for $\xi \ll -1$ and a $\frac{1}{4}\lambda\phi^4$ potential, the slow-roll conditions are met and inflation takes place. This we will show now explicitly. Moreover, in section 4.1.3 we will solve the full field equation (without making any approximations) numerically and we will see that inflation is a success.

To continue, we will now introduce a potential

$$V(\phi) = \frac{1}{4}\lambda\phi^4,$$  

(4.15)

such that the equations for $\phi$ and $H^2$ become

$$3H\dot{\phi} \approx \frac{-M_p^2\lambda\phi^3}{(M_p^2 - \zeta(1 - 6\zeta)\phi^2)}$$

$$H^2 \approx \frac{\lambda\phi^4}{12(M_p^2 - \zeta\phi^2)} \left[1 - \frac{8M_p^2\xi}{(M_p^2 - \zeta(1 - 6\zeta)\phi^2)}\right].$$  

(4.16)

Now we will assume large negative nonminimal coupling,

$$\xi \ll -1,$$  

(4.17)
such that we get for Eqs. (4.18)

\[ \dot{\phi} \approx -\frac{M^2_P \lambda \phi}{3H\xi(1-6\xi)}, \]
\[ H^2 \approx -\frac{\lambda \phi^2}{12\xi}. \] (4.18)

From the second equation we find that

\[ H \approx \sqrt{-\frac{\lambda}{12\xi}} \phi, \] (4.19)

which we can substitute in the first equation and we obtain

\[ \dot{\phi} \approx -\frac{4M^2_P}{1-6\xi} \sqrt{-\frac{\lambda}{12\xi}} \phi. \] (4.20)

It is now easy to see that the slow-roll conditions (4.12) are satisfied. We can easily solve Eq. (4.20) and we get

\[ \phi(t) = \phi_{in} - \frac{4M^2_P}{1-6\xi} \sqrt{-\frac{\lambda}{12\xi}} t, \] (4.21)

where \( \phi_{in} = \phi(t = 0) \) is the initial value of the field. The Hubble rate is now also easily solved by using Eq. (4.19),

\[ H(t) = H_{in} - \frac{4M^2_P H_{in}^2}{(1-6\xi)\phi_{in}^2} t, \] (4.22)

with

\[ H_{in} = \sqrt{-\frac{\lambda}{12\xi}} \phi_{in}. \]

By using that \( H = \frac{\dot{a}}{a} \), we can solve Eq. (4.22) for the scale factor. This gives

\[ a(t) = a_{in} \exp\left[H_{in} t - \frac{2M^2_P H_{in}^2}{(1-6\xi)\phi_{in}^2} t^2 \right], \] (4.23)

where again \( a_{in} = a(t = 0) \). We can also put an estimate on the initial value of the field \( \phi \) by looking at the number of e-folds \( N \). We calculate

\[ N = \int H dt = \int_{\phi_{in}}^{\phi_f} \frac{H}{\dot{\phi}} d\phi = -\int_{\phi_{in}}^{\phi_f} \frac{1-6\xi}{4M^2_P} \phi d\phi = -\frac{1-6\xi}{8} \frac{1}{M^2_P} (\phi_f^2 - \phi_{in}^2). \] (4.24)

If we now want the number of e-folds to be \( \geq 70 \) and we assume that inflation ends when the value of the field at the final time \( \phi_f \approx 0 \), we find that for successful inflation we need

\[ -\frac{6\xi}{8} \left( \frac{\phi_{in}}{M_P} \right)^2 \geq 70. \] (4.25)

We see that more negative \( \xi \) is, the smaller the constraint on the initial value for the scalar field \( \phi \). For our choice \( \xi = -10^3 \), the initial value of the field to solve the flatness and horizon puzzles must be \( \phi_i \geq 0.3 M_P \).
4.1.3 Numerical solutions of the field equation

In this section I will show numerical solutions of the field equation (4.11) and $H$ solved from Eq. (4.7). In Fig. 4.1 we show the evolution of the inflaton field $\phi$ as a function of time and the number of e-folds $N$ for $\xi = -10^3$. We can easily see that $\dot{\phi}$ is constant during inflation, such that $\ddot{\phi} \approx 0$. Furthermore, we see that with an initial value of $\phi = 0.4\, M_P$ the number of e-folds $N > 70$, thus inflation is successful. At the end of inflation the scalar inflaton field starts to oscillate at the minimum of the potential and the slow-roll approximation is no longer valid. In this oscillatory regime preheating and reheating of the universe will occur, which we will briefly discuss at the end of this section.

As a comparison we have calculated the evolution of $\phi$ and the number of e-folds when $\xi = 0$ in Fig. 4.2. We see that the initial value of the field $\phi$ must be much larger in order to have successful inflation. The inflation rolls down the potential well in an extremely short period of time after which it starts to oscillate rapidly. The difference with the nonminimal case can be explained by noticing that the $\xi R \phi$ term in the field equation (4.9) acts as a mass term with negative mass. This effectively slows down inflation, i.e. the inflaton rolls down the potential well much slower than in the minimal case. Thus the inflationary period is elongated through the nonminimal coupling and inflation is successful for relatively small initial values of $\phi$.

The effect of the nonminimal coupling term can be seen more clearly when looking at the Hubble parameter $H$. First of all, the proportionality of $H$ with the inflaton field $\phi$ (see Eq. (4.19)) is verified with great accuracy. This justifies the use of the slow-roll approximations. When $\phi$ enters the oscillatory regime, we notice an interesting behavior of the Hubble parameter. In Fig. 4.4a we can see that the Hubble parameter makes sharp drops to (almost) zero in the oscillatory regime. This behavior can be explained by looking at the expression for $H^2$ in Eq. (4.7) and writing this as

$$H^2 = A - BH,$$  \hspace{1cm} (4.26)
Figure 4.2: Numerical solution of the field $\phi$ and number of e-folds $N$ for $\xi = 0$. The self-coupling $\lambda = 10^{-3}$. The initial value of the field is $\phi_{in} = 25$, in units of $M_p$. The initial velocity of the field is taken to be zero. Important differences with the nonminimal case are the observation that the field $\phi$ starts to oscillate almost immediately and that the initial value of the field must be much larger in order to meet the condition $N \geq 70$ which is necessary for successful inflation.

\[
A = \frac{\frac{1}{2} \dot{\phi}^2 + V(\phi)}{3(M_p^2 - \xi \phi^2)} \quad (4.27)
\]
\[
B = -\frac{6 \xi \phi \dot{\phi}}{3(M_p^2 - \xi \phi^2)} \quad (4.28)
\]

Note that $A$ is always positive and $B > 0$ during inflation since $\xi < 0$ and $\phi, \dot{\phi} > 0$. Thus we can write for $H$

\[
H = -\frac{B}{2} + \sqrt{\left(\frac{B}{2}\right)^2 + A}, \quad (4.29)
\]

where we have chosen only the positive solution for $H$. Now we can make certain approximations. Suppose that $A \gg (B/2)^2$. This roughly happens when $\lambda \phi^2 \gg (\xi \dot{\phi})^2$, i.e. when the fields are large and its derivatives small (slow-roll regime). Then we can approximate $H$ by

\[
H = -\frac{B}{2} + \sqrt{\frac{B}{2}^2 + 4A} \approx \sqrt{A} \quad \text{when} \quad A \gg \left(\frac{B}{2}\right)^2. \quad (4.30)
\]

Thus for large $\xi$,

\[
H \approx \sqrt{\frac{V(\phi)}{-3 \xi \phi^2}} = \sqrt{-\frac{\lambda \phi^2}{12 \xi}}, \quad (4.31)
\]

which is of course the same result as we got previously for $H$ in the inflationary regime. The situation changes however when the fields are very close to the minimum of the potential. Then $\lambda \phi^2 \ll (\xi \dot{\phi})^2$, which means $(B/2)^2 \gg A$. We can then approximate $H$ by

\[
H = -\frac{B}{2} + \frac{B}{2} \sqrt{1 + \frac{4A}{B^2}} \\
\approx -\frac{B}{2} + \frac{B}{2} \left(1 + \frac{2A}{B^2}\right) \\
= -\frac{B}{2} + \frac{|B|}{2} \frac{A}{|B|} \quad \text{when} \quad A \ll \left(\frac{B}{2}\right)^2. \quad (4.32)
\]
Figure 4.3: Numerical solution of the Hubble parameter $H$ with the same constants and initial conditions as in Fig. 4.1. $H$ is given in units of $M_P$ and $t$ in units of $M_P^{-1}$. In Fig. 4.3b we approximate $H$ during inflation as in Eq. (4.19). The proportionality of $H$ with $\phi$ is numerically verified in Fig. 4.3a. When $\phi$ enters the oscillatory regime, $H$ drops to (almost) zero.

Figure 4.4: Numerical solution of the Hubble parameter $H$ with the same constants and initial conditions as in Fig. 4.1 in the oscillatory regime. When the field gets close to the minimum of the potential, the Hubble rate drops to almost zero. The approximation of $H$ as in Eq. (4.32) is numerically verified.

Note that $B$ is positive as long as the field is rolling down the potential well, since then the sign of $\phi$ and $\dot{\phi}$ is the same. In Fig. 4.4b the approximation in Eq. (4.32) is confirmed. The fact that the Hubble rate drops to almost zero, a so-called loitering phase where the field $\phi$ nonadiabatically changes, has interesting consequences for the preheating of the nonminimally coupled inflaton. We will not discuss the preheating of the nonminimally coupled inflaton here, but instead give the main results of the research by Tsujikawa, Maeda and Torii in [25]. They find that the nonminimally coupled inflaton alters the preheating process in two ways. First of all the frequency oscillations of the inflaton field decreases with respect to the minimal case. We can see this when we compare Figs. 4.1a and 4.2a. This causes a delay in the growth of quantum fluctuations of the inflaton. Also, the structure of resonance is different compared to the minimal case. For sufficiently large negative $\zeta$, the energy transfer of the classical inflaton field to the quantum fluctuations is much more efficient and reaches its maximum around $\zeta = -200$. For $\zeta < -200$, the final variance of the fluctuations decreases because the initial value of the inflaton field becomes smaller. This concludes our short discussion on preheating of the nonminimally coupled inflaton field. We will shortly turn to the idea of Bezrukov and Shaposhnikov [4] that the Higgs boson could be the inflaton in nonminimal models. First we must introduce the conformal transformation.
4.2 The conformal transformation

The conformal transformation is defined by a transformation of the metric

\[ g_{\mu\nu}(x) \rightarrow \bar{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x). \]  

(4.32)

This changes several quantities that appear in general relativity, see appendix B.1. First of all we have the transformations

\[ \bar{g}^{\mu\nu}(x) = \Omega^{-2}(x)g^{\mu\nu}(x) \]

\[ \sqrt{-\bar{g}} = \Omega^D \sqrt{-g}. \]  

(4.33)

The Christoffel connection, defined in Eq. (B.3), then transforms as

\[ \bar{\Gamma}^a_{\mu\nu} = \Gamma^a_{\mu\nu} + \left( \delta^a_{\mu} \partial_{\nu} + \delta^a_{\nu} \partial_{\mu} - g_{\mu\nu} g^{a\lambda} \partial_{\lambda} \right) \ln \Omega. \]  

(4.34)

The Ricci tensor is defined in Eq. (B.6). After a little effort we find that the Ricci tensor in \( D \) dimensions becomes after a conformal transformation

\[ \bar{R}_{\mu\nu} = R_{\mu\nu} + \left( (D-2) \partial_{\mu} \partial_{\nu} - g_{\mu\nu} \Box \right) \ln \Omega \]

\[ + (D-2) \left( \partial_{\mu} \ln \Omega \partial_{\nu} \ln \Omega - g_{\mu\nu} \bar{g}^{\rho\lambda} \partial_{\rho} \ln \Omega \partial_{\lambda} \ln \Omega \right), \]  

(4.35)

where the \( \Box \) operator is defined as

\[ \Box = g^{\rho\lambda} \nabla_\rho \nabla_\lambda. \]  

(4.36)

with

\[ g^{\rho\lambda} \nabla_\rho \nabla_\lambda \phi = g^{\rho\lambda} (\partial_\rho \partial_\lambda \phi - \Gamma^a_{\rho\lambda} \partial_\sigma \phi). \]  

(4.37)

Finally the Ricci scalar from Eq. B.7 becomes

\[ \bar{R} = \Omega^{-2} \left[ R - 2(D-1) \Box \ln \Omega - (D-2)(D-1) g^{\rho\lambda} (\partial_\rho \ln \Omega \partial_\lambda \ln \Omega) \right]. \]  

(4.38)

We now want to see how an action transforms under a conformal transformation. Let us consider the action of a nonminimally coupled scalar field in \( D \) dimensions, see Eq. (4.1). After the conformal transformation the action takes the form

\[ S = \int d^Dx \sqrt{-\bar{g}} \left( -\frac{1}{2} \Omega^{2-D} \bar{g}^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} \xi \Omega^{2-D} \bar{R} \phi^2 - \Omega^{-D} V(\phi) \right) \]

\[ - (D-1) \Omega^{2-D} \phi^2 \left( \Box \ln \Omega + \frac{1}{2} (D-2) \bar{g}^{\rho\lambda} (\partial_\rho \ln \Omega \partial_\lambda \ln \Omega) \right), \]  

(4.39)

where \( \Box \) now contains the metric \( \bar{g}_{\mu\nu} \). Note that we can bring this closer to the original form by rescaling the field itself as

\[ \phi \rightarrow \hat{\phi} = \Omega^{\frac{D}{2}} \phi. \]  

(4.40)

The derivative of \( \phi \) then becomes

\[ \partial_\mu \phi = \partial_\mu \left( \Omega^{\frac{D-2}{2}} \hat{\phi} \right) = \Omega^{\frac{D-2}{2}} (\partial_\mu \hat{\phi} + \hat{\phi} \partial_\mu \ln \Omega) \]  

(4.41)
such that the kinetic term transforms as

\[ \sqrt{-g} g^{\mu \nu} (\partial_\mu \phi)(\partial_\nu \phi) = \sqrt{-\tilde{g}} \tilde{g}^{\mu \nu} (\partial_\mu \phi)(\partial_\nu \phi) + \frac{1}{4}(D-2)^2(\partial_\mu \ln \Omega)(\partial_\nu \ln \Omega) \]

\[ + \frac{D-2}{2}(\phi(\partial_\mu \phi)\partial_\nu \ln \Omega + \phi(\partial_\nu \phi)\partial_\mu \ln \Omega) \]

\[ = \sqrt{-g} g^{\mu \nu} (\partial_\mu \phi)(\partial_\nu \phi) + \frac{1}{4}(D-2)^2\phi^2(\partial_\mu \ln \Omega)(\partial_\nu \ln \Omega) \]

\[ + \frac{D-2}{2}(\partial_\mu \phi^2)\partial_\nu \ln \Omega \]

\[ = \frac{1}{2}(D-2)\tilde{g}^{\mu \nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{D-2}{2} \sqrt{-\tilde{g}} \phi^2 \left[ \Box \ln \Omega + \frac{1}{2}(D-2)\tilde{g}^{\mu \nu} (\partial_\mu \ln \Omega)(\partial_\nu \ln \Omega) \right]. \]

In the third step I have partially integrated the last term and dropped the boundary term. In the final step we have used that

\[ \frac{1}{\sqrt{-\tilde{g}}} \partial_\rho \sqrt{-\tilde{g}} \tilde{g}^{\rho \lambda} \partial_\lambda = \Box = \tilde{g}^{\rho \lambda} \tilde{\nabla}_\rho \tilde{\nabla}_\lambda. \quad (4.42) \]

Using the new rescaled fields we find that the action is

\[ S = \int d^D x \sqrt{-\tilde{g}} \left\{ -\frac{1}{2} \tilde{g}^{\mu \nu} (\partial_\mu \tilde{\phi})(\partial_\nu \tilde{\phi}) - \frac{1}{2} \tilde{R} \tilde{\phi}^2 - \Omega^2 V(\Omega \tilde{\phi}) \right\} \]

\[ + \left\{ \frac{D-2}{4} - (D-1)\zeta \right\} \tilde{\phi}^2 \left[ \Box \ln \Omega + \frac{1}{2}(D-2)\tilde{g}^{\mu \nu} (\partial_\mu \ln \Omega)(\partial_\nu \ln \Omega) \right]. \quad (4.43) \]

Now we insert a general potential into the action

\[ V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4, \quad (4.44) \]

such that the action is

\[ S = \int d^D x \sqrt{-\tilde{g}} \left\{ -\frac{1}{2} \tilde{g}^{\mu \nu} (\partial_\mu \tilde{\phi})(\partial_\nu \tilde{\phi}) - \frac{1}{2} \tilde{R} \tilde{\phi}^2 - \Omega^2 V(\Omega \tilde{\phi}) \right\} \]

\[ + \left\{ \frac{D-2}{4} - (D-1)\zeta \right\} \tilde{\phi}^2 \left[ \Box \ln \Omega + \frac{1}{2}(D-2)\tilde{g}^{\mu \nu} (\partial_\mu \ln \Omega)(\partial_\nu \ln \Omega) \right]. \quad (4.45) \]

If \( \zeta \) now has the special value

\[ \zeta = \frac{D-2}{4(D-1)}, \quad (4.46) \]

we find that the action is invariant in \( D \neq 4 \) dimensions if \( m^2 = 0 \) and \( \lambda = 0 \). However, in 4 dimensions we see that the action is invariant under a conformal transformation when \( m^2 = 0 \) and \( \zeta = \frac{1}{6} \). Thus, the action Eq. (4.1) with a \( \lambda \phi^4 \) potential is conformally invariant if \( \zeta = \frac{1}{6} \).

As an example we will now consider the conformal FLRW metric, see section B.3. The metric has the form \( g_{\mu \nu}(x) = a^2(\eta) \eta_{\mu \nu} \), so we can recognize \( a(t) = \Omega(x) \). Eqs. (B.21), (B.23) and (B.25) can now easily be derived if we appreciate that in a flat Minkowsky space time the Christoffel connection vanishes (in Cartesian coordinates). We have seen that the action in
Eq. (4.44) is invariant under a conformal transformation for a massless scalar field and for the special value \( \xi = \frac{1}{6} \). We say that such a scalar field is conformally coupled. This means that for a massless conformally coupled scalar field we transform back to the Minkowski metric by a conformal transformation and the action will be the same with this new metric. Note that we only considered the matter part of the action, but we also have to include the usual the Einstein-Hilbert action. This part is not invariant under a conformal transformation. However, since the field \( \phi \) does not occur in this part, the dynamics of \( \phi \) (the equation of motion) are not altered by this part. So the equation of motion for a massless conformally coupled scalar is in an expanding FLRW universe equal to the equation in a Minkowski space time. As a consequence, a massless conformally coupled scalar field does not feel the expansion of the universe. Other examples are the photon in 4 dimensions or the massless fermion in \( D \)-dimensions. The latter we will actually show in chapter 6.

This concludes our general introduction of the conformal transformation. Let us for a moment return to the statements in the introduction to this chapter. We mentioned that Fakir and Unruh calculated the amplitude of density perturbations in Ref. [16]. In the minimal inflationary model this amplitude is found to be \( \delta \rho / \rho \propto \sqrt{\lambda} \), see Eq. (3.20). However, in the nonminimal case we also have to include the \( \xi \mathcal{R} \phi^2 \) term in the potential, which acts as a mass term. The power spectrum in Eq. (3.20) was calculated for a minimally coupled inflaton field. The question is how to calculate the spectrum of density perturbations in the nonminimal case. Fortunately we can do a trick such that we can still use the power spectrum for a minimally coupled inflaton field. We perform a specific conformal transformation that removes the coupling of the inflaton field to gravity. In the next section we will actually perform this conformal transformation, and we will see that we can write our action again in the minimally coupled form, but the potential will be modified. From this potential one can then easily calculate the amplitude of density perturbations, which then gives the relation \( \delta \rho / \rho \propto \sqrt{\lambda / \xi^2} \). For more on this see for example [17, 16, 26]. Let us now quickly go to the next section and apply the conformal transformation to rewrite our original action.

### 4.3 The Higgs boson as the inflaton

In this section we will explain the idea by Bezrukov and Shaposhnikov [4]. As mentioned in the previous section, a scalar field is an essential ingredient for inflationary models. The only known scalar particle in the Standard Model is the Higgs boson. The potential for the Higgs field is

\[
V(H) = \lambda (H^\dagger H)^2
\]  

(4.47)

where \( H \) is the Higgs doublet

\[
H = \begin{pmatrix} \phi^0 \\ \phi^+ \end{pmatrix},
\]  

(4.48)

where \( \phi^0 \) and \( \phi^+ \) are complex scalar fields. We can fix a gauge (the unitary gauge) in which

\[
H = \begin{pmatrix} \phi \\ \sqrt{2} \end{pmatrix},
\]  

(4.49)

where \( \phi \) is now a real scalar field with vacuum expectation value (VEV) \( \langle \phi \rangle = v \). The potential then becomes the familiar "Mexican hat" potential

\[
V(\phi) = \frac{1}{4} \lambda (\phi^2 - v^2)^2
\]  

(4.50)
We now consider a chaotic inflationary scenario, where the Higgs field $\phi$ has a large initial value of $\mathcal{O}(M_P)$. Since $v \ll M_P$, we can safely neglect the Higgs VEV $v$ in the potential (4.50). Thus, the Higgs potential during chaotic inflation is effectively a quartic potential with self-coupling $\lambda$. The value of $\lambda$ is not yet known, but we can calculate an allowed range. Taking the quadratic part of the Higgs potential (4.50), we find that the Higgs mass is $m_H = \sqrt{\lambda v^2}$. The Higgs mass is expected to lie in the range $[114-185]$ GeV. The lower bound is set by experiments, whereas the upper bound is needed for stability of the Standard Model all the way up to the Planck scale. We can infer from this that the quartic self-coupling of the Higgs boson must be $\lambda \sim 10^{-1}$.

For successful inflation in Linde’s chaotic inflation scenario, the self-coupling of the scalar field must be very small $\lambda \approx 10^{-12}$ in order to produce the correct amplitude of density fluctuations. We calculated this in section 3.6 where we found that $\delta \rho/\rho \propto \sqrt{\lambda}$. Such a small self-coupling excludes the Higgs boson with $\lambda$ as a candidate for the inflaton.

In this section we include a nonminimal coupling term in the Higgs action. This changes the potential of our theory and therefore the constraints on $\lambda$. As we will see, the amplitude of density perturbations $\delta \rho/\rho$ is proportional to $\sqrt{\lambda}\xi$, such that the self-coupling can be much larger if $\xi$ is sufficiently large. Specifically the quartic self-coupling can be $\lambda \sim 10^{-1}$ if $\xi \sim -10^4$. This allows the Higgs boson to be the inflaton, which is a very nice feature of nonminimal inflation, because we can now explain inflation from the Standard Model.

The total action we thus consider consist of the Standard Model action, the Einstein-Hilbert action and the nonminimal coupling term.

$$S = \int d^4x \sqrt{-g} \left\{ \mathcal{L}_M + \frac{1}{2} M_P^2 R - \xi H^\dagger H R \right\},$$

(4.51)

where $\mathcal{L}_M$ is the Standard Model Lagrangian. If we again take the unitary gauge for the Higgs doublet $H$ and neglect all the gauge and fermion interactions for now, we find the action for the Higgs boson

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} (M_P^2 - \xi \phi^2) R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2 \right\},$$

(4.52)

where again the Higgs VEV appears with a value $v = 246$ GeV. This is the action in the so-called Jordan frame. We can now get rid of the nonminimal coupling of $\phi$ to gravity by making a conformal transformation to the Einstein frame. The reason that we do this is to calculate the power spectrum from Eq. (3.20), which gives us constraints on the inflaton potential. The power spectrum was derived for a minimally coupled inflaton field, and therefore we will perform a conformal transformation that removes the nonminimal coupling term from the action. To do the conformal transformation we first write our action as

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} M_P^2 \Omega^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2 \right\},$$

(4.53)

where

$$\Omega^2 = \frac{M_P^2 - \xi \phi^2}{M_P^2}.$$  

(4.54)

We see that if we now perform the conformal transformation to the new metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, the action becomes (see also Eq. (4.39)),

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} M_P^2 \tilde{R} - \frac{1}{2} \tilde{\Omega}^{-2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2 \right\} - 3M_P^2 \left\{ \square \ln \Omega + \tilde{g}^{\mu\nu} (\partial_\mu \ln \Omega)(\partial_\nu \ln \Omega) \right\}. $$

(4.55)
Note that we have succeeded in removing the coupling of the Higgs field to gravity, but an additional term has appeared. Let us simplify this term by making use of $\nabla_{A} \Omega = \partial_{A} \Omega$, since $\Omega$ only contains a scalar field. Then we get

$$-3M_{P}^{2} \sqrt{g} \left[ \square \ln \Omega + \bar{g}^{\rho\lambda}(\partial_{\rho} \ln \Omega)(\partial_{\lambda} \ln \Omega) \right]$$

$$= -3M_{P}^{2} \sqrt{g} \bar{g}^{\rho\lambda} \left[ \nabla_{\rho} \left( \frac{1}{\Omega} \right) \partial_{\lambda} \Omega + (\frac{1}{\Omega}) \partial_{\rho} \Omega(\partial_{\lambda} \Omega) \right]$$

$$= -3M_{P}^{2} \sqrt{g} \bar{g}^{\rho\lambda} \left[ -\frac{1}{\Omega^{2}} \partial_{\rho} \partial_{\lambda} \Omega + \frac{1}{\Omega} \nabla_{\rho} \partial_{\lambda} \Omega + \frac{1}{\Omega^{2}} \partial_{\rho} \Omega \partial_{\lambda} \Omega \right]$$

$$= 3M_{P}^{2} \nabla_{\rho} \left( \sqrt{g} \bar{g}^{\rho\lambda} \frac{1}{\Omega} \right) \partial_{\lambda} \Omega$$

$$= -3 \frac{1}{\Omega^{2}} M_{P}^{2} \sqrt{-\bar{g}} \bar{g}^{\rho\lambda} \partial_{\rho} \partial_{\lambda} \Omega,$$

(4.56)

where in the last step I have used metric compatibility $\nabla_{\rho} \bar{g}_{\mu\nu} = 0$. Now I use the specific form of $\Omega^{2}$ from Eq. (4.54) and I find

$$-3 \frac{1}{\Omega^{2}} M_{P}^{2} \sqrt{-\bar{g}} \bar{g}^{\rho\lambda} \partial_{\rho} \Omega \partial_{\lambda} \Omega = -3 \frac{1}{\Omega^{4}} \xi^{2} \phi^{2} M_{P}^{2} \sqrt{-\bar{g}} \bar{g}^{\rho\lambda} \partial_{\rho} \phi \partial_{\lambda} \phi.$$  

(4.57)

We see that this combines neatly with the kinetic term in the action (4.55) and we get

$$S = \int d^{4}x \sqrt{-\bar{g}} \left( \frac{1}{2} M_{P}^{2} \bar{R} - \frac{1}{2} M_{P}^{2} \Omega^{2} + 6 \xi^{2} \phi^{2} \bar{g}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - \Omega^{-4} \lambda \frac{4}{4} \phi^{2} - \nu^{2} \right)^{2} \right).$$

We see that we have removed the coupling to gravity by doing the conformal transformation, but as a consequence we created a non-trivial kinetic term. We can actually write this again as a canonical kinetic term by defining a new field $\chi(\phi)$, such that we have

$$\frac{1}{2} \bar{g}^{\mu\nu} \partial_{\mu} \chi(\phi) \partial_{\nu} \chi(\phi) = \frac{1}{2} \left( \frac{d\chi}{d\phi} \right)^{2} \bar{g}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi.$$  

(4.58)

When

$$\frac{d\chi}{d\phi} = \sqrt{\frac{M_{P}^{2} \Omega^{2} + 6 \xi^{2} \phi^{2}}{M_{P}^{2} \Omega^{4}}},$$

(4.59)

we retrieve our non-trivial kinetic term. In terms of the new field $\chi$ the action becomes

$$S = \int d^{4}x \sqrt{-\bar{g}} \left( \frac{1}{2} M_{P}^{2} \bar{R} - \frac{1}{2} \bar{g}^{\mu\nu} \partial_{\mu} \chi \partial_{\nu} \chi - \Omega^{-4} \lambda \frac{4}{4} (\phi(\chi)^{2} - \nu^{2} \right)^{2} \right).$$

We could solve Eq. (4.59) for the new field $\chi$ in the case where $\xi < 0$. However, it is more insightful to solve the differential equation in two different regimes. First consider the regime where the field $\phi$ is small, i.e.

$$\phi \ll \frac{M_{P}}{\sqrt{-\xi}} \quad \text{(small field approximation).}$$

(4.60)

Take a typical value $\xi = -10^{4}$, then we can safely say that the small field approximation is valid when the Higgs field $\phi < 10^{-2} M_{P}$. So this approximation works well up to an energy scale of $\sim 10^{16}$ GeV. In the small field approximation, $\Omega^{2} \approx 1$ and

$$\frac{d\chi}{d\phi} \approx 1, \quad \chi \approx \phi.$$  

(4.61)
Therefore, the potential in terms of the field $\chi$ becomes

$$V(\chi) \simeq \frac{\lambda}{4} (\chi^2 - v^2)^2,$$  \hspace{1cm} (4.62)

which is the well-known Mexican hat potential that leads to the Higgs mechanism through which the $W$ and $Z$ bosons become massive. In Fig. 4.5a we show the effective Higgs potential in the Einstein frame for small field values.

However, if we now consider another regime where

$$\phi \gg \frac{M_P}{\sqrt{-\xi}},$$  \hspace{1cm} (4.63)

we find that $\Omega^2 \simeq \frac{\xi \phi^2}{M_P^2}$, and that

$$\frac{d\chi}{d\phi} \simeq \sqrt{6} \frac{M_P}{\phi}, \quad \chi \simeq \sqrt{6} M_P \ln \left( \frac{\sqrt{-\xi} \phi}{M_P} \right).$$  \hspace{1cm} (4.64)

Again for a typical value $\xi = 10^4$, we are in the large field approximation when $\phi \gtrsim 10^{-1} M_P$. We write for the field $\phi$

$$\phi(\chi) \simeq \frac{M_P}{\sqrt{-\xi}} \exp \left( \frac{\chi}{\sqrt{6} M_P} \right).$$  \hspace{1cm} (4.65)

Note that in the regime where $\phi \gg \frac{M_P}{\sqrt{-\xi}}$ the new field $\chi \gg \sqrt{6} M_P$. The effective Higgs potential in the Einstein frame then becomes

$$V(\chi) \simeq \frac{\lambda M_P^4}{4 \xi^2} \frac{1}{\left( 1 + \exp \left( -\frac{2 \chi}{\sqrt{6} M_P} \right) \right)^2}.$$  \hspace{1cm} (4.66)

Note that we have neglected $v$ since $v \ll \phi$. In Fig. 4.5b we show the form of the potential for large values of $\phi$. The flatness of the potential makes sure that the slow-roll approximation is valid and inflation is successful. The large field regime where $\phi \gtrsim 10^{-1} M_P$ is precisely the chaotic inflationary regime, and

**Figure 4.5**: Effective Higgs potential in the Einstein frame. For small field values the effective potential reduces to the original Mexican hat Higgs potential. However, for large field values the effective potential becomes very flat and asymptotically reaches the value $\frac{\lambda M_P^4}{4 \xi^2}$. The flatness of the potential ensures that inflation is possible. Inflation ends when the fields become very small, such that $\chi \ll \sqrt{6} M_P$. 

---

The diagrams illustrate the effective potential in the Einstein frame for small and large field values. The left diagram shows the potential for small field values $\phi \ll M_P/\sqrt{-\xi}$, where the potential is nearly flat. The right diagram shows the potential for large field values $\phi \gg M_P/\sqrt{-\xi}$, where the potential becomes very flat and asymptotically reaches the value $\frac{\lambda M_P^4}{4 \xi^2}$.
Figure 4.6: WMAP measurements of the spectral index $n_s$ and the tensor to scalar ratio $r$ at a pivot wavelength $k = 0.002\text{Mpc}^{-1}$. The scale-free Harrison-Zeldovich spectrum is excluded by two standard deviations, as is the $\lambda \phi^4$ inflaton potential. However, the spectral index $n_s$ and the tensor to scalar ratio $r$ for the nonminimally coupled Higgs boson lie within the $1 \sigma$ contour of the WMAP measurements, and this is therefore still a candidate model for inflation.

therefore the potential (4.66) is the effective inflationary potential for the Higgs field. In this case the Higgs boson is minimally coupled to gravity, and we can therefore use our previous knowledge from section 3.6 to calculate the constraints on this effective potential. We found that $\delta \rho/\rho \propto \sqrt{V}$, where $V$ is the inflaton potential at the first horizon crossing. Note that the effective inflaton potential in Eq. (4.66) has approximately the value $\lambda/\xi^2$ during inflation, and therefore the fractional density perturbations $\delta \rho/\rho \propto \sqrt{\lambda/\xi^2}$. From the CMBR we find that $\delta \rho/\rho \sim 10^{-4}$. For the nonminimally coupled inflaton field the value of the quartic self-coupling can therefore be relatively large $\lambda \sim 10^{-1}$ if $\xi \sim 10^4$. This is the result that we already mentioned at the beginning of this section, and precisely this feature allows the Higgs boson to be the inflaton.

In section 3.6 it was said that not only the amplitude of density perturbations, but also the scaling of this amplitude with wave number $k$ can be measured from the CMBR. This scaling was expressed by the spectral index $n_s$ in Eq. (3.23), which is 1 for a scale invariant power spectrum. The slow-roll parameters $\epsilon$ and $\eta$ however cause deviations of the spectral index from 1, and this constrains the inflaton potential since the slow-roll parameters contain derivatives of the potential. It was shown in Fig. 3.2 that the quartic $\lambda \phi^4$ potential was excluded by CMBR measurements. For the nonminimally coupled inflaton field, the effective inflationary potential in Eq. (4.66) is not anymore a quartic potential, and therefore the slow-roll parameters and constraints will be different. The results are shown in Fig. 4.6. It is shown that the nonminimally coupled Higgs boson is not excluded as candidate for the inflaton by the CMBR measurements.

Radiative corrections to the effective potential (4.66) could spoil the flatness of the potential and therefore inflation through the nonminimally coupled Higgs boson. However in Refs. [20][23][22] it is shown that inflation is still successful if one includes one- and two-loop radiative corrections. The authors find approximately the same range of the allowed Higgs mass $[124–194] \text{GeV}$ in order for inflation to work. Recently the authors in [24] have obtained the same results in the Jordan frame where the Higgs boson is nonminimally coupled to the Ricci scalar $R$. 
4.4 The two-Higgs doublet model

In this section we will consider again the Higgs action with nonminimal coupling, but this time we will consider the simplest supersymmetric extension: the two-Higgs doublet model. As the name suggests, in this model there are two Higgs doublets, $H_1$ and $H_2$. The motivation for the use of this model is the fact that the extra Higgs doublet naturally appears in supersymmetric models. It also introduces more couplings and four extra degrees of freedom, e.g. two extra complex fields. As we will see, of the two Higgs doublets one has a real expectation value, whereas the other has an additional phase $\theta$ in the vacuum. Precisely this phase provides a source of CP violation, which is an essential ingredient for baryogenesis. We will come back to this in chapter 6. For some good literature on the two-Higgs doublet model, see for example [27].

Again, considering only the Higgs sector of the action and leaving the gauge and fermion interactions aside for now, we can write the most general matter action

$$ S = \int d^4x \sqrt{-g} \left( - \sum_{i,j=1,2} \omega_{ij} \bar{g}^{\mu\nu}(\partial_\mu H_i)^\dagger(\partial_\nu H_j) - V(H_1, H_2) - \sum_{i,j=1,2} \xi_{ij} R H_i^\dagger H_j \right). $$

We have allowed for an off-diagonal nonminimal term and an off-diagonal kinetic term that mixes the derivatives of $H_1$ and $H_2$. The nonminimal couplings $\xi_{ij}$ and the constants $\omega_{ij}$ in front of the kinetic term can in principle be complex quantities. However, if we demand our action to be real, i.e $S^\dagger = S$, we find that the diagonal terms $\omega_{11}, \omega_{22}$ and $\xi_{11}, \xi_{22}$ must be real and the off-diagonal terms are related as $\omega_{12} = \omega_{21}^*$ and $\xi_{12} = \xi_{21}^*$. The most general potential we can write for the two Higgs doublet model is (see e.g. [27]),

$$ V(H_1, H_2) = \lambda_1 (H_1^\dagger H_1 - v_1^2)^2 + \lambda_2 (H_2^\dagger H_2 - v_2^2)^2 + \lambda_3 [(H_1^\dagger H_1 - v_1^2) + (H_2^\dagger H_2 - v_2^2)]^2 + \lambda_4 [(H_1^\dagger H_1)(H_2^\dagger H_2) - (H_1^\dagger H_2)(H_2^\dagger H_1)] \nonumber $$

$$ + \lambda_5 [\text{Re}(H_1^\dagger H_2) - v_1 v_2 \cos \theta]^2 + \lambda_6 [\text{Im}(H_1^\dagger H_2) - v_1 v_2 \sin \theta]^2 + \lambda_7. \quad (4.67) $$

The couplings $\lambda_i, \ (i = 1, ..., 7)$ are real because of hermiticity of the action. The constant $\lambda_7$ only appears such that the minimum of the potential is $V = 0$ and has no physical meaning. We will take $\lambda_7 = 0$ for now. $v_1$ and $v_2$ are the real expectation values of the Higgs doublets at zero temperature, and $\theta$ is the phase difference between the doublets in the vacuum.

Define the two Higgs doublets as

$$ H_1 = \begin{pmatrix} \phi_1^0 \\ \phi_1^+ \end{pmatrix}, \quad H_2 = \begin{pmatrix} \phi_2^0 \\ \phi_2^+ \end{pmatrix}, \quad (4.68) $$

where the $\phi^{+0}$ are charged and neutral complex scalar fields. We can fix a gauge such that

$$ H_1 = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} \phi_2 \\ 0 \end{pmatrix}, \quad (4.69) $$

where $\phi_1$ and $\phi_2$ are complex scalar fields with expectation values $\langle \phi_1 \rangle = v_1$ and $\langle \phi_2 \rangle = v_2 e^{i\theta}$. For simplicity we take both $\phi_1$ and $\phi_2$ to be complex, but we remember that only their relative phase is important. If we now also take $\omega_{11} = \omega_{22} = 1$ and $\omega_{12} = \omega$ (making the diagonal kinetic terms canonical) we find that we can write our matter Lagrangian as

$$ \mathcal{L} = \sqrt{-g} [g^{\mu\nu} \partial_\mu \bar{\Phi}^* \Omega \partial_\nu \Phi - R \bar{\Phi}^* \Xi \Phi - V(\phi_1, \phi_2)], \quad (4.70) $$

where we have defined

$$ \bar{\Phi} = \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & \omega \\ \omega^* & 1 \end{pmatrix}, \quad \Xi = \begin{pmatrix} \xi_{11} & \xi_{12}^* \\ \xi_{12} & \xi_{22} \end{pmatrix}. \quad (4.71) $$
Note that the matrices $\Omega$ and $\Xi$ are hermitean. Suppose now that we are in the inflationary regime where $\phi_1, \phi_2 \gg v_1, v_2$ as in the ordinary Higgs model of the previous section. It is easy to see that we can then write our potential (4.67) as

$$V(\phi_1, \phi_2) = \lambda_1(\phi_1^* \phi_1)^2 + \lambda_2(\phi_2^* \phi_2)^2 + \lambda_3(\phi_1^* \phi_1)(\phi_2^* \phi_2),$$

where I have actually redefined $\lambda_3 + \lambda_5 \cos^2 \theta + \lambda_6 \sin^2 \theta - \lambda_3$. Thus we have obtained the Lagrangian (4.70) with a relatively simple potential.

As a consequence of the off-diagonal kinetic terms the equations of motions for $\phi_1$ and $\phi_2$ will be coupled through the derivative terms. In order to solve the equations of motion, we want each equation of motion to contain only derivatives of one field. Therefore we want to diagonalize the kinetic term. This was also done by Garbrecht and Prokopec[28] for a similar Lagrangian. In that case they were able to solve the field equations analytically. We will try to do the same here. First we want to diagonalize the kinetic term by solving the eigenvalue equation

$$\Omega U = UD,$$

where $D$ is a $2 \times 2$ diagonal matrix with the eigenvalues of $\Omega$ on the diagonal, and $U$ is the corresponding matrix with the two eigenvectors as columns. Solving the eigenvalue matrix from Eq. (4.73) gives

$$D = \begin{pmatrix} 1 + |\omega| & 0 \\ 0 & 1 - |\omega| \end{pmatrix},$$

with $|\omega| = \sqrt{\omega^\dagger \omega}$. Multiplying both sides of Eq. (4.73) from the right with $U^{-1}$, we see that we can write

$$\Omega = UD U^{-1},$$

where the matrices are given by

$$U = \begin{pmatrix} \sqrt{\frac{\omega}{\omega}} & -\sqrt{\frac{\omega}{\omega}} \\ 1 & 1 \end{pmatrix},$$

$$U^{-1} = \frac{1}{2} \begin{pmatrix} \sqrt{\frac{\omega}{\omega}} & 1 \\ -\sqrt{\frac{\omega}{\omega}} & 1 \end{pmatrix}.$$

Now we plug Eq. (4.75) into the derivative term of the kinetic term of the Lagrangian (4.70) and we write

$$\mathcal{L}_{kin} = \sqrt{-g} g^{\mu \nu} \partial_\mu \Phi^T U D U^{-1} \partial_\nu \Phi = \sqrt{-g} g^{\mu \nu} \partial_\mu \Phi^T \partial_\nu \Phi,$$

where I have defined new fields

$$\Phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \sqrt{D} U \Phi^T = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + |\omega|} \sqrt{\frac{\omega}{\omega}} \phi_1 + \phi_2 \\ \sqrt{1 - |\omega|} \sqrt{\frac{\omega}{\omega}} \phi_1 + \phi_2 \end{pmatrix},$$

$$\Phi^T = \begin{pmatrix} \phi_+^* \\ \phi_-^* \end{pmatrix}^T = \frac{1}{\sqrt{2}} \Phi^T U \sqrt{D} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + |\omega|} \sqrt{\frac{\omega}{\omega}} \phi_1^* + \phi_2^* \\ \sqrt{1 - |\omega|} \sqrt{\frac{\omega}{\omega}} \phi_1^* + \phi_2^* \end{pmatrix}^T.$$
Implicitly we have defined two new fields $\phi_+$ and $\phi_-$, and as a matter of clarity we give these fields again

$$
\phi_+ = \frac{1}{\sqrt{2}} \sqrt{1 + |\omega|} \left( \sqrt{\frac{\omega^*}{\omega}} \phi_1 + \phi_2 \right)
$$

$$
\phi_- = \frac{1}{\sqrt{2}} \sqrt{1 - |\omega|} \left( -\sqrt{\frac{\omega^*}{\omega}} \phi_1 + \phi_2 \right).
$$

(4.80)

We can now also invert these relations and express $\phi_1$ and $\phi_2$ in terms of $\phi_+$ and $\phi_-$, and we obtain

$$
\phi_1 = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\omega^*}{\omega}} \phi_+ - \frac{\phi_-}{\sqrt{1 + |\omega|}} \right)
$$

$$
\phi_2 = \frac{1}{\sqrt{2}} \left( \frac{\phi_+}{\sqrt{1 + |\omega|}} + \frac{\phi_-}{\sqrt{1 - |\omega|}} \right).
$$

(4.81)

If we now substitute these fields into the kinetic term of the Lagrangian, this becomes

$$
L_{\text{der}} = p - g \left[ g_{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + g_{\mu\nu} \partial_\mu \phi_- \partial_\nu \phi_- \right]
$$

$$
= -g \left[ g_{\mu\nu} \partial_\mu \Phi^+ \partial_\nu \Phi \right],
$$

(4.82)

Thus we have succeeded in diagonalizing and canonically normalizing the kinetic term of the Lagrangian.

Of course we also want to express the other parts of the Lagrangian in terms of the new fields $\phi_+$ and $\phi_-$. Let us first focus on the nonminimal coupling term of the Lagrangian, $L_\xi$. We can write

$$
L_\xi = -R \Phi^+ \Xi \Phi
$$

$$
= -R \Phi^+ U \sqrt{D} D^{-1} U^{-1} \Xi U \sqrt{D} D^{-1} \Phi
$$

$$
= -R \Phi^+ \sqrt{D}^{-1} U^{-1} \Xi U \sqrt{D}^{-1} \Phi
$$

$$
= -R \Phi^+ \Xi \Phi,
$$

(4.83)

where I have defined

$$
\sqrt{D}^{-1} U^{-1} \Xi U \sqrt{D}^{-1} \equiv \Xi \equiv \left( \begin{array}{cc} \xi_{++} & \xi_{+-} \\ \xi_{-+} & \xi_{--} \end{array} \right).
$$

(4.84)

Explicitly, the matrix terms of $\Xi$ are

$$
\xi_{++} = \frac{1}{2(1 + |\omega|)} \left( \xi_{11} + \xi_{22} - \xi_{12} \sqrt{\frac{\omega^*}{\omega}} - \xi_{12} \sqrt{\frac{\omega^*}{\omega}} \right)
$$

$$
\xi_{--} = \frac{1}{2(1 - |\omega|)} \left( \xi_{11} + \xi_{22} + \xi_{12} \sqrt{\frac{\omega^*}{\omega}} + \xi_{12} \sqrt{\frac{\omega^*}{\omega}} \right)
$$

$$
\xi_{+-} = \frac{1}{2 \sqrt{1 - |\omega|^2}} \left( \xi_{22} - \xi_{11} - \xi_{12} \sqrt{\frac{\omega^*}{\omega}} + \xi_{12} \sqrt{\frac{\omega^*}{\omega}} \right)
$$

$$
\xi_{-+} = \frac{1}{2 \sqrt{1 - |\omega|^2}} \left( \xi_{22} - \xi_{11} + \xi_{12} \sqrt{\frac{\omega^*}{\omega}} - \xi_{12} \sqrt{\frac{\omega^*}{\omega}} \right).
$$

(4.85)

Now we write the complex couplings $\omega$ and $\xi_{12}$ as

$$
\omega = |\omega| e^{i\theta_\omega} \quad \xi_{12} = |\xi_{12}| e^{i\theta_{12}},
$$

(4.86)
and we can write

\[
\begin{align*}
\xi_{++} &= \frac{1}{2(1 + |\omega|)}(\xi_{11} + \xi_{22} - 2|\xi_{12}|\cos\theta_\xi) \\
\xi_{--} &= \frac{1}{2(1 - |\omega|)}(\xi_{11} + \xi_{22} + 2|\xi_{12}|\cos\theta_\xi) \\
\xi_{+-} &= \frac{1}{2\sqrt{1 - |\omega|^2}}(\xi_{22} - \xi_{11} + 2i|\xi_{12}|\sin\theta_\xi) \\
\xi_{-+} &= \frac{1}{2\sqrt{1 - |\omega|^2}}(\xi_{22} - \xi_{11} - 2i|\xi_{12}|\sin\theta_\xi).
\end{align*}
\]  

(4.87)

where the phase \(\theta_\xi\) is defined as

\[
\theta_\xi = \theta_\omega - \theta_{12}.
\]  

(4.88)

It is now easy to see that the new nonminimal coupling matrix \(\Xi\) is hermitean, i.e. \(\Xi^\dagger = \Xi\). Thus we have also redefined the nonminimal term in our Lagrangian in terms of the new fields. Our result is a new nonminimal matrix \(\Xi\) that has the same properties as the original \(\tilde{\Xi}\). Therefore in this section of the Lagrangian there is effectively no change.

The final sector of the Lagrangian contains the potential term \(V(\phi_1, \phi_2)\) from Eq. (4.72). We can also rewrite this potential in terms of \(\phi_+\) and \(\phi_-\) by using Eq. (4.81). It is not hard to see that in addition to terms \(\phi_+^4\), \(\phi_-^4\) and \(\phi_+^2\phi_-^2\) there will be terms \(\phi_-\phi_+^3\) and \(\phi_+\phi_-^3\). This is a rough sketch of the modified potential, because in fact the fields are complex, although the potential is real. We will not write down this potential explicitly, but instead we will write the Lagrangian in terms of the new fields \(\phi_+\) and \(\phi_-\) in the following way, nonminimal couplings

\[
\mathcal{L} = \sqrt{-g} \left[ g^\mu\nu \partial_\mu \phi_+^* \partial_\nu \phi_+ + g^\mu\nu \partial_\mu \phi_-^* \partial_\nu \phi_- \\
- R \xi_{++} \phi_+^* \phi_+ - R \xi_{--} \phi_-^* \phi_- - R \xi_{+-} \phi_+^* \phi_- - R \xi_{-+} \phi_-^* \phi_+ - V(\phi_+, \phi_-) \right].
\]  

(4.89)

If we compare this Lagrangian to the original Lagrangian in Eq. (4.70) with \(\Omega = \text{diag}(1,1)\) (thus a diagonal kinetic term), we see that the Lagrangians are almost the same. By a redefinition of the fields only the potential term is changed and now contains \(\phi_-\phi_+^3\) and \(\phi_+\phi_-^3\) terms. Numerically we have checked that these potential terms do not qualitatively influence the dynamics of inflation for typical coupling values \(\lambda_i \sim 10^{-3}\). Therefore, from now on we simply take our kinetic term to be diagonal from the start and simply work with the fields \(\phi_1\) and \(\phi_2\).

Of course we have not taken interaction terms of the Higgs boson with the gauge bosons or fermions into account. These have the form \(f \phi_i \bar{\psi} \psi\). If we consider Eqs. (4.81), we see that the \(\phi_1\) fields feature a term \(e^{i\theta_\omega}\). However, this constant phase can be absorbed in the fermion fields and it will not change the fermion action. The situation is different when the phase is not constant, which is the case for the phase \(\theta\) between the Higgs bosons. We will come back to this in chapter 6. For now we will focus on the Higgs sector of the Lagrangian without interactions. As in section 4.1 we will first derive the field equations, then try to make approximations and see if inflation is successful. Finally we will show numerical solutions of the field equations and actually see that inflation is a success.

### 4.4.1 Dynamical equations

We now derive the constraint equations for \(H^2\) and \(H\) and the field equations as we did in section 4.1. For simplicity we take our kinetic term of the Lagrangian (4.70) to be diagonal
such that the potential term is given by (4.72). As shown in the previous section, if we would have an off-diagonal kinetic term, we can remove this by a redefinition of our fields. The only consequence is extra potential terms. We have numerically verified that these extra potential terms have no significant influence on the dynamics of inflation. Therefore, we take our action to be

\[ S = \int d^4x \sqrt{-g} \left( - \sum_{i=1,2} g^\mu{}^\nu (\partial_\mu \phi_i)^* (\partial_\nu \phi_i) - \sum_{i,j=1,2} \xi_{ij} R \phi_i^* \phi_j - V(\phi_1, \phi_2) \right), \tag{4.90} \]

where the potential \( V(\phi_1, \phi_2) \) is given in Eq. (4.72). By a variation of the action with respect to the metric \( g^\mu{}^\nu \) we find the Einstein tensor

\[ G_{\mu\nu} = \frac{1}{M_p^2 - 2 \sum_{i,j=1,2} \xi_{ij} \phi_i^* \phi_j} \left\{ \sum_{i=1,2} |\phi_i|^2 + V(\phi_1, \phi_2) + 6H \sum_{i,j=1,2} \xi_{ij} [\phi_i^* \phi_j + \phi_j^* \phi_i] \right\}. \tag{4.91} \]

We can compare this to Eq. (4.3) and we see that this is indeed the extension from one real field to two complex fields. Note the factors of 2, which would disappear if we would write the complex fields in terms of real fields as \( \phi_i = \frac{1}{\sqrt{2}} (\phi_i^R + i \phi_i^I) \). If we again assume a homogeneous and isotropic background, we find \( H^2 \) from the 00 component of the Einstein tensor as

\[ H^2 = \frac{1}{3(M_p^2 - 2 \sum_{i,j=1,2} \xi_{ij} \phi_i^* \phi_j)} \left\{ \sum_{i=1,2} |\phi_i|^2 + V(\phi_1, \phi_2) + 6H \sum_{i,j=1,2} \xi_{ij} [\phi_i^* \phi_j + \phi_j^* \phi_i] \right\}. \tag{4.92} \]

Similarly we find the equation for \( \dot{H} \) from the diagonal spatial components of \( G_{ij} \) and by using the equation for \( H^2 \). This gives

\[ \dot{H} = \frac{1}{M_p^2 - 2 \sum_{i,j=1,2} \xi_{ij} \phi_i^* \phi_j} \left\{ - \sum_{i=1,2} |\phi_i|^2 - \sum_{i,j=1,2} \xi_{ij} [-(\phi_i^* \dot{\phi}_j + \phi_j^* \dot{\phi}_i) + H(\phi_i^* \phi_j + \phi_j^* \phi_i) - 2(\phi_i^* \phi_i)] \right\}. \tag{4.93} \]

Finally the equation of motion for \( \phi_i \) is

\[ \dot{\phi}_i + 3H \phi_i + R \sum_k \xi_{ik} \phi_k + \frac{\partial V(\phi_1, \phi_2)}{\partial \phi_i^*} = 0. \tag{4.94} \]

The equation of motion for \( \phi_i^* \) is obtained by complex conjugation. We can again eliminate \( \dot{\phi}_i \) from the expression for \( \dot{H} \) by using the equation of motion. Then we find the Ricci scalar \( R = 6[H + \dot{H}^2] \) in terms of \( \phi_i \) and \( \dot{\phi}_i \) as

\[ R = \frac{1}{M_p^2 - 2 \sum_{i,j,k=1,2} \xi_{ij} \phi_k^* (\delta_{jk} - 6 \xi_{jk}) \phi_k} \left\{ -2 \sum_{i,j=1,2} (\delta_{ij} - 6 \xi_{ij}) (\dot{\phi}_i^* \dot{\phi}_j) \right\} - 6 \sum_{i,j=1,2} \xi_{ij} [\phi_i^* \frac{\partial V(\phi_1, \phi_2)}{\partial \phi_i^*} + \frac{\partial V(\phi_1, \phi_2)}{\partial \phi_i} \phi_j] + 4V(\phi_1, \phi_2). \tag{4.95} \]

Note the conformal coupling value of the nonminimal matrix, i.e. \( \xi_{ij} = \frac{1}{6} \delta_{ij} \). If we take our inflationary potential to be (4.72), we see that \( R \) vanishes for the conformal coupling.
value of $\xi$. With the explicit expression for the Ricci scalar in Eq. (4.94) the field equation becomes,

$$\dot{\phi}_i + 3H\dot{\phi}_i + \frac{\sum_{k=1,2} \xi_{ik} \phi_k}{M_p^2 - 2\sum_{k,l,m=1,2} \phi^*_k \xi_{kl}(\delta_{lm} - 6\xi_{lm})\phi_m} \left[ -2 \sum_{l,m=1,2} (\delta_{lm} - 6\xi_{lm})(\phi^*_l \phi^*_m) \right] - \frac{\partial V}{\partial \phi_i} \left( \frac{\partial V}{\partial \phi^*_i} \right),$$

where $V = V(\phi_1, \phi_2)$. We have written the potential terms on the righthand side of the equation of motion. In the slow-roll approximation these potential terms dominate over the kinetic terms. As in the single field case, we assume the slow-roll conditions (4.12). This basically means we neglect the terms proportional to $\ddot{\phi}$ and $\dot{\phi}^2$ but keep the term $3H\dot{\phi}_i$ and the potential terms. Thus in the slow-roll approximation the field equation becomes

$$3H\dot{\phi}_i \approx \frac{\sum_{k=1,2} \xi_{ik} \phi_k}{M_p^2 - 2\sum_{k,l,m=1,2} \phi^*_k \xi_{kl}(\delta_{lm} - 6\xi_{lm})\phi_m} \left[ 6 \sum_{l,m=1,2} \xi_{lm}(\phi^*_l \frac{\partial V}{\partial \phi^*_m} + \frac{\partial V}{\partial \phi_m}) - 4V \right] - \frac{\partial V}{\partial \phi_i} \left( \frac{\partial V}{\partial \phi^*_i} \right).$$

Then we substitute this expression into the energy constraint equation (4.91) and also use the slow-roll approximation to find

$$H^2 \approx \frac{1}{3(M_p^2 - 2\sum_{i,j=1,2} \xi_{ij} \phi^*_i \phi_j)} \left[ V + \frac{2}{M_p^2 - 2\sum_{k,l,m=1,2} \phi^*_k \xi_{kl}(\delta_{lm} - 6\xi_{lm})\phi_m} \right.\left. \times \left[ (M_p^2 - 2\sum_{k,l=1,2} \phi^*_k \xi_{kl} \phi_l) - \sum_{i,j=1,2} \xi_{ij}(\phi^*_i \frac{\partial V}{\partial \phi^*_j} + \phi_j \frac{\partial V}{\partial \phi_i}) \right] - 8V \sum_{i,j,k=1,2} \phi^*_i \xi_{ij} \phi^*_j \phi_k \right].$$

If we now substitute the potential (4.72) and take the large negative nonminimal coupling limit $\xi_{ii} \ll 0$, we find a complicated expression for $H^2$ and the equations of motion. However, we can solve these equations numerically, which we will do in the next section.

### 4.4.2 Numerical solutions of the field equations

In Fig. 4.7 we have numerically calculated solutions for the fields $\phi_1$ and $\phi_2$ from the field equation (4.95). As a matter of simplicity we have chosen the fields and nonminimal couplings to be real. All the nonminimal couplings have typical values $\xi_{ii} \sim -10^3$ and the quartic couplings $\lambda_i \sim 10^{-3}$. One interesting observation is that inflation does not happen immediately, i.e. the fields do not roll down from their initial value to their minimum straight away. Instead, they interact (through the interaction terms with couplings $\xi_{12}, \xi_{21}$ and $\lambda_3$) and start to oscillate rapidly. This is shown in Fig. 4.8. After a short time the oscillations are damped out and the fields acquire a "fixed" value. Then the fields do roll down to their minimum and inflation works. The numerical solutions for $\phi_1$ and $\phi_2$ in Fig. 4.7 show that during inflation the fields are proportional to each other. This suggest that we can take one linear combination of the two fields that is the inflaton. The other linear combination is an oscillating mode that quickly decays. This could then be a source for particle production and perhaps baryogenesis.

In Fig. 4.9a we show the growth of the number of e-folds in time. For the typical nonminimal model as mentioned above, the maximum number of e-folds that is reached at the end of inflation is $N \approx 1200$. The condition necessary to solve the flatness and horizon puzzles is $N \geq 70$, so this condition is easily satisfied. We could therefore relax the initial values of the fields or the values of the nonminimal couplings. Thus, our nonminimal couplings could be much smaller, or the initial values of the fields could be lower. The Hubble parameter
Figure 4.7: Numerical solutions of the fields $\phi_1$ and $\phi_2$ in the two-Higgs doublet model. The nonminimal couplings are real and have typical values in the order of $10^3$. For the simulations above, $\xi_{11} = 10^3$, $\xi_{22} = 1.2 \times 10^3$, $\xi_{12} = \xi_{21} = 10^3$. The quartic couplings also have typical values $10^{-3}$. In this case $\lambda_1 = 0.8 \times 10^{-3}$, $\lambda_2 = 1.2 \times 10^{-3}$ and $\lambda_3 = 2 \times 10^{-3}$. The initial values for $\phi_1$ and $\phi_2$ are 0.4 and 0.8 respectively, in units of $M_P$. The time $t$ is given in units of $M_P^{-1}$. After some complicated interaction between the two fields at early times (shown in Fig. 4.8), the fields reach a certain value and slowly roll down to their minimum. During the inflationary period the fields are proportional to each other, which suggests that we can find linear combinations of the two fields such that one of the linear combinations is the actual inflaton, whereas the other combination is an oscillating field that rapidly decays and could lead to particle creation.

is also shown in Fig. 4.9b. Again we see that the Hubble parameter is decreasing linearly in time and is proportional to both $\phi_1$ and $\phi_2$. This also suggests that we have only one linear combination of the fields that is the inflaton. This linear combination is then also proportional to $H$. At the end of inflation, the potential term in the equation for $H^2$ (4.91) is close to zero and the term containing the nonminimal coupling dominates. We again see the behavior typical to nonminimal inflationary models, the Hubble parameter makes sharp drops to almost zero.

Our conclusion from our numerical simulations is that inflation also works in a nonminimally coupled two-Higgs doublet model. However, only one particular combination of the two fields is the actual inflaton. The other linear combination is an oscillating mode that quickly decays and could be a source for particle production and baryogenesis. In this thesis we will focus our attention on the Higgs boson as the inflaton and leave the oscillating mode aside for now. In future work we hope to take a closer look at the oscillating mode and the reheating process.

We can comment on our own numerical results. First of all, we have shown numerical results for the fields $\phi_1$ and $\phi_2$ with the potential (4.72). As we have mentioned earlier this section, we could also have a kinetic term that couples derivatives of $\phi_1$ and $\phi_2$. This we could then diagonalize by defining new fields $\phi_+$ and $\phi_-$, which again gave us a hermitean nonminimal coupling matrix. The only difference is in the potential term, that now also contains terms $\phi_+ \phi_3^2$ and $\phi_- \phi_3^2$. We have numerically verified that these extra potential terms do not change our results significantly, although we will not show the results here. The numerical results look the same as those in Figs. 4.7, 4.8 and 4.9, but some of the numerical factors will be different.

A second comment on our own results is that we have calculated the numerical results for real nonminimal couplings and fields. However, the nonminimal coupling $\xi_{12}$ is in principle a complex quantity, and the fields are complex as well (although only their relative phase is a physical degree of freedom). Of course we would have to include this fact into our numer-
Figure 4.8: Numerical solution of the $\phi_1$ and $\phi_2$ with the same constants and initial conditions as in Fig. 4.7 for early times, i.e. $\phi_1(0) = 0.4$ and $\phi_2(0) = 0.8$ in units of $M_P$. This is the pre-inflationary region where the two fields interact and rapidly oscillate until the oscillations damp out and the fields reach a constant value. After this region, the fields slowly roll down to their minimum. These figures also suggest that we can find one linear combination of the fields that is the oscillating mode, whereas the other linear combination is the inflaton. One can easily see that roughly the combination $\phi_2 - \phi_1$ is the oscillating mode, whereas the combination $\phi_1 + \phi_2$ is the inflaton.

4.5 Summary

In this chapter we have done an extensive investigation of inflationary models with a scalar field that is nonminimally coupled to gravity. We have seen in section 4.1 that inflation is successful for a real scalar field in a quartic potential that has a strong negative nonminimal coupling to gravity. We have actually been able to solve the field equations analytically in the slow-roll approximation by taking the limit of large negative nonminimal coupling. Numerically we have verified these analytical results and we have found agreement between the numerical and analytical results.

The large negative nonminimal coupling has the additional advantage that the constraint on the smallness of the quartic self-coupling in a minimal model, imposed by a calculation of density perturbations, can be relaxed by many orders of magnitude. This allows the Higgs boson to be the inflaton. In section 4.2 we have introduced the conformal transformation of the metric. In section 4.3 we have performed a specific conformal transformation of the metric that removes the nonminimal coupling term from the Higgs Lagrangian, but introduces an effective potential containing the nonminimal couplings. The small-field effective potential reduces to our ordinary Higgs potential, but for large field values the potential becomes asymptotically flat. Precisely this feature makes sure that slow-roll inflation works. For the effective inflaton potential we could easily calculate the constraints on the nonminimally coupled Higgs boson, and we have found that it is a viable candidate for the inflaton.

Finally in section 4.4 we have introduced the two-Higgs doublet model, which is perhaps the
Figure 4.9: Numerical solution of the number of e-folds $N$ and the Hubble parameter $H$ as a function of time with the same constants and initial conditions as in Fig. 4.7 for early times. The number of e-folds reaches a maximum value of $\sim 1200$, which obviously satisfies the condition $N \geq 70$ to solve the flatness and horizon puzzles. The Hubble parameter is also shown to be decreasing linearly in time. This again shows that the Hubble parameter is proportional to both fields, and is therefore also proportional to a linear combination that is the inflaton. At the end of inflation, we again see the typical behavior of nonminimal models, that is, the Hubble parameter suddenly drops to almost zero. This happens when the nonminimal coupling terms in the expression for $H^2$ (4.91) start to dominate over the potential term.

The simplest supersymmetric extension of the Standard Model. Compared to the original Higgs doublet model, there are two additional degrees of freedom: one extra field and a phase between the fields. The most general two-Higgs doublet model contains kinetic terms and nonminimal couplings that couple the two Higgs doublets. We have shown that we can diagonalize the kinetic terms, at the expense of introducing extra potential terms. These do not significantly influence the dynamics inflation and we therefore took our kinetic term to be diagonal from the start. Again we have derived the field equations, and numerical calculations for the two real scalar fields show that inflation is still successful in this model. One linear combination of the fields serves as the inflaton, whereas the other is an oscillating mode that quickly decays and leads to reheating. The third degree of freedom, the phase between the fields, we did not include in our numerical simulations, but it can provide a source for CP violation and perhaps baryogenesis. This remains to be investigated in future work.

All calculations in this chapter were solved classically and we have completely ignored quantum field theory. In the next chapter we will first introduce the concept of quantum field theory in curved spaces, which turns out to be nontrivial and has very interesting physical consequences. We will then be able to quantize our theory and include quantum effects in our model, which we will apply in chapter 6.
Chapter 5

Quantum field theory in an expanding universe

In the previous chapter we have given an elaborate discussion on inflation through a non-minimally coupled scalar field. We have shown that nonminimal inflation works for a theory with a quartic potential when the nonminimal coupling is negative and large. An additional benefit is that the constraint on the quartic coupling $\lambda$ can be relaxed by precisely this large nonminimal coupling. This allows the Higgs boson to be the inflaton and reproduce the correct spectrum of density fluctuations. We have extended this to a two-Higgs doublet model and have shown that inflation also works in that case. The reader should have seen that up till now we have considered only classical scalar fields. The solutions of the field equations describe the classical evolution of the scalar field. We have not yet included quantum corrections and its effect on the dynamics of the scalar field.

As is well known, a unifying theory of gravity and quantum field theory has not yet been constructed. The problem is that gravity is a non-renormalizable theory, which suggests that we need some new physics to get rid of this problem. So it seems we cannot treat our classical action, consisting of the Einstein-Hilbert action and the matter action, as an action with quantum fields. However, quantum gravity only becomes important when the size of the universe was smaller than or comparable to the typical quantum mechanical length scale, the Planck length $l_p \approx 1.6 \times 10^{-35}$ m. This happened at the characteristic energy scale $M_P \approx 2.4 \times 10^{18}$ GeV. As long as we are sufficiently below this energy scale, gravity and quantum mechanics decouple and we do not see any quantum gravitational effects. As an analogue, gravity is described by Einsteins theory of general relativity, but in our daily life we do not see any general relativistic effects and Newtonian gravity works perfectly well. So even though we do not know a unifying theory of gravity and quantum field theory, we treat our action as a low energy effective theory at which we do not see any of the effects of the high energy theory.

The argument above suggests that we are allowed to treat our fields in the action as quantum fields. As is usually done, one can see the quantum field as a classical field plus some small quantum perturbation. The classical field, i.e. the solution of the classical equations of motion that minimizes the action, gives the dominant contribution to the dynamics of the system. Quantum (or radiative) corrections to the evolution of the system are small ($\mathcal{O}(\hbar)$). This approach to incorporating quantum field theory with gravity has proved extremely useful in the theory of cosmological perturbations, see for example [11]. The small quantum fluctuations are frozen in on super-Hubble scales and are enhanced during inflation by the rapid expansion of the universe, see section 3.6. For density fluctuations, this leads to the creation of stars and galaxies and eventually the birth of our own planet. One of the
great triumphs of inflation is that it predicts a nearly scale invariant spectrum of density fluctuations. Indeed, the WMAP mission has observed the spectrum of density perturbations, and the result is that the spectrum is almost scale invariant, but not quite. This near-scale invariance is a generic feature of inflation, but the actual size of the deviations of a scale invariant spectrum is model dependent. The deviations from scale invariance are proportional to the slow-roll parameters, which thus constrain potentials, but do not give a specific form of the potential.

We now return to our nonminimal inflationary model. Our goal in this chapter is to successfully quantize the nonminimal inflationary model. We will apply this in chapter ?? to calculate one-loop radiative corrections to the fermion propagator. However, as we will see in the next sections, quantizing a field theory is straightforward in Minkowski space, but quite subtle in an expanding universe. For good literature on quantum field theory in curved backgrounds, see [30] and [31]. One of the main problems is that the vacuum at one point in time is not the same vacuum at some later time. This leads to the very special consequence that particles can be created through the expansion of the universe.

The outline of this chapter will be the following. In section 5.1 we will quickly introduce quantum field theory in Minkowski space. We will quantize our theory by introducing creation and annihilation operators, define a vacuum and show that this remains the state of lowest energy by a specific choice of quantizing our fields. In section 5.2 we will quantize the nonminimally coupled inflaton field and we will see that there is no unique choice for the vacuum in an expanding universe. However, in an inflationary universe we can make a natural vacuum choice in certain regimes which allows us to quantize the field uniquely and calculate the scalar propagator. Finally in section 5.3 we will extend our findings and quantize the two-Higgs doublet model. We will see that we can again solve the equations of motion exactly which allows us to calculate the propagator for the two-Higgs doublet model in quasi de Sitter space.

5.1 Quantum field theory in Minkowski space

We start with a simple action with one real scalar field with a mass $m$,

$$
S = \int d^4x \left( -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right) = \int d^3x dt \left( \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 \right). \tag{5.1}
$$

Minimizing this action gives the equation of motion for the scalar field $\phi$, which in this case is the Klein-Gordon equation,

$$
0 = -\eta^{\mu\nu} \partial_\mu \partial_\nu \phi + m^2 \phi = \ddot{\phi} - \nabla^2 \phi + m^2 \phi. \tag{5.2}
$$

To simplify this equation we can perform a Fourier expansion of the field, i.e.

$$
\phi(x,t) = \int \frac{d^3k}{(2\pi)^3} \phi_k(t) e^{ik \cdot x}, \tag{5.3}
$$

where $k = |k|$. If the field $\phi$ is real, we have the additional condition that $\phi_k^* = \phi_{-k}$. Substituting this expansion in our action (5.1), we find

$$
S = \int dt \frac{d^3k}{(2\pi)^3} \left( \frac{1}{2} |\phi_k|^2 - \frac{1}{2} (k^2 + m^2) |\phi_k|^2 \right). \tag{5.4}
$$
with equation of motion
\[ 0 = \ddot{\hat{\phi}}_k + \omega_k^2 \hat{\phi}_k, \quad \omega_k^2 = k^2 + m^2 \] (5.5)
where \( k^2 = \mathbf{k} \cdot \mathbf{k} \). Thus, each mode function \( \hat{\phi}_k \) satisfies the equation of motion for a harmonic oscillator with frequency \( \omega_k \). The general solution of the mode functions is
\[ \hat{\phi}_k(t) = \alpha_k e^{i \omega_k t} + \beta_k e^{-i \omega_k t}, \] (5.6)
where \( \alpha_k \) and \( \beta_k \) are constants. The total solution \( \hat{\phi}(\mathbf{x}, t) \) is therefore the sum of an infinite amount of harmonic oscillators. We can quantize each harmonic oscillator separately by imposing the canonical quantization condition
\[ [\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)] = i \delta^3(\mathbf{x} - \mathbf{x}'), \] (5.7)
where
\[ \hat{\pi}(\mathbf{x}, t) = \frac{\delta L}{\delta \hat{\phi}(\mathbf{x}, t)} = \dot{\phi}(\mathbf{x}, t). \] (5.8)
If we would again substitute the mode expansion (5.3) we would find that the modes satisfy the commutation relation
\[ [\hat{\phi}_k(t), \hat{\pi}_k(t)] = i(2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}'), \] (5.9)
with
\[ \hat{\pi}_k(t) = \frac{\delta L}{\delta \hat{\phi}_k(t)} = \dot{\phi}_k(t). \] (5.10)
A convenient way to quantize the fields is now to use the mode expansion
\[ \hat{\phi}_k(t) = \frac{1}{\sqrt{2\omega_k}} \left( \hat{\alpha}_k^- e^{-i \omega_k t} + \hat{\alpha}_k^+ e^{i \omega_k t} \right), \] (5.11)
where \( \hat{\alpha}_k^+ \) and \( \hat{\alpha}_k^- \) are the creation and annihilation operators. Basically, we have replaced the \( \alpha_k \) and \( \beta_k \) in the general solution (5.6) by operators. Note that the condition \( \hat{\phi}_k^\dagger = \hat{\phi}_{-k} \) gives \( (\hat{\alpha}_k^-)^\dagger = \hat{\alpha}_k^+ \). The conjugate momentum is
\[ \hat{\pi}_k(t) = i \sqrt{\frac{\omega_k}{2}} \left( \hat{\alpha}_k^- e^{-i \omega_k t} + \hat{\alpha}_k^+ e^{i \omega_k t} \right). \] (5.12)
We could substitute the expansion (5.11) into the Fourier decomposition (5.3), and we find
\[ \hat{\phi}(\mathbf{x}, t) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left( \hat{\alpha}_k^- e^{-i \omega_k t} + \hat{\alpha}_k^+ e^{i \omega_k t} \right) e^{i \mathbf{k} \cdot \mathbf{x}} \]
\[ = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left( \hat{\alpha}_k^- e^{i (\mathbf{k} \cdot \mathbf{x} - \omega_k t)} + \hat{\alpha}_k^+ e^{-i (\mathbf{k} \cdot \mathbf{x} - \omega_k t)} \right), \] (5.13)
thus we expand the field \( \phi \) in terms of the plane wave solutions of the Klein-Gordon equation, with operator-valued integration constants. If we now want the mode expansion (5.11) to satisfy the canonical commutator (5.9), we find that the creation and annihilation operators must satisfy
\[ [\hat{\alpha}_k^-, \hat{\alpha}_k^+] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \] (5.14)
\[ [\hat{\alpha}_k^+, \hat{\alpha}_k^-] = [\hat{\alpha}_k^-, \hat{\alpha}_k^-] = 0. \]
We now define the vacuum state that is annihilated by all annihilation operators, i.e.
\[ \hat{a}^-|0\rangle = 0. \] (5.15)

Excited states are made by acting with the creation operator on the vacuum, for example
\[ \hat{a}^+|0\rangle = |1_k\rangle, \hat{a}^+ \hat{a}^+|0\rangle = |2_k\rangle, \hat{a}^+ \hat{a}^+|0\rangle = |1_k, 1_k\rangle. \]
In general we can define a state as
\[ |n\rangle \equiv |n_{k_1}, n_{k_2}, n_{k_3}, \ldots\rangle = \prod_{s} \left( \hat{a}^+_{k_s} \right)^{n_{k_s}} \sqrt{n_{k_s}!} |0\rangle. \] (5.16)

These states span up the Hilbert space, and all states are orthonormal. The factors in the denominators are there because
\[ \hat{a}^+|n_{k}\rangle = \sqrt{n + 1}(n + 1)_{k}\rangle \]
\[ \hat{a}^-|n_{k}\rangle = \sqrt{n}(n - 1)_{k}\rangle. \] (5.17)

The physical interpretation is that the creation operator \( \hat{a}^+ \) now creates a particle in the Fourier mode with momentum \( k \), whereas the annihilation operator annihilates a particle in such a Fourier mode. We also define the particle number operator \( \hat{N}_k \).
\[ \hat{N}_k \equiv \hat{a}^+ \hat{a}^- \] (5.18)

The reason for this becomes clear if we apply the number operator (5.18) on a general state \( |n\rangle \) from Eq. (5.16), which gives
\[ \hat{N}_k |n\rangle = \hat{a}^+ \hat{a}^- |n\rangle = n_{k}|n\rangle. \] (5.19)

So the number operator counts the number of particles with momentum \( k \). Note that the operator that gives the total number of particles is
\[ \hat{N} = \int \frac{d^3k}{(2\pi)^3} \hat{N}_k = \int \frac{d^3k}{(2\pi)^3} \hat{a}^+ \hat{a}^- \] (5.20)

This gives \( \hat{N}|n\rangle = N|n\rangle \), where \( N \) is the total number of particles. Let us now calculate the energy of our system. First we calculate the Hamiltonian,
\[ H = \int \frac{d^3k}{(2\pi)^3} \left[ \pi_{k}\dot{\phi}_{k} - L \right] \]
\[ = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2} \pi_{k}\pi_{-k} + \omega_k^2 \phi_{k}\phi_{-k} \right]. \] (5.21)

Now we quantize this Hamiltonian system by substituting the expansions for \( \hat{\phi}_k \) and \( \hat{\pi}_k \) from Eqs. (5.11) and (5.12). This gives the Hamiltonian operator
\[ \hat{H} = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{2} \left[ \hat{\pi}_k \hat{\phi}_k + \hat{\phi}_k \hat{\pi}_k \right] \]
\[ = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{2} \left[ 2\hat{\phi}_k \hat{\phi}_k + (2\pi)^3 \delta^3(0) \right]. \] (5.22)

where in the third line I have used the commutation relation (5.15). The infinite term \( \delta^3(0) \) is a consequence of the fact that we are working in an infinite space volume. If we would work in a box, this would be the volume \( V \) of the box. Dividing by this term then gives us the energy and particle number densities. We can recognize the number operator \( \hat{N}_k \) from
Eq. (5.18). If we now act with the Hamiltonian operator (5.22) on a general state $|n\rangle$ from Eq. (5.16) we find
\[ \hat{H}|n\rangle = \left( \int \frac{d^3k}{(2\pi)^3} \omega_k n_k + \int \frac{d^3k}{2} \omega_k \right) |n\rangle. \] (5.23)
This expression shows that if we add a particle with momentum $k$ to our system, the energy of our system increases with a factor $\omega_k$. The curious infinite term
\[ E_0 = \int d^3k \frac{\omega_k}{2} \] (5.24)
is the vacuum energy of our system, which we can see by acting with the Hamiltonian (5.22) on the vacuum from Eq. (5.15),
\[ \hat{H}|0\rangle = E_0|0\rangle. \] (5.25)
This infinite vacuum energy is fortunately not important, because we are only interested in energy excitations ("particles") with respect to the vacuum. So this vacuum we can just subtract from our Hamiltonian and has no physical effect.
To summarize this section, we have successfully quantized our action (5.1) by imposing the canonical commutator (5.7) and expanding our field in creation and annihilation operators as in Eq. (5.13). Our fields and physical quantities are now operators that act on wave functions (5.16). We have seen that the energy is quantized and that there is an infinite amount of energy in the vacuum. Energy excitations with respect to this vacuum are interpreted as particles.

### 5.1.1 Bogoliubov transformation

In Eq. (5.11) we have expanded our field in terms of the two fundamental solutions of the equation of motion Eq. (5.5) with operator-valued prefactors. The two fundamental solutions (including normalization) are
\[ u_k(t) = \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t} \]
\[ u_k^*(t). \] (5.26)
We can define two other linearly independent solutions as
\[ v_k(t) = \alpha_k u_k(t) + \beta_k u_k^*(t) \]
\[ v_k^*(t). \] (5.27)
Why would we not expand our field $\phi_k$ in terms of these solutions $v_k(t)$ instead of the fundamental solution $u_k$? Well, we have no good reason for either of the choices, so let us just expand our field in terms of the solutions $v_k(t)$ from Eq. (5.27). This gives
\[ \hat{\phi}_k(t) = \hat{a}_{-k} v_k(t) + \hat{a}_{-k}^* v_k^*(t). \] (5.28)
If we now impose the canonical commutation relation (5.9), we find that our solutions satisfy this canonical commutator if we demand Eq. (5.15) and additionally are normalized as
\[ v_k^* v_k - v_k^* v_k = i. \] (5.29)
The expression on the left-hand side is the well known Wronskian, which is used a lot to determine linear independence of solutions. The Wronskian is time independent, which can easily be checked by acting with a time derivative on (5.29) and using the equations...
of motion (5.4). If we now substitute our solution for \( v_k \) from Eq. (5.27) with \( u_k \) from Eq. (5.26) into (5.29), we find the condition
\[
|\alpha_k|^2 - |\beta_k|^2 = 1.
\] (5.30)
Note that we could have written our expansion (5.28) in terms of the solutions \( u_k \) as
\[
\hat{\phi}_k(t) = \hat{b}_k u_k(t) + \hat{b}_k^* u_k^*(t).
\] (5.31)
A simple exercise shows that the new creation and annihilation operators \( \hat{b}_k \) and \( \hat{b}_k^* \) can be expressed in terms of the "old" creation and annihilation operators as
\[
\hat{b}_k = \alpha_k \hat{a}_k - \beta_k \hat{a}_k^* \quad \text{and} \quad \hat{b}_k^* = \alpha_k \hat{a}_k^* + \beta_k \hat{a}_k.
\] (5.32)
This is a Bogoliubov transformation. If the commutation relations (5.15) are satisfied for \( \hat{a}_k \) and \( \hat{a}_k^* \), then the same commutators are satisfied for \( \hat{b}_k \) and \( \hat{b}_k^* \) if the relation (5.30) is fulfilled. If we now act with the total number operator, which in this case is \( \int d^3k |\beta_k|^2 \), on the vacuum defined in Eq. (5.15), we find
\[
\langle 0 | \int d^3k \hat{b}_k^* \hat{b}_k | 0 \rangle = \langle 0 | \int d^3k |\beta_k|^2 | 0 \rangle = \int d^3k |\beta_k|^2.
\] (5.33)
So instead of finding zero particles in the vacuum, we now find \( \int d^3k |\beta_k|^2 \) particles. Since we only have the condition (5.30) on these coefficients, \( |\beta_k|^2 \) can run from zero to infinity. Our particle number apparently depends on the choice of coefficients in the mode expansion in Eq. (5.28)! The question now remains: can we now, based on physical arguments, make some choice for \( \alpha_k \) and \( \beta_k \)? The answer is: yes, we can. We calculate again the energy by acting with the Hamiltonian on the vacuum of Eq. (5.15). This time the Hamiltonian is expressed in terms of the fields defined in Eq. (5.28) (or Eq. (5.31) together with Eq. (5.32)). We find after a small calculation that the energy of the vacuum is
\[
E_{\text{vac}} = \langle 0 | \hat{H} | 0 \rangle = \langle 0 | \int d^3k (1 + 2|\beta_k|^2) \frac{\omega_k}{2} | 0 \rangle = \int d^3k (1 + 2|\beta_k|^2) \frac{\omega_k}{2}.
\] (5.34)
Now we can use our physical argument: if we want our vacuum to be the state of lowest energy, then the only choice we have is to take \( \beta = 0 \), which reduce our lowest energy to \( E_{\text{vac}} = E_0 \) from Eq. (5.24). In this case, also our particle number does not depend on the choice of the vacuum.
Another important observation is that if the vacuum is the state of lowest energy at one point in time, then it will also be the state of lowest energy at some later time. The reason is that the Hamiltonian is constant in time, which can easily be seen when concerning the expression for \( H \) from Eq. (5.21). The creation and annihilation are constants in time, and so is our frequency \( \omega_k \). Another way to see that the Hamiltonian is time independent is by considering the (classical) expression for \( H \) and taking the time derivative
\[
\frac{d}{dt} H = \frac{d}{dt} \int d^3k \left( \frac{1}{2} |\alpha_k|^2 + \frac{1}{2} (k^2 + m^2) |\beta_k|^2 \right)
= \int d^3k \left( \frac{1}{2} \phi_k \dot{\phi}_k + \frac{1}{2} (k^2 + m^2) \phi_k \dot{\phi}_k - \frac{1}{2} \dot{\phi}_k \phi_k + \frac{1}{2} \phi_k \dot{\phi}_k + \frac{1}{2} (k^2 + m^2) \phi_k \phi_k \right)
= 0,
\]
where we have used the equation of motion Eq. (5.5) in the final step. So we conclude that the state of lowest energy is given by the vacuum (5.15) when we expand our field as in Eq.
(5.28) with \(a = 1\) and \(\beta = 0\), and it remains the lowest energy state if our system evolves in time.

The situation changes of course when our Hamiltonian in Eq. (5.21) is not time independent. For example, the frequency of the harmonic oscillator could be time dependent \(\omega_k = \omega_k(t)\). As we will see in the section 5.2, this is precisely what happens in an expanding universe. As we have shown in this section, a harmonic oscillator with a constant frequency allows us to define a unique vacuum state that is the state of lowest energy at all times. If the frequency is on the other hand time dependent, such a unique vacuum does not exist. We have also seen that the notion of particles is not well defined in these cases. Therefore in an expanding universe, we will abandon the whole particle concept and only work with the propagator, which is still well-defined. Before we do this however, let us consider a toy model of a harmonic oscillator with a varying frequency.

5.1.2 In and out regions

Suppose we have a harmonic oscillator with a varying frequency, i.e.

\[ \ddot{\phi}_k + \omega_k^2(t)\phi_k = 0. \tag{5.35} \]

In principle we have a different solution for the fields \(\phi_k\) at each instant of time. This will give us a different expansion in creation and annihilation operators every time, and our vacuum will be different at each instant of time. So it seems this is as far as we can go.

We can make our lives easier however by defining the so-called \(\text{in}\) and \(\text{out}\) regions. In these regions the frequency is approximately constant and we can find solutions of Eq. (5.35). As a preview to the next section, in an expanding universe we can consider regions where the universe is asymptotically flat, i.e. the scale factor becomes constant in these regions.

Suppose now that we have some constant frequency \(\omega_k^{\text{in}}\) for \(t < t_0\) (the \(\text{in}\) region) and a constant frequency \(\omega_k^{\text{out}}\) for \(t > t_1\) (the \(\text{out}\) region). We can now write our field expansion in the \(\text{in}\) region as

\[ \hat{\phi}(x, t) = \int \frac{d^3k}{(2\pi)^3} \left( \hat{a}_{-k}^{\text{in}} u_{k}^{\text{in}} + \hat{a}_{-k}^{\text{out}} (u_{k}^{\text{in}^*})^* \right) e^{i k \cdot x}, \tag{5.36} \]

where the fields \(u_{k}^{\text{in}}\) are the fundamental solutions of Eq. (5.35) as in Eq. (5.26) with frequency \(\omega_k = \omega_k^{\text{in}}\). As we have seen, with this expansion the vacuum defined by \(\hat{a}_{-k}^{\text{in}}|0_{\text{in}}\rangle = 0\) is also the state of lowest energy. We could do the same for the \(\text{out}\) region, and write

\[ \hat{\phi}(x, t) = \int \frac{d^3k}{(2\pi)^3} \left( \hat{b}_{-k}^{\text{out}} u_{k}^{\text{out}} + \hat{b}_{-k}^{\text{in}} (u_{k}^{\text{out}^*})^* \right) e^{i k \cdot x}. \tag{5.37} \]

Again, the annihilation operators in the \(\text{out}\) region also define a vacuum by \(\hat{b}_{-k}^{\text{in}}|0_{\text{out}}\rangle = 0\), which is the state of lowest energy in the \(\text{out}\) region. We are now interested in the energy difference between the \(\text{in}\) and \(\text{out}\) vacua. The energy difference will basically tell us that particles have been created. To relate the vacua, we first have to express \(u_{k}^{\text{in}}\) in terms of \(u_{k}^{\text{out}}\). First we define

\[ u_{k}^{\text{in}} = u_{k}^{\text{in}^*} e^{i k \cdot x} = \frac{1}{\sqrt{2\omega_k^{\text{in}}}} e^{i (k \cdot x - \omega_k^{\text{in}} t)}, \quad (u_{k}^{\text{in}})^* \]

\[ u_{k}^{\text{out}} = u_{k}^{\text{out}^*} e^{i k \cdot x} = \frac{1}{\sqrt{2\omega_k^{\text{out}}}} e^{i (k \cdot x - \omega_k^{\text{out}} t)}, \quad (u_{k}^{\text{out}})^*. \tag{5.38} \]
This allows us to write the field expansions (5.36) and (5.37) as

\[
\hat{\phi}(x, t) = \int \frac{d^3k}{(2\pi)^3} \left( \hat{a}_k u_k^{in} + \hat{a}_k^+ (u_k^{in})^* \right),
\]

\[
\hat{\phi}(x, t) = \int \frac{d^3k}{(2\pi)^3} \left( \hat{b}^- u_k^{out} + \hat{b}_k^+ (u_k^{out})^* \right). \tag{5.39}
\]

Note the change of sign of \( k \) in the creation operators \( \hat{a}_k^+ \) and \( \hat{b}_k^+ \). Since the combinations \( u_k^{in} \) and \( (u_k^{in})^* \), as well as the combinations \( u_k^{out} \) and \( (u_k^{out})^* \) form a complete set of solutions, we are allowed to express the \( in \) modes in terms of the \( out \) modes as

\[
u_k^{in} = \int \frac{d^3k'}{(2\pi)^3} \left( a_{kk'} u_k^{out} + \beta_{kk'} (u_k^{out})^* \right). \tag{5.40}
\]

Imposing the canonical commutator (5.9) on the field in Eq. (5.36), we find the following condition on the coefficients

\[
\int \frac{d^3j}{(2\pi)^3} \left( a_{kj} \hat{a}_{kj}^* - \beta_{kj} \beta_{kj}^* \right) = \delta_{kk'}. \tag{5.41}
\]

Substituting this expansion into Eq. (5.36), we find that we can express the creation and annihilation operators in the \( out \) region as

\[
\hat{b}_k^+ = \int \frac{d^3k'}{(2\pi)^3} \left( a_{kk'}^* \hat{a}_{kk'}^+ + \beta_{kk'} \hat{a}_{kk'}^* \right),
\]

\[
\hat{b}_k^- = \int \frac{d^3k'}{(2\pi)^3} \left( a_{kk'} \hat{a}_{kk'}^+ + \beta_{kk'}^* \hat{a}_{kk'}^+ \right). \tag{5.42}
\]

This is a generalized Bogoliubov transformation as in Eq. (5.32). Consider the following situation: initially we are in the \( in \) region where we have the vacuum \( |0_{in}\rangle \) with zero particle number and energy \( E_0 \). We are now in a position to calculate the particle number in the \( out \) region by using the new creation and annihilation operators from Eq. (5.42). For the particle number, this gives

\[
\langle 0_{in} | N_k^{out} | 0_{in}\rangle = \langle 0_{in} | \hat{b}_k^+ \hat{b}_k^- | 0_{in}\rangle = \int \frac{d^3k'}{(2\pi)^3} |\beta_{kk'}|^2. \tag{5.43}
\]

Thus we see that although the particle number is zero in the \( in \) region, it becomes nonzero in the \( out \) region. The important requirement is the time dependence of the frequency of the harmonic oscillator. This happens for example for a uniform accelerated observer in Minkowski space. Even though the fields are in the zero particle vacuum state, the uniform accelerated observer will detect a distribution of particles similar to a thermal bath of blackbody radiation. This effect is called the Unruh effect.

Another example of a scalar field theory with a time dependent frequency is a scalar field in an expanding universe. The vacuum can be the zero particle state at some instant of time, but at a later time we would find particles in this vacuum. Therefore, we can say that particles are created by the expansion of the universe. In the next section we will look more closely at quantum field theory in an expanding universe. We will quantize our nonminimal scalar inflationary model, and see that we can again write this as a harmonic oscillator with a time dependent frequency. The choice of the vacuum state will therefore be difficult because there is not a unique choice, but we will see that we can define a natural vacuum state in quasi de Sitter space, the so-called Bunch-Davies vacuum. With this vacuum choice we can eventually derive the scalar propagator. In section 5.3 we will generalize our findings to the two-Higgs model. We will quantize this theory and find the scalar propagator, which will be useful in calculating quantum effects of the inflaton on the dynamics of fermions in chapter 6.
5.2 Quantization of the nonminimally coupled inflaton field

We consider again the action for a real scalar field that is nonminimally coupled to gravity. For convenience, we will give the explicit action again

\[ S = \int d^4x \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - V(\phi) - \frac{1}{2} \xi R \phi^2 \right). \]  

(5.44)

The field equation for \( \phi \) is

\[-\Box \phi + \xi R \phi + \frac{dV(\phi)}{d\phi} = 0, \]

where

\[ \Box \phi = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \phi. \]  

(5.46)

In quantum field theory, our field \( \phi \) is a quantum field. The main contribution is the classical homogeneous inflaton field, but on top of that there are small quantum fluctuations. We therefore consider the following field

\[ \phi(t, \mathbf{x}) = \phi_0(t) + \delta \phi(t, \mathbf{x}), \]  

(5.47)

where the expectation value of the field is the classical inflaton field \( \phi_0 \), i.e. \( \langle \phi(t, \mathbf{x}) \rangle = \phi_0(t) \), whereas the fluctuations satisfy \( \langle \delta \phi(t, \mathbf{x}) \rangle = 0 \). We expand our action to quadratic order in fluctuations and derive the field equations for the classical inflaton field and the quantum fluctuations. First of all, the expansion of the action (5.44) to linear order in fluctuations vanishes, since this is precisely what we do when we want to find the classical field equation for the background field \( \phi_0 \). If we use the quartic potential \( V(\phi) = \frac{1}{4} \lambda \phi^4 \) from Eq. (4.15) and the FLRW metric (B.11) the classical field equation is

\[ \ddot{\phi}_0 + 3H \dot{\phi}_0 + \xi R \phi_0 + \lambda \phi_0^3 = 0. \]  

(5.48)

The classical equation of motion (5.48) determines the dynamics of the inflaton field and is responsible for inflation, which we have discussed extensively in section 4.1. The Lagrangian density for the quantum fluctuations is then

\[ \mathcal{L}_Q = \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} (\partial_\mu \delta \phi)(\partial_\nu \delta \phi) - \frac{1}{2} \left( 3 \lambda \phi_0^2 + \xi R \right) \delta \phi^2 \right] + \mathcal{O}(\delta \phi^3), \]  

(5.49)

where I have used the index \( Q \) to indicate that this Lagrangian contains quantum fluctuations. We see that the classical inflaton field acts as a mass term for the quantum fluctuations, so I will define a mass term

\[ m^2 = 3\lambda \phi_0^2. \]  

(5.50)

Again with the substitution of the FLRW metric we find the field equation for the fluctuations

\[ \delta \ddot{\phi} + 3H \delta \dot{\phi} - \nabla^2 \delta \phi + \xi R \delta \phi + m^2 \delta \phi = 0. \]  

(5.51)

The quantum fluctuations, described by Eq. (5.51), are the origin of density perturbations, whose spectrum we can measure from the CMBR. Also, quantum fluctuations can influence the dynamics of fermions, which we will discuss in chapter 6. For now, let us continue with the quantization of our inflaton field.
We could also use the conformal FLRW metric by making the substitution $dt = a(\eta)d\eta$, i.e. $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$, such that the field equations (5.48) and (5.51) become

$$
\phi''_0 + 2 \frac{a'}{a} \phi'_0 + a^2 \xi R \phi_0 + a^2 \lambda \phi^3_0 = 0
$$

$$
\delta \phi'' + 2 \frac{a'}{a} \delta \phi' - \nabla^2 \phi + a^2 (\xi R + m^2) \delta \phi = 0.
$$

(5.52)

Now we perform a conformal rescaling of our fields, $\phi_0 \rightarrow \phi_0 = a \phi_0$ and $\delta \phi \rightarrow \delta \phi = a \delta \phi$, which allows us to write

$$
\phi''_0 + \left( a^2 \xi R - \frac{a''}{a} \right) \phi_0 + \lambda \phi^3_0 = 0
$$

$$
\delta \phi'' + \left( -\nabla^2 + a^2 \xi R - \frac{a''}{a} + a^2 m^2 \right) \delta \phi = 0.
$$

(5.53)

The field $\phi_0 = a \phi_0$ is a classical field, so we do not need to quantize this field. To quantize the quantum fluctuation $\delta \phi = \delta \phi / a$ we first calculate the Hamiltonian by using the Lagrangian density (5.49) and the conformal FLRW metric

$$
H = \int d^3 x \left[ \pi \delta \phi' - L' \right]
$$

$$
= \int d^3 x \left\{ \frac{1}{2} \pi^2 + a^2 (\nabla \delta \phi)^2 + \frac{1}{2} a^4 \left( m^2 + \xi R \right) \delta \phi^2 \right\}, \quad (5.54)
$$

where the conjugate momentum is

$$
\pi(\eta, x) = \frac{\partial L}{\partial \delta \phi} = a^2 \delta \phi'.
$$

(5.55)

We quantize the field $\delta \phi$ by imposing the canonical commutator

$$
[\delta \phi(\eta, x), \pi(\eta, x')] = i \delta^3(x - x'),
$$

(5.56)

The canonical commutator is satisfied if we expand the quantum fluctuation $\delta \phi$ in the following way

$$
\delta \phi(\eta, x) = \int \frac{d^3 k}{(2\pi)^3} \left( a_k^\dagger \delta \phi_k(\eta) + a_k \delta \phi_k^*(\eta) \right) e^{ikx},
$$

(5.57)

and

$$
\pi(\eta, x) = a^2 \int \frac{d^3 k}{(2\pi)^3} \left( a_k^\dagger \delta \phi_k'(\eta) + a_k \delta \phi_k''(\eta) \right) e^{ikx}.
$$

(5.58)

Substituting Eq. (5.57) and Eq. (5.58) into the commutator (5.56), we find that the commutation relation is satisfied when

$$
[a_k^\dagger, a_k^+ ] = (2\pi)^3 \delta^3(k - k'),
$$

(5.59)

and when the modes satisfy the normalization condition

$$
W[\delta \phi_k, \delta \phi_k^*] \equiv \delta \phi_k \delta \phi_k^* - \delta \phi_k^* \delta \phi_k = \frac{i}{a^2},
$$

(5.60)

where $W$ is the Wronskian between the two modes. Now we rewrite the expansion (5.57) with the conformally rescaled field $\delta \phi = a \delta \phi$, such that the mode functions become $\delta \phi_k = a \delta \phi_k$. So we have the expansion

$$
\delta \phi(\eta, x) = \int \frac{d^3 k}{(2\pi)^3} \left( a_k^\dagger \delta \phi_k(\eta) + a_k^+ \delta \phi_k^*(\eta) \right) e^{ikx},
$$

(5.61)
If we now substitute the expansion (5.61) with the conformally rescaled fields \( \delta \varphi_k = a \delta \phi_k \) into (5.53) and also recognize \( R \) from Eq. (B.27), we find that the modes \( \varphi_k(\eta) \) (and also \( \varphi_k^*(\eta) \)) satisfy

\[
\delta \varphi_k'' + \omega_k^2(\eta) \delta \varphi_k = 0,
\]

with

\[
\omega_k^2(\eta) = k^2 + a^2 \left[ R \left( \xi - \frac{1}{6} \right) + m^2 \right].
\]

We can see that \( \omega_k = k^2 \) for a massless scalar field that is conformally coupled. In other words the massless conformally coupled scalar obeys the same field equation as a massless scalar in Minkowski space. This is another confirmation of our discussion in section 4.2. Note that it seems that for \( \xi \ll 0 \), \( \omega_k^2 \) in Eq. (5.63) can become negative. This would lead to exponential instead of plane wave solutions of the field equation (5.62) and exponential growth of the fluctuations. However, if we remember that \( R = 6(2H^2 + \dot{H}) = 6H^2(2 - \epsilon) \), and that during inflation the Hubble parameter is proportional to the inflaton field \( \phi_0 \) (see Eq. (4.19)), we find that the negative contribution of \( \xi R \) to the frequency is canceled by the \( a^2m^2 = 3a^2 \lambda \phi_0^2 \) term. So during inflation the fluctuations remain small compared to the inflaton field \( \phi_0 \).

From Eq. (5.62) we see that the quantum fluctuation modes \( \varphi_k \) are harmonic oscillators with a frequency \( \omega_k(\eta) \). This frequency from Eq. (5.63) is not constant in time, because the scale factor \( a \), Ricci scalar \( R \) and classical inflaton background field \( \varphi_0 = a \phi_0 \) are all time dependent. As we have seen in the previous section, this means that if we define our mode functions \( \varphi_k \) in Eq. (5.57) such that the vacuum defined by \( \hat{a}_k^\dagger |0 \rangle = 0 \) is also the state of lowest energy at some instant of time, it will not be the state of lowest energy at a later time. This can be seen more clearly if we write the Hamiltonian (5.54) in terms of the conformally rescaled fields \( \delta \varphi \),

\[
H = \int d^3x \left[ \frac{1}{2} (\delta \varphi')^2 + (\nabla \delta \varphi)^2 + \frac{1}{2} a^2 \left( m^2 + R(\xi - \frac{1}{6}) \right) \delta \varphi^2 \right].
\]

Now we construct the Hamiltonian operator by inserting the expansion for \( \delta \varphi \) from Eq. (5.61). This gives

\[
\hat{H} = \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{2} \left[ (\delta \varphi')^2 + \omega_k^2(\eta)(\delta \varphi_k)^2 \right] \left[ \hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger \right] \right. \\
+ \frac{1}{2} \left[ (\delta \varphi')^2 + \omega_k^2(\eta)(\delta \varphi_k)^2 \right] \hat{a}_k^\dagger \hat{a}_k - \frac{1}{2} \left[ (\delta \varphi_k')^2 + \omega_k^2(\eta)(\delta \varphi_k^*)^2 \right] \hat{a}_k^\dagger \hat{a}_k^\dagger \right\}.
\]

We can as usual calculate the vacuum energy with respect to the vacuum that is annihilated by \( \hat{a}_k^\dagger \),

\[
E = \langle 0 | \hat{H} | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \left[ (\delta \varphi_k')^2 + \omega_k^2(\eta)(\delta \varphi_k)^2 \right] \delta^3(0).
\]

The problem of defining a good vacuum is now clear. If we choose the mode functions such that the vacuum energy is minimized at some instant of time, it will not be the lowest energy state at a later time if the frequency \( \omega_k \) is time dependent. The extra energy with respect to the old vacuum is then caused by particle production. This prevents us from defining a natural vacuum state with zero particles and energy. However, in some cases we can define a natural vacuum state, and in the following section we will show when this happens.
5.2.1 Adiabatic vacuum

Suppose now we have a certain regime where the frequency \( \omega_k \) from Eq. (5.63) is slowly changing in time. One can imagine that even though the Hamiltonian is not constant in time, particle production is very small because the frequency changes very slowly. To be more quantitative about changing 'slowly', we demand that the relative change of \( \omega_k \) during one period of oscillation is negligible, which gives the condition

\[
\frac{\omega_k'}{\omega_k} \ll 1. \tag{5.67}
\]

This is the adiabaticity condition, and the regime where this condition is satisfied is called the adiabatic regime. Suppose now we have a time \( \eta_0 \) where we have done a mode expansion in terms of creation and annihilation operators that minimizes the energy (5.66) at this time. This means that

\[
\delta \varphi_k(\eta_0) = \frac{1}{\sqrt{2\omega_k(\eta_0)}} \quad \delta \varphi_k'(\eta_0) = i \sqrt{\frac{\omega_k(\eta_0)}{2}} = i \omega_k(\eta_0) \frac{1}{\sqrt{2\omega_k(\eta_0)}}. \tag{5.68}
\]

This gives a (minimized) energy in the modes of \( \delta \varphi_k \) of

\[
E_k(\eta_0) = \frac{1}{2} \left( |\delta \varphi_k'|^2 + \omega_k^2 |\delta \varphi_k|^2 \right) \bigg|_{\eta=\eta_0} = \frac{1}{2} \omega_k. \tag{5.69}
\]

However, the usual vacuum that we define from the annihilation operators at this time is not a good vacuum, in the sense that the vacuum is not the lowest energy state at a later time. We should therefore construct a better vacuum. If we are in the adiabatic regime, the field equation (5.62) gives approximate solutions at a later time \( \eta \) by using the WKB approximation,

\[
\delta \varphi_k^{\text{approx}}(\eta) = \frac{1}{\sqrt{2\omega_k(\eta)}} \exp \left[ i \int_{\eta_0}^{\eta} \omega_k(\eta') d\eta' \right]. \tag{5.70}
\]

These mode functions define the adiabatic vacuum at some time \( \eta \) in the adiabatic regime. We would like to stress that at time \( \eta_0 \) this is not the true vacuum state that minimizes the energy. We would like to see what the energy of these mode functions with respect to the adiabatic vacuum at time \( \eta_0 \) is compared to the minimal energy (5.69) of the mode functions (5.68) with respect to the true vacuum at time \( \eta_0 \). Therefore we calculate the value and derivative of the functions \( \delta \varphi_k^{\text{approx}}(\eta) \) at time \( \eta_0 \). This is

\[
\delta \varphi_k^{\text{approx}}(\eta_0) = \frac{1}{\sqrt{2\omega_k(\eta_0)}} \quad \delta \varphi_k'^{\text{approx}}(\eta_0) = \left( i \omega_k(\eta_0) - \frac{1}{2} \frac{\omega_k'(\eta_0)}{\omega_k(\eta_0)} \right) \frac{1}{\sqrt{2\omega_k(\eta_0)}}. \tag{5.71}
\]

This yields an energy in the mode \( \delta \varphi_k \) with respect to the adiabatic vacuum of

\[
E_k(\eta_0) = \frac{1}{2} \left( |\delta \varphi_k'|^2 + \omega_k^2 |\delta \varphi_k|^2 \right) \bigg|_{\eta=\eta_0} = \frac{1}{2} \omega_k + \frac{1}{16} \frac{\omega_k^2}{\omega_k} \approx \frac{1}{2} \omega_k. \tag{5.72}
\]

So the energy in the adiabatic vacuum is approximately equal to the energy in the true vacuum at time \( \eta_0 \). The adiabatic vacuum however is a 'good' vacuum for every time \( \eta \) in.
the adiabatic regime, whereas the true vacuum is only the state of lowest energy at time \( \eta_0 \).

We would like to see that the WKB approximation of the modes in Eq. (5.70) becomes exact. This is the case when the frequency becomes constant. This happens for the frequency (5.63) in quasi de Sitter space, with the scale factor from Eq. (2.23). The frequency then is

\[
\omega_k^2(\eta) = k^2 + \frac{1}{\eta^2} \frac{m^2 + R(\zeta - \frac{1}{6})}{H^2(1 - c)^2}.
\]  

(5.73)

We see immediately that the frequency becomes a constant in the limit when \( \eta \to -\infty \).

Also the adiabaticity condition in Eq. (5.67) gives \( \frac{\omega_k^2}{\omega_k} \to 0 \). With our knowledge about the adiabatic vacuum, we now conclude that if we take \( \eta_0 \to -\infty \), we can define a vacuum from the mode functions at time \( \eta_0 \) that minimizes that energy at \( \eta_0 \). Note that this limit only defines a natural vacuum state if \( k^2 > \frac{1}{\eta^2(1 - c)^2} \). This true vacuum will remain the lowest energy state at a later time in the adiabatic regime to a very good approximation.

Our conclusion from this section is that in quasi de Sitter space we can define a natural vacuum for early times that is the state of lowest energy. This natural vacuum state in the limit \( \eta \to -\infty \) is called the Bunch-Davies vacuum. Physically, it corresponds zero particles in the infinite past.

### 5.2.2 Solutions of the mode functions

Let us now return to the field equation for the fluctuations (5.62). In general this equation cannot be exactly solved, but in some cosmological space times it can be done. One of those is quasi de Sitter space, with the scale factor from Eq. (2.23). The field equation (5.62) then becomes

\[
\delta \varphi_k'' + \left( k^2 + \frac{1}{\eta^2} \frac{m^2 + R(\zeta - \frac{1}{6})}{H^2(1 - c)^2} \right) \delta \varphi_k = 0.
\]  

(5.74)

We now write this equation as

\[
\frac{d^2}{d(-k\eta)^2} + \left( \frac{1}{4} - \frac{\nu^2}{(-k\eta)^2} \right) \delta \varphi_k = 0,
\]  

(5.75)

with

\[
\nu^2 = \frac{1}{4} \frac{m^2 + R(\zeta - \frac{1}{6})}{H^2(1 - c)^2} = \frac{1}{4} \frac{m^2}{H^2(1 - c)^2} - \frac{6(2 - c)(\zeta - \frac{1}{6})}{(1 - c)^2} = \left( \frac{3 - c}{2(1 - c)} \right)^2 - \frac{m^2}{H^2(1 - c)^2} - \frac{6(2 - c)\zeta}{(1 - c)^2}.
\]  

(5.76)

Eq. (5.75) is Bessel’s equation when the index \( \nu \) is constant. Fortunately, in our large negative nonminimal coupling model, the index \( \nu \) is constant. First of all, \( \epsilon \) is a constant, and also the ratio \( m^2/H^2 = 3\lambda \phi_0^2/H^2 \) is constant since the Hubble parameter is proportional to the background field, see Eq. (4.19) and for example [32]. The two solutions of Eq. (5.75) are

\[
\sqrt{-k\eta}J_\nu(-k\eta), \quad \sqrt{-k\eta}N_\nu(-k\eta),
\]  

(5.77)

where \( J_\nu(-k\eta) \) and \( N_\nu(-k\eta) \) are the Bessel functions of the first and second kind respectively. The reason why we chose the variable of the Bessel functions to be \(-k\eta\), is that in an
expanding universe with $\epsilon \ll 1$ the conformal time $\eta$ runs from $-\infty$ to 0, see the discussion below Eq. (2.21). We could also write the two independent solutions as a linear combination of $J_\nu(-k\eta)$ and $N_\nu(-k\eta)$,

$$
H_\nu^{(1)}(-k\eta) = J_\nu(-k\eta) + iN_\nu(-k\eta)
$$
$$
H_\nu^{(2)}(-k\eta) = J_\nu(-k\eta) - iN_\nu(-k\eta),
$$

(5.78)

where $H_\nu^{(1,2)}(-k\eta)$ are the Bessel functions of the third kind, called Hankel functions. These Hankel functions prove to be useful in our analysis, as we will see shortly. Combining the above, we find that the two fundamental solutions of Eq. (5.62) are

$$
\delta \phi^{(1)}_k(\eta) = \frac{1}{\sqrt{2k}} \sqrt{\pi} \eta H_\nu^{(1)}(-k\eta) = \sqrt{-\frac{\pi\eta}{4}} H_\nu^{(1)}(-k\eta)
$$
$$
\delta \phi^{(2)}_k(\eta) = \sqrt{-\frac{\pi\eta}{4}} H_\nu^{(2)}(-k\eta).
$$

(5.79)

In the first line I have explicitly written out the normalization factors, which have been chosen such that the Wronskian of the two solutions satisfies

$$
W[\delta \phi^{(1)}_k(\eta), \delta \phi^{(2)}_k(\eta)] = \delta \phi^{(1)}_k(\eta)\delta \phi^{(2)}_k(\eta) - \delta \phi^{(2)}_k(\eta)\delta \phi^{(1)}_k(\eta) = i.
$$

(5.80)

In general we have the relation $\left(H_\nu^{(1)}(z)\right)^* = H_\nu^{(2)}(z^*)$. Since in this case both the index $\nu$ and the coordinate $z = -k\eta$ are real, we find that $\delta \phi^{(2)}_k(\eta) = \delta \phi^{(1)*}_k(\eta)$. Of course this will be different when for example $\nu$ is complex, as we will see when we quantize the two-Higgs doublet model. If we now look at the original mode fluctuations $\delta \phi_k$, we can write the two fundamental solutions as

$$
\delta \phi^{(1)}_k(\eta) = \frac{\delta \phi^{(1)}_k(\eta)}{a(\eta)} = \frac{1}{a} \sqrt{-\frac{\pi\eta}{4}} H_\nu^{(1)}(-k\eta)
$$
$$
\delta \phi^{(2)}_k(\eta) = \frac{\delta \phi^{(2)}_k(\eta)}{a(\eta)} = \delta \phi^{(1)*}_k(\eta).
$$

(5.81)

The Wronskian between these two solutions is then simply

$$
W[\delta \phi^{(1)}_k(\eta), \delta \phi^{(2)}_k(\eta)] = \frac{i}{a^2}.
$$

(5.82)

The general mode functions in Eq. (5.57) can now be expressed as linear combinations of the fundamental solutions in Eq. (5.81) in the following way

$$
\delta \phi_k = a_k \delta \phi^{(1)}_k + \beta_k \delta \phi^{(2)}_k
$$
$$
\delta \phi^*_k = a_k^* \delta \phi^{(2)*}_k + \beta_k^* \delta \phi^{(1)*}_k,
$$

(5.83)

where I have used that $\delta \phi^{(2)*}_k = \delta \phi^{(1)}_k$. These mode functions must satisfy the normalization condition in Eq. (5.60), and by using the relation (5.82) we find the condition

$$
|a_k|^2 - |eta_k|^2 = 1.
$$

(5.84)

The question is now: can we make some natural choice for the parameters $a_k$ and $\beta_k$? In general we cannot do this, because the lowest energy state at one instant of time will not be
the lowest energy state at a later time in an expanding universe. However, in the previous section we showed that in the limit $\eta \to -\infty$ we can define a natural vacuum state, the Bunch-Davies vacuum, that minimizes the energy and remains the lowest energy state to a very good approximation. So let us take the limit $\eta \to -\infty$. The modes $\delta \varphi_k$ from Eq. (5.79) then become

$$
\delta \varphi_k^{(1)}(\eta \to -\infty) = \frac{1}{\sqrt{2k}} e^{-ik\eta} \\
\delta \varphi_k^{(2)}(\eta \to -\infty) = \frac{1}{\sqrt{2k}} e^{ik\eta}. 
$$

The energy in the mode $\delta \varphi_k$ from Eq. (5.66) then is

$$
E_k(\eta \to -\infty) = \frac{1}{2} \left( |\delta \varphi_k^{(2)}|^2 + \omega_k^2 |\delta \varphi_k^{(1)}|^2 \right) \bigg|_{\eta \to -\infty} \\
= \frac{1}{2} k (|\alpha_k|^2 + |\beta_k|^2) \\
= \frac{1}{2} k (1 + 2|\beta_k|^2). 
$$

So, the energy is minimized when $\alpha_k = 1$, $\beta_k = 0$. Precisely these conditions lead to the Bunch-Davies vacuum, which corresponds to zero particles and energy in the infinite asymptotic past.

We note that the adiabatic analysis we made in this section is only valid for those modes with $k > Ha = \frac{1}{\eta(1-\epsilon)}$. If $k < Ha$, the state is nonadiabatic and we can no longer define the Bunch-Davies vacuum to be the natural vacuum state.

### 5.2.3 Scalar propagator

In the previous sections we have looked at particle production due to the expansion of the universe. We calculated the particle number with respect to the vacuum, and we have seen that we cannot define a unique zero particle vacuum. However, in the adiabatic regime, which is the infinite asymptotic past, we showed that particle production is minimal. We could define a natural vacuum state, the Bunch-Davies vacuum, with zero particles as $\eta \to -\infty$.

Outside of the adiabatic regime this analysis breaks down and we cannot define a natural vacuum state anymore. There is however another important quantity, which is the amplitude of field fluctuations. This quantity is well defined even for quantum states where we cannot unambiguously define the notion of a particle. The amplitude of field fluctuations with respect to some state is described by the expression

$$
\langle n | \delta \hat{\phi}(x, \eta) \delta \hat{\phi}(x', \eta) | n \rangle. 
$$

For simplicity we take the state to be the vacuum state $|n\rangle = |0\rangle$. Note that Eq. (5.87) is the equal time scalar propagator,

$$
\langle 0 | \delta \hat{\phi}(x) \delta \hat{\phi}(x') | 0 \rangle 
$$

The propagator describes the probability for a particle to travel from a space time point $x = (x, \eta)$ to another space time point $x' = (x', \eta')$. Note that even though the vacuum contains zero particles, particle-antiparticle pairs can be created from the vacuum by the Heisenberg uncertainty principle. This means that the amplitude of field fluctuations can be nonzero if we calculate (5.87) with respect to the vacuum $|0\rangle$. Indeed, if we substitute the expansion


\[
\langle 0|T[\delta \phi(x)\delta \phi(x')]|0\rangle = \theta(\eta - \eta')\langle 0|\delta \phi(x)\delta \phi(x')|0\rangle + \theta(\eta' - \eta)\langle 0|\delta \phi(x')\delta \phi(x)|0\rangle,
\]

(5.89)

where \( \theta \) is the Heaviside \( \theta \)-function. The Feynman propagator is Lorentz invariant, and hence the physical propagator. We are actually able to calculate this propagator explicitly if we insert the expansion (5.61) with the mode functions (5.63). For our vacuum state we will take the Bunch-Davies vacuum with \( \alpha = 1, \beta = 0 \). The Feynman propagator in Eq. (5.89) then becomes

\[
i\Delta(x, x') = \pi \frac{\sqrt{m\eta'}}{4aa'} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}(\vec{x} - \vec{x}')} \times \left\{ \theta(\eta - \eta')H_\nu^{(1)}(-k\eta)H_\nu^{(2)}(-k\eta') + \theta(\eta' - \eta)H_\nu^{(1)}(-k\eta')H_\nu^{(2)}(-k\eta) \right\}.
\]

This integration has been performed in [33] in a \( D \)-dimensional FLRW universe. The importance of this becomes clear when we calculate one loop fermion self energy diagrams in chapter 6. We will use dimensional regularization to renormalize the infinite self energy, and therefore we need the scalar and fermion propagators in \( D \) dimensions. To calculate the propagator in \( D \) dimensions we need the solutions for the field fluctuations \( \delta \phi_k \) in \( D \) dimensions, and only a few things will change with respect to the four dimensional case. We can still write the field equation for the fluctuations in the form (5.74), but we have new rescaling for the fields, \( \varphi = a^{2(1-\epsilon)} \phi \). Furthermore we can recognize the conformal coupling factor \( \frac{1}{\xi} \), but in \( D \) dimensions this is \( \frac{D(2-D)}{4(2-D)} \), see Eq. (4.46). Finally the Ricci scalar is given by Eq. (B.25) in \( D \) dimensions. If we now again write the field equation for \( \delta \phi_k \) in the form of Bessel’s equation (5.75), we find that the index \( \nu \) is now

\[
\nu_D^2 = \left( \frac{D-1-\epsilon}{2(1-\epsilon)} \right)^2 + \frac{1}{(1-\epsilon)^2} \frac{\xi R_D + m^2}{H^2} = \left( \frac{D-1-\epsilon}{2(1-\epsilon)} \right)^2 + \frac{(D-1)(D-2\epsilon)}{(1-\epsilon)^2} \xi + \frac{1}{(1-\epsilon)^2} \frac{m^2}{H^2}.
\]

(5.91)

In four dimensions we retrieve the \( \nu \) from Eq. (5.76). We find that the solutions for \( \delta \phi_k \) are again Hankel functions, but this time the index is \( \nu_D \). In \( D \) dimensions the scalar propagator then takes the form

\[
i\Delta(x, x') = \frac{\pi}{4} \frac{\sqrt{\eta\eta'}}{(aa')^{\frac{D}{2}-1}} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}(\vec{x} - \vec{x}')} \times \left\{ \theta(\eta - \eta')H_\nu^{(1)}(-k\eta)H_\nu^{(2)}(-k\eta') + \theta(\eta' - \eta)H_\nu^{(1)}(-k\eta')H_\nu^{(2)}(-k\eta) \right\}.
\]

(5.92)

Again, in 4 dimensions the mass term precisely cancels against the main contribution coming from the \( \xi R \) term. The integration (5.92) can now be performed to give the propagator in \( D \) dimensions,

\[
i\Delta(x, x')_\infty = \frac{[(1-\epsilon)^2HH']^{\nu_D-1} \Gamma(D-1)\Gamma(D-1)\Gamma(D-1-\nu_D)}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)} \times \text{e}_2\text{F}_1\left\{ \begin{array}{c} \frac{D-1}{2}+\nu_D, \frac{D-1}{2}-\nu_D; \frac{D}{2};1-\frac{\nu}{4} \end{array} \right\}.
\]

(5.93)
where
\[ y = y_{\pm}(x, x') = -\left(\frac{\eta - \eta'}{-i\epsilon} + i(x - x')^2\right) = \frac{\Delta x^2(x, x')}{\eta \eta'}. \] (5.94)

Note that \( \epsilon \) is used for the pole description, whereas \( \epsilon \) is defined as \(-\hat{H}\hat{H}^2\). The pole prescription is such that we calculate the Feynman propagator. Let us stress that the propagator is strictly speaking infinite (hence the index \( \infty \)), because in the infrared (for small values of \( k \) in Eq. (5.92)) the propagator diverges. These infrared divergencies can be removed by different procedures and this has been done in [33] and [34]. For this thesis we leave the discussion of infrared divergencies aside and continue to bring the propagator to a simpler form. We use the series expansion of the hypergeometric function
\[ 2F_1(a, b, c, d) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)} \frac{z^n}{n!} \] (5.95)

and the transformation formula
\[ 2F_1 \left( \frac{D-1}{2} + \nu D, \frac{D-1}{2} - \nu D; \frac{D}{2}; 1 - \frac{y}{4} \right) = \frac{\Gamma(D/2)\Gamma(D/2-1)}{\Gamma(\nu D)\Gamma(\nu D-2) + \nu D} \frac{1}{2} \Gamma \left( \frac{1}{2} + \nu D, \frac{D-1}{2} - \nu D; 2 - \frac{D}{2}; \frac{y}{4} \right) \] (5.96)

Also we use properties of the gamma function such as \( x\Gamma(x) = \Gamma(1 + x) \) to show that
\[ \Gamma(1 - \frac{D}{2})\Gamma \left( \frac{D}{2} \right) = -\Gamma(\frac{D}{2} - 1)\Gamma (2 - \frac{D}{2}). \] (5.97)

We then get for the propagator
\[ i\Delta(x, x')_{\infty} = \frac{(1 - e)^2HH'}{2} \frac{\Gamma(D/2 - 1)}{\Gamma(1/2 + \nu D)\Gamma(1/2 - \nu D)} \times \left[ \sum_{n=0}^{\infty} \frac{\Gamma(1/2 + \nu D + n)\Gamma(1/2 - \nu D + n)}{\Gamma(2 - D/2 + n)\Gamma(n + 1)} \left( \frac{y}{4} \right)^n \right] \] (5.98)

Now, in the first sum, we change our summation variable \( n \to n + 1 \), such that the sum now runs from \( n = -1 \) to \( \infty \), and we split off the \( n = -1 \) term. The reason for this will become clear in a moment. The propagator can then be written as
\[ i\Delta(x, x')_{\infty} = \frac{(1 - e)^2HH'}{2} \frac{\Gamma(D/2 - 1)}{\Gamma(1/2 + \nu D)\Gamma(1/2 - \nu D)} \times \left[ \sum_{n=0}^{\infty} \frac{\Gamma(3/2 + \nu D + n)\Gamma(3/2 - \nu D + n)}{\Gamma(3 - D/2 + n)\Gamma(n + 2)} \left( \frac{y}{4} \right)^{n - 2} \right] - \frac{\Gamma(1/2 + \nu D + n)\Gamma(1/2 - \nu D + n)}{\Gamma(1/2 + n)\Gamma(n + 1)} \left( \frac{y}{4} \right)^n \right]. \] (5.99)
Note that for $D = 4$, the factor $\Gamma(2 - \frac{D}{2})$ is singular and the propagator seems to be infinite (apart from the singularity at $x = x'$). However, we will see that the propagator is finite when we expand the propagator up to linear order in $D - 4$. The first part without the sums is finite and we do not need to expand here. For the second part, we first expand $\Gamma(2 - \frac{D}{2})$ around $D = 4$,

$$\Gamma(2 - \frac{D}{2}) = -\frac{2}{D-4} - \gamma_E + \Theta((D-4)),$$

where $\gamma_E \equiv -\psi(1)$ is the Euler constant and $\psi(z) \equiv \frac{d\ln \Gamma(z)}{dz}$ is the digamma function. For the non-singular Gamma functions in the sums, we use the following expansion

$$\Gamma(a + b(D-4)) = \Gamma(a) + b\Gamma'(a)(D-4) + \Theta((D-4)^2) = \Gamma(a)(1 + \psi(a)(D-4)) + \Theta((D-4)^2).$$

Furthermore, we need to expand $v_D$ to linear order in $(D-4)$, which gives

$$v_D = \sqrt{\left(\frac{D-1 - \epsilon}{2(1-\epsilon)}\right)^2 + \frac{(D-1)(D-2\epsilon)}{(1-\epsilon)^2} \xi + \frac{1}{(1-\epsilon)^2} \frac{m^2}{H^2}}$$

$$= \sqrt{\left(\frac{3 - \epsilon}{2(1-\epsilon)}\right)^2 + \frac{R\xi + m^2}{H^2(1-\epsilon)^2}} + \left(\frac{3 - \epsilon}{2(1-\epsilon)^2} - \frac{3 + 4 - 2\epsilon}{(1-\epsilon)^2} \xi\right)(D-4) + \Theta(D-4)^2$$

$$= v\left[1 + \frac{1}{2} v^2 (1-\epsilon)^2 \left(\frac{3 - \epsilon}{2} + (-7 + 2\epsilon)\xi\right)(D-4) + \Theta(D-4)^2\right]$$

$$= v + \frac{1}{2} v C(D-4) + \Theta(D-4)^2, \quad C = \frac{\frac{3}{2} - \frac{\epsilon}{2} + (-7 + 2\epsilon)\xi}{(\frac{1}{2}(3-\epsilon))^2 - 3(4-2\epsilon)\xi}.$$

In the second line I have used that $R \equiv R_{D=4} = 3(4-2\epsilon)H^2$. In the third line I have used the expression for $v \equiv v_{D=4}$ from Eq. (5.76). In the last line I have implicitly defined the constant $C$. When $\epsilon \ll 1$ and $\Xi \gg 1$, this constant is of order unity. Now we can expand the different Gamma functions that appear in the sums of the propagator (5.99)

$$\Gamma\left(\frac{3}{2} + v_D + n\right) = \Gamma\left(\frac{3}{2} + v + n\right)\left(1 + \frac{1}{2} v C(D-4)\psi\left(\frac{3}{2} + v + n\right)\right) + \Theta(D-4)^2$$

$$\Gamma\left(\frac{3}{2} - v_D + n\right) = \Gamma\left(\frac{3}{2} + v + n\right)\left(1 - \frac{1}{2} v C(D-4)\psi\left(\frac{3}{2} - v + n\right)\right) + \Theta(D-4)^2$$

$$\Gamma\left(\frac{1}{2} + v_D\right) = \Gamma\left(\frac{1}{2} + v\right)\left(1 + \frac{1}{2} v C(D-4)\psi\left(\frac{1}{2} + v\right)\right) + \Theta(D-4)^2$$

$$\Gamma\left(\frac{1}{2} - v_D\right) = \Gamma\left(\frac{1}{2} - v\right)\left(1 - \frac{1}{2} v C(D-4)\psi\left(\frac{1}{2} - v\right)\right) + \Theta(D-4)^2$$

$$\Gamma\left(\frac{D-1}{2} + v_D + n\right) = \Gamma\left(\frac{3}{2} + v + n\right)\left(1 + \frac{1}{2} v C(D-4)\psi\left(\frac{3}{2} + v + n\right)\right) + \Theta(D-4)^2$$

$$\Gamma\left(\frac{D-1}{2} - v_D + n\right) = \Gamma\left(\frac{3}{2} - v + n\right)\left(1 + \frac{1}{2} v C(D-4)\psi\left(\frac{3}{2} - v + n\right)\right) + \Theta(D-4)^2$$

$$\Gamma\left(3 - \frac{D}{2} + n\right) = \Gamma(1 + n)\left(1 - \frac{1}{2} (D-4)\psi(1 + n)\right) + \Theta(D-4)^2$$

$$\Gamma\left(\frac{D}{2} + n\right) = \Gamma(1 + n)\left(1 + \frac{1}{2} (D-4)\psi(1 + n)\right) + \Theta(D-4)^2,$$

and also

$$\left(\frac{y}{4}\right)^{-\frac{D}{2} + 2} = \left(\frac{y}{4}\right)^n \left(1 - \frac{1}{2} \ln \left(\frac{y}{4}\right) (D-4)\right) + \Theta(D-4)^2.$$
Now we plug all these expansions into the sums of Eq. (5.99). One thing we can immediately see is that the zeroth order expansions cancel against each other in the sums, and therefore the terms that are summed over are only linear in \(D - 4\). This is the reason why we wrote the scalar propagator as in (5.99). Our final result for the finite propagator (up to linear order in \((D - 4)\)) is

\[
i \Delta(x, x')_\infty = \frac{[(1 - \epsilon)^2HH']^{\nu - 1}}{(4\pi)^{\nu/2}} \Gamma \left( \frac{D}{2} - 1 \right) \left( \frac{y}{4} \right)^{1 - \nu/2} + \frac{(1 - \epsilon)^2HH'}{(4\pi)^2} \sum_{n=0}^\infty \frac{\Gamma \left( \frac{3}{2} + \nu + n \right) \Gamma \left( \frac{3}{2} - \nu - n \right) - 1}{\Gamma \left( \frac{3}{2} + \nu \right) \Gamma \left( \frac{3}{2} - \nu \right) - 1} \frac{1}{\Gamma \left( 1 + n \right) \Gamma \left( 2 + n \right)} \left( \frac{y}{4} \right)^n \\
\times \left[ \ln \left( \frac{y}{4} \right) + \psi \left( \frac{3}{2} + \nu + n \right) + \psi \left( \frac{3}{2} - \nu + n \right) - \psi \left( 1 + n \right) - \psi \left( 2 + n \right) \right] + \mathcal{O}(D - 4).
\]

For convenience I will define

\[
\Psi_n = \psi \left( \frac{3}{2} + \nu + n \right) + \psi \left( \frac{3}{2} - \nu + n \right) - \psi \left( 1 + n \right) - \psi \left( 2 + n \right).
\]

Next we use the relation \(x \Gamma(x) = \Gamma(x + 1)\) to get

\[
\sum_{n=0}^\infty \frac{\Gamma \left( \frac{3}{2} + \nu + n \right) \Gamma \left( \frac{3}{2} - \nu + n \right)}{\Gamma \left( \frac{3}{2} + \nu \right) \Gamma \left( \frac{3}{2} - \nu \right)} = \left( \frac{1}{2} + \nu \right) \Gamma \left( \frac{1}{2} + \nu \right) \Gamma \left( \frac{1}{2} - \nu \right) + \left( \frac{1}{2} + \nu \right) \Gamma \left( \frac{3}{2} + \nu \right) \Gamma \left( \frac{1}{2} - \nu \right) + \ldots
\]

\[
= \Gamma \left( \frac{1}{2} + \nu \right) \Gamma \left( \frac{1}{2} - \nu \right) \left[ \left( \frac{1}{4} - \nu^2 \right) + \left( \frac{1}{4} - \nu^2 \right) \frac{9}{4} - \nu^2 \right] + \ldots
\]

\[
= \sum_{n=0}^\infty \left[ \Gamma \left( \frac{1}{2} + \nu \right) \Gamma \left( \frac{1}{2} - \nu \right) \left[ \frac{\Psi_n}{n!} \prod_{k=0}^{n-1} \left( \frac{2k + 1}{2} \right)^2 - \nu^2 \right] \right].
\]

The propagator then takes the form

\[
i \Delta(x, x')_\infty = \frac{[(1 - \epsilon)^2HH']^{\nu - 1}}{(4\pi)^{\nu/2}} \Gamma \left( \frac{D}{2} - 1 \right) \left( \frac{y}{4} \right)^{1 - \nu/2} + \frac{(1 - \epsilon)^2HH'}{(4\pi)^2} \sum_{n=0}^\infty \left[ \left( \frac{y}{4} \right)^n \Psi_n \prod_{k=0}^{n-1} \left( \frac{2k + 1}{2} \right)^2 - \nu^2 \right] + \mathcal{O}(D - 4). \tag{5.101}
\]

Now we take another look at the expression for \(\nu\) from Eq. (5.76) and define a quantity

\[
s = \frac{\xi R + m^2}{H^2}, \quad s \ll 1. \tag{5.102}
\]

\(s \ll 1\) because the mass term precisely cancels the main contribution coming from the \(\xi R\) term during inflation. This happens because the classical inflaton field \(\phi_0 \propto H\) during inflation, see Eq. (4.19). We can make an expansion for \(s \ll 1\) in the expression for \(\nu\). This gives

\[
\nu = \frac{1}{1 - \epsilon} \left( \frac{3 - \epsilon}{2} + \frac{s}{3 - \epsilon} \right). \tag{5.103}
\]

Now we also expand for \(\epsilon \ll 1\), which is also true during inflation. We get

\[
\nu = \frac{3}{2} + \frac{s}{3} + \epsilon + \mathcal{O}(s\epsilon). \tag{5.104}
\]
When we now look at the scalar propagator (5.105), we see that all terms with \( n \geq 1 \) contain a term \( (3/2)^2 - \nu^2 \), which is proportional to \( s \) and the slow-roll parameter \( \epsilon \). Therefore, all these terms are suppressed since \( s \ll 1 \) and \( \epsilon \ll 1 \). This means we can only take the term with \( n = 0 \) of the sum in the propagator. This gives for the propagator

\[
\Delta(x,x')_\infty = \frac{1-(1-\epsilon)^2HH'}{(4\pi)^2} \Gamma (\frac{D}{2}-1) \left( \frac{v}{4} \right)^{1-g} + \frac{(1-\epsilon)^2HH'}{(4\pi)^2} \Psi_0 \left( \frac{1}{4} - v^2 \right) + \Theta (\frac{s}{3} + \epsilon),
\]

(5.105)

where \( \Psi_0 \) is

\[
\Psi_0 = \psi \left( \frac{3}{2} + \nu \right) + \psi \left( \frac{3}{2} - \nu \right) - \psi (1) - \psi (2).
\]

(5.106)

Using that \( \psi (1) = -\gamma_E \), \( \psi (2) = 1 - \gamma_E \), \( \psi (3) = \frac{3}{2} - \gamma_E \), with \( \gamma_E = 0.57... \) the Euler constant, and the expansion around \( \nu = \frac{3}{2} \) from Eq. (5.104),

\[
\psi \left( \frac{3}{2} - \nu \right) = \frac{1}{\nu + \epsilon} - \gamma_E + \Theta \left( \frac{s}{3} + \epsilon \right),
\]

(5.107)

we find that

\[
\Psi_0 = \log \left( \frac{\gamma}{4} \right) + \frac{1}{2} + \frac{1}{\nu + \epsilon} + \Theta \left( \frac{s}{3} + \epsilon \right).
\]

(5.108)

We stress that in higher order terms of \( \Psi_n \) the \( \frac{1}{\nu + \epsilon} \) term does not appear because we do not have to expand the digamma-function \( \psi (z) \) around \( z = 0 \) for \( n \geq 1 \). To summarize, we have obtained a relatively simple expression for the scalar propagator in the limit where \( s, \epsilon \ll 1 \). This scalar propagator in Eq. (5.105) we will use in chapter 6 to calculate the one loop fermion self energy.

To summarize the previous sections, we have succeeded to quantize the nonminimally coupled inflaton field. We have seen that in general we cannot define a unique zero particle vacuum state in an expanding universe. However, in quasi de Sitter space we can define a natural vacuum state, the Bunch-Davies vacuum. This vacuum state is defined in the infinite asymptotic past, and in this adiabatic regime particle production is minimal. Finally we calculated the scalar propagator in \( D \) dimensions and expanded this infinite propagator around \( D = 4 \) to find our final result (5.105). In the next section we repeat the quantization procedure for the two-Higgs doublet model. We will see that there are only some small changes with respect to the model for one real scalar field, i.e. the standard Higgs model.

### 5.3 Quantization of the two-Higgs doublet model

First we will give again the Lagrangian for the two-Higgs doublet model from Eq. (4.90)

\[
S = \int d^4x \sqrt{-g} \left( - \sum_{i=1,2} g^{\mu \nu} (\partial_\mu \phi_i)^* (\partial_\nu \phi_i) - \sum_{i,j=1,2} \xi_{ij} R \phi_i^* \phi_j - V(\phi_1, \phi_2) \right),
\]

where the potential \( V(\phi_1, \phi_2) \) is given in Eq. (4.72). We mention again that we could also have chosen a different kinetic term that mixes the derivatives of \( \phi_1 \) and \( \phi_2 \), but we can always rewrite such an action to the form above by taking linear combinations of the fields. There will only be minor changes in the potential term, but as we mentioned in section 4.4 this does not influence the dynamics of inflation. Again we consider quantum fluctuations
of the field and expand the action up to quadratic order in fluctuations, which allows us to write the action for the quantum fluctuations

\[ S_Q = \int d^4 x \sqrt{-g} \left( - \sum_{i=1,2} g^{\mu \nu} (\partial_\mu \delta \phi_i) (\partial_\nu \delta \phi_i) - \sum_{i,j=1,2} \xi_{ij} R \delta \phi_i^* \delta \phi_j \right. \]

\[ \left. - \sum_{i,j=1,2} \left[ \delta \phi_i^* \frac{\partial V}{\partial \phi_i} \delta \phi_j + \frac{1}{2} \delta \phi_i \frac{\partial V}{\partial \phi_i} \delta \phi_j + \frac{1}{2} \delta \phi_i^* \frac{\partial V}{\partial \phi_i^*} \delta \phi_j^* \right] \right) \],

where \( \phi_i(x) = \phi_i^0(t) + \delta \phi_i(x, t) \), with \( \phi_i^0(t) \) the classical inflaton field and \( \delta \phi_i(x, t) \) the quantum fluctuation. The potential terms might suggest that we have some nontrivial terms \( (\delta \phi^*)^2 \) and \( \propto \delta \phi^2 \). These terms are in fact real, because our potential is also real, i.e. it contains only terms \( \phi_i^* \phi_i \). We can also write these terms as \( \propto \delta \phi^* \delta \phi \) by writing the complex field \( \phi = |\phi| \exp(i\theta) \) and move the phase into the potential derivatives. The action we can therefore write as

\[ S_Q = \int d^4 x \sqrt{-g} \left( - \sum_{i=1,2} g^{\mu \nu} (\partial_\mu \delta \phi_i) (\partial_\nu \delta \phi_i) - \sum_{i,j=1,2} \xi_{ij} R \delta \phi_i^* \delta \phi_j - \sum_{i,j=1,2} \delta \phi_i^* m_{ij}^2 \delta \phi_j \right), \tag{5.109} \]

where

\[ m_{ij}^2 = 2 \frac{\partial V}{\partial \phi_i^* \partial \phi_j}. \tag{5.110} \]

Note that \( (m_{ij}^2)^* = m_{ji}^2 \). The mass term (5.110) contains only the classical inflaton fields \( \phi_i^0 \) because higher order fluctuation terms vanish. So we can say that the classical inflaton fields \( \phi_i^0 \) effectively generate a mass for the quantum fluctuations \( \delta \phi_i \). From now on we will omit the \( \delta \)'s in the quantum fields for notational convenience. Since we only work with the quantum fluctuations in this section, we do not experience any notational problems. Just remember that from this point on the fields \( \phi_i \) are actually quantum fluctuations \( \delta \phi_i \).

We can write the quantum action (5.109) in matrix notation as

\[ S_Q = \int d^4 x \sqrt{-g} \left( -g^{\mu \nu} \partial_\mu \Phi^\dagger \partial_\nu \Phi - R \Phi^\dagger \Xi \Phi - \Phi^\dagger M^2 \Phi \right), \tag{5.111} \]

where

\[ \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \Xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{12} & \xi_{22} \end{pmatrix}, \quad M^2 = \begin{pmatrix} m_{11}^2 & m_{12}^2 \\ (m_{12}^2)^* & m_{22}^2 \end{pmatrix}. \tag{5.112} \]

The field vector \( \Phi \) contains the quantum fluctuations \( \phi_i \). The matrices nonminimal coupling matrix and the mass matrix are hermitean, i.e. \( \Xi^\dagger = \Xi, (M^2)^\dagger = M^2 \).

Let us now derive the field equations for the quantum fluctuations from the action (5.111). This gives

\[ -\Box \Phi + (R \Xi + M^2) \Phi = 0. \]

The field equation for the complex conjugate fields \( \phi_i^* \) is obtained by complex conjugation of this equation. Substituting the conformal FLRW metric \( g_{\mu \nu} = a^2(\eta) \eta_{\mu \nu} \), we get

\[ \frac{1}{a^2} \left( a^2 \Phi' \right)' - \nabla^2 \Phi + a^2 (R \Xi + M^2) \Phi = 0. \]

Again we can do a conformal rescaling of the fields to get

\[ \left( \delta^2_\eta - \nabla^2 + a^2 \left[ R(\Xi - \frac{1}{6}) + M^2 \right] \right)(a \Phi) = 0. \tag{5.113} \]
Note that this is a matrix equation since $\Xi$ and $M^2$ are matrices, thus one should keep in mind that all the other terms are multiplied by the $2 \times 2$ unit matrix. Comparing this to Eq. (5.53), we see that we have the same field equations for the two-Higgs doublet model as for the real scalar field. There are only two differences. 1) Instead of one field we now have two scalar fields. 2) The scalar fields are complex instead of real fields. We now take these two differences into account when we quantize the two-Higgs doublet model.

We now perform canonical quantization for the complex scalar fields $\phi_a$ and $\phi_b$ by imposing the commutator

$$[\hat{\phi}_i(\vec{x}, \eta), \hat{\pi}_j(\vec{x}', \eta)] = i\delta_{ij}\delta^2(\vec{x} - \vec{x}'), \quad (i, j = 1, 2),$$

(5.114)

where the canonical momentum is by definition

$$\hat{\pi}_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} = a^2 \hat{\phi}_i^*$$,

(5.115)

We could also write

$$\hat{\Pi} = \begin{pmatrix} \hat{\pi}_1 \\ \pi_2 \end{pmatrix} = a^2 \begin{pmatrix} \hat{\phi}_1^* \\ \phi_2^* \end{pmatrix} = a^2 \hat{\Phi}^*$$.

(5.116)

The system of field equations for the complex fields $\phi_1$ and $\phi_2$ that we want to solve is linear (Eq. (5.57)), and since the nonminimal coupling and mass matrices are hermitean, we can express the complex fields in terms of annihilation and creation operators as

$$\hat{\phi}_i(x) = \sum_l \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{e^{ik·x} + e^{-ik·x}} \phi_{k,il}(\eta) \hat{a}^+_{k,l} + \frac{1}{e^{ik·x} - e^{-ik·x}} \phi_{k,il}^*(\eta) \hat{b}^+_{k,l} \right],$$

$$\hat{\phi}_i(x) = \sum_l \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{e^{ik·x} + e^{-ik·x}} \phi_{k,il}(\eta) \hat{a}^+_{k,l} + \frac{1}{e^{ik·x} - e^{-ik·x}} \phi_{k,il}^*(\eta) \hat{b}^+_{k,l} \right],$$

$$\hat{\phi}_i(x) = \sum_l \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{e^{ik·x} + e^{-ik·x}} \phi_{k,il}(\eta) \hat{a}^+_{k,l} + \frac{1}{e^{ik·x} - e^{-ik·x}} \phi_{k,il}^*(\eta) \hat{b}^+_{k,l} \right],$$

(5.117)

Notice the differences with the expansion for the single real scalar field in Eq. (5.57). First of all there are now creation and annihilation operators for the fields $\phi_1$ and $\phi_2$, since we have the canonical commutation relation for both fields, see the $\delta_{ij}$ in Eq. (5.114). Secondly we defined two sets of creation and annihilation operators $a^\pm$ and $b^\pm$ and two mode functions $\phi_k$ and $\psi_k$. We did this because the fields are complex, i.e. $\phi_i^* \neq \phi_i$. Physically, excitations of the field $\phi$ describes particles, whereas excitations of its complex conjugate describes antiparticles. The operators $a^\pm$ create or annihilate a particle, whereas the operators $b^\pm$ create or annihilate an antiparticle.

From the expansion (5.117) we can find the conjugate momentum

$$\hat{\pi}_j(x) = a^2 \hat{\phi}_j^*$$

$$\hat{\pi}_j(x) = a^2 \sum_m \int \frac{d^3k}{(2\pi)^3} \left[ \phi_{k,jm}(\eta) \hat{a}^+_{k,m} + \phi_{k,jm}^*(\eta) \hat{b}^+_{k,m} \right],$$

$$\hat{\pi}_j(x) = a^2 \sum_m \int \frac{d^3k}{(2\pi)^3} \left[ \phi_{k,jm}(\eta) \hat{a}^+_{k,m} + \phi_{k,jm}^*(\eta) \hat{b}^+_{k,m} \right],$$

(5.118)

The equations for $\hat{\phi}_i$ and $\hat{\pi}_j$ can also be written in matrix form as

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} e^{ik·x} \left[ \phi_k(\eta) \cdot \hat{A}_k - \Psi^*_k(\eta) \cdot \hat{B}_k^+ \right]$$

$$\hat{\pi}(x) = a^2 \int \frac{d^3k}{(2\pi)^3} e^{-ik·x} \left[ \phi_k^*(\eta) \cdot \hat{A}_k^+ + \Psi_k(\eta) \cdot \hat{B}_k^\pm \right],$$

(5.119)
where
\[ \Phi_k = \left( \begin{array}{cc} \phi_{k,aa} & \phi_{k,ab} \\ \phi_{k,ba} & \phi_{k,bb} \end{array} \right), \quad \Psi_k = \left( \begin{array}{cc} \varphi_{k,aa} & \varphi_{k,ab} \\ \varphi_{k,ba} & \varphi_{k,bb} \end{array} \right), \quad \hat{A}_k^- = \left( \begin{array}{c} a_{k,a} \\ \bar{a}_{k,b} \end{array} \right), \quad \hat{B}_k^- = \left( \begin{array}{c} \hat{b}_{k,a} \\ \hat{b}_{k,b} \end{array} \right) \]

The creation and annihilation operators \( a^\pm \) and \( b^\pm \) form two independent sets and obey the commutation relations
\[
\begin{align*}
[a^-_{k,i}, a^+_{k',j}] &= \delta_{ij}(2\pi)^3 \delta(k - k') \\
[b^-_{k,i}, b^+_{k',j}] &= \delta_{ij}(2\pi)^3 \delta(k - k') \\
[a^-_{k,i}, b^+_{k',j}] &= \delta_{ij}(2\pi)^3 \delta(k - k').
\end{align*}
\]

Again we have the relations \( a^+ = (a^-)^* \) and \( b^+ = (b^-)^* \). Using the expansion of the fields \( \hat{\phi}_i \) in Eq. (5.117) and the conjugate momenta \( \hat{\pi}_j \) in Eq. (5.118) together with these commutation relations we can write the canonical commutation relation from Eq. (5.114) as
\[
\left[ \hat{\phi}_i(x, \eta), \hat{\pi}_j(x', \eta) \right] = a^2 \sum_m \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot (x - x')} \left( \phi_{k,im}(\eta) \phi^*_{k,jm}(\eta) - \phi^*_{k,im}(\eta) \phi_{k,jm}(\eta) \right)
\]
\[
= a^2 \sum_m \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot (x - x')} \left( \phi_{k}(\eta) \cdot \Phi_{k}^T(\eta) - \Phi_{k}(\eta) \cdot \phi_{k}(\eta) \right)_{ji}
\]
\[
= i \delta_{ij} \delta^3(\vec{x} - \vec{x}'), \quad (i, j = a, b),
\]

where I have imposed the canonical commutation relation in the last line. This implies that
\[
\Phi_k(\eta) \cdot \Phi_k^T(\eta) - \Phi_k^*(\eta) \cdot \Phi_k^T(\eta) = \Phi_k(\eta) \cdot \Phi_k^T(\eta) - \left( \Psi_k^*(\eta) \cdot \Psi_k^T(\eta) \right)^T = \frac{i}{a^2}. \tag{5.121}
\]

Now we substitute the mode expansion for \( \Phi \) from Eq. (5.119) into the field equation Eq. (5.113) to get
\[
\left( \partial^2_\eta + k^2 + a^2 \left( R(\Xi - \frac{1}{6}) + M^2 \right) \right)(a \Phi_k(\eta)) = 0
\]
\[
\left( \partial^2_\eta + k^2 + a^2 \left( R(\Xi - \frac{1}{6}) + M^2 \right) \right)(a \Psi_k^*(\eta)) = 0, \tag{5.122}
\]

and similarly for the complex conjugates of the fields \( \Phi_k \) and \( \Psi_k^* \). We see that we obtain the same equations as for the real scalar field, Eq. (5.62). Thus, again we can solve this equation in quasi de Sitter space and find the propagator. This is exactly what we will do in the next section.

### 5.3.1 Propagator for the two-Higgs doublet model

In quasi de Sitter space with the scale factor (2.23), the general solutions of Eq. (5.122) are
\[
\Phi_k = \Phi_k^{(1)} \cdot \alpha + \Phi_k^{(2)} \cdot \beta, \quad \Psi_k^* = \Phi_k^{(1)} \cdot \gamma + \Phi_k^{(2)} \cdot \delta, \tag{5.123}
\]

where \( \alpha, \beta, \gamma, \delta \) are \( 2 \times 2 \) matrices and the functions \( \Phi_k^{(1)} \) and \( \Phi_k^{(2)} \) are
\[
\Phi_k^{(1)}(\eta) = \frac{1}{a} \sqrt{-\frac{\pi \eta}{4}} H_{\nu}^{(1)}(-k\eta),
\]
\[
\Phi_k^{(2)}(\eta) = \frac{1}{a} \sqrt{-\frac{\pi \eta}{4}} H_{\nu}^{(2)}(-k\eta). \tag{5.124}
\]
The functions $\Phi_k^{(1)}$ and $\Phi_k^{(2)}$ are $2 \times 2$-matrices because the index $\nu$ is a constant $2 \times 2$-matrix, and in 4 dimensions it is

$$\nu^2 = \left( \frac{3 - \epsilon}{2(1 - \epsilon)} \right)^2 - \frac{M^2}{H^2(1 - \epsilon)^2} - \frac{6(2 - \epsilon)\Xi}{(1 - \epsilon)^2}.$$  

Because the nonminimal coupling and mass matrices are hermitean, the index $\nu$ is also hermitean, $\nu^\dagger = \nu$. This useful property allows us to derive the important relation

$$\left( \Phi_k^{(1)}(\eta) \right)^\dagger = \Phi_k^{(2)}(\eta)$$  

(5.125)

The normalization of the mode functions $\Phi_k^{(1)}$ and $\Phi_k^{(2)}$ has been chosen such that the Wronskian of the two solutions satisfies

$$W \left[ \Phi_k^{(1)}(\eta), \Phi_k^{(2)}(\eta) \right] = \Phi_k^{(1)}(\eta) \cdot \Phi_k^{(2)\dagger}(\eta) - \Phi_k^{(2)}(\eta) \cdot \Phi_k^{(1)\dagger}(\eta)$$

$$= i \frac{1}{a^2}.$$  

(5.126)

The canonical commutation relation Eq. (5.116) is satisfied only when condition Eq. (5.121) is fulfilled. Using the Wronskian between the two fundamental solutions from Eq. (5.126) we get the following conditions on the matrices

$$\alpha \cdot \alpha^\dagger - \gamma \cdot \gamma^\dagger = 1$$

$$\beta \cdot \beta^\dagger - \delta \cdot \delta^\dagger = -1$$

$$\alpha \cdot \beta^\dagger - \gamma \cdot \delta^\dagger = 0,$$  

(5.127)

where $1$ denotes the $2 \times 2$ unit matrix. A particular solution for the matrices is $\alpha = \delta = \text{diag}(1,1)$ and $\beta = \gamma = 0$. This corresponds to an initial vacuum state with zero particles as $\eta \rightarrow -\infty$, our familiar Bunch-Davies vacuum. In this special case the two general solutions are simply $\Phi_k = \Phi_k^{(1)}$ and $\Psi_k^* = \Phi_k^{(2)}$, such that the Wronskian between the two solutions is $W(\Phi_k, \Psi_k^*) = i/a^2$.

In the Bunch-Davies vacuum we can again calculate the propagator for the two-Higgs doublet model. There will now be four propagators because we have two fields that can propagate into each other. To be exact, the time ordered/Feynman propagator is defined as

$$i \Delta(x,x') = \langle 0\mid T(\Phi(x)\Phi^\dagger(x'))\mid 0 \rangle$$

$$= \langle 0\mid \theta(t - t')\Phi(x)\Phi^\dagger(x') + \theta(t' - t)\Phi(x')\Phi^\dagger(x)\mid 0 \rangle.$$  

(5.128)

Note that we use terms like $\Phi \Phi^\dagger$ to make sure that the propagator is actually a hermitean $2 \times 2$-matrix. To be more precise

$$i \Delta(x,x') = \langle 0\mid \left[ \begin{array}{cc} T[\phi_1(x)\phi_1^\dagger(x')] & T[\phi_1(x)\phi_2^\dagger(x')] \\ T[\phi_2(x)\phi_1^\dagger(x')] & T[\phi_2(x)\phi_2^\dagger(x')] \end{array} \right] \mid 0 \rangle$$

$$= \langle 0\mid \theta(t - t') \left[ \begin{array}{cc} \phi_1(x)\phi_1^\dagger(x') & \phi_1(x)\phi_2^\dagger(x') \\ \phi_2(x)\phi_1^\dagger(x') & \phi_2(x)\phi_2^\dagger(x') \end{array} \right] \mid 0 \rangle$$

$$+ \langle 0\mid \theta(t' - t) \left[ \begin{array}{cc} \phi_1(x')\phi_1^\dagger(x) & \phi_1(x')\phi_2^\dagger(x) \\ \phi_2(x')\phi_1^\dagger(x) & \phi_2(x')\phi_2^\dagger(x) \end{array} \right] \mid 0 \rangle.$$  

(5.129)

Choosing the Bunch-Davies vacuum with $\alpha = \delta = 1$ and $\beta = \gamma = 0$ we find that the Feynman propagator is

$$i \Delta(x,x') = \frac{\pi \sqrt{\eta \eta'}}{4a^2} \int \frac{d^3k}{(2\pi)^3} e^{ik(x-x')}$$

$$\times \left\{ \theta(\eta - \eta') H_v^{(1)}(-k\eta) H_v^{(2)}(-k\eta') + \theta(\eta' - \eta) H_v^{(1)}(-k\eta') H_v^{(2)}(-k\eta) \right\}.$$
This again exactly the same expression as in the previous section, the difference being that $\nu$ is now a hermitean matrix. We can simply copy the final result for the propagator from the previous section, Eq. (5.105), which is

$$i \Delta(x,x')_{\infty} = \frac{[(1 - \epsilon)^2 H H']^{\frac{3}{2} - 1}}{(4\pi)^D} \Gamma\left(\frac{D}{2} - 1\right) \left(\frac{y}{4}\right)^{1 - \frac{D}{2}} + \frac{(1 - \epsilon)^2 H H'}{(4\pi)^2} \Psi_0 \left(\frac{1}{4} - \nu^2\right) + \mathcal{O}(s^3 + \epsilon).$$

(5.130)

The matrix form of the propagator is hidden in the fact that the parameter $\nu$ is a $2 \times 2$-matrix. The small parameter $s$ is now

$$s \equiv \frac{\Xi R + M^2}{H^2}, \quad ||s|| \ll 1.$$

(5.131)

Again, $R = 6H^2(2 - \epsilon)$ and the mass matrix $M^2$ is proportional to $H^2$, such that it precisely cancels the main contribution coming from the $\xi R$ term. We have numerically verified that $s$ is actually proportional to the slow-roll parameter $\epsilon$.

This concludes our discussion on the quantization of the two-Higgs doublet model. In the next chapter we will use the propagator for the two-Higgs doublet model to calculate the effect of the inflaton on the dynamics of fermions during inflation.

5.4 Summary

In this chapter our goal was to quantize the nonminimally coupled inflaton field. In section 5.1 we repeated the usual quantization procedure in Minkowski space. We have shown that we could write our action as an infinite sum of harmonic oscillators with frequency $\omega_k$. This allowed us to expand our field in mode functions with operator valued constants, the creation and annihilation operators, which in turn defined a vacuum that was annihilated by all annihilation operators. In section 5.1.1 we showed that the vacuum is not unique. We could have expanded our field in different mode functions, with different creation and annihilation operators and thus a different vacuum. The different creation and annihilation operators were related through a so-called Bogoliubov transformation. The fact that there is an infinite amount of choices for the vacuum makes it impossible to define the notion of a particle, which is defined as an excitation with respect to the vacuum. Fortunately, in Minkowski space we can make a unique choice for the vacuum that is the state of lowest energy and zero particles. An important fact is that this vacuum is also invariant under time translations, i.e. the vacuum will remain the state of lowest energy as time proceeds. The reason is that the Hamiltonian is constant in time.

This analysis breaks down when the Hamiltonian is not time independent, which happens when the frequency of the harmonic oscillator is time dependent. In section 5.1.2 we showed that we can still solve the field equation if there are in and out regions where the frequency is approximately constant. We showed that by defining a vacuum in the in region, we could make a simple estimate of the amount of particles produced in the out region with respect to the in vacuum. We concluded that for a time dependent harmonic oscillator particles are produced as time proceeds, and that therefore the notion of a particle is not well defined. Physical examples are a uniform accelerated observer in Minkowski space that will see a distribution of particles similar to a thermal bath of blackbody radiation, the Unruh effect. Another example is quantum field theory in an expanding universe in general. In section 5.2 we split our nonminimally coupled inflaton field in a classical background field and a
small quantum fluctuation. We quantized the fluctuation field and showed that we could again write down the field equation for a harmonic oscillator. The frequency however was time dependent because of the time dependent scale factor that appeared in the field equation. This makes it impossible to define a unique vacuum that remains the lowest energy state for all time. However, there are certain situations where we can still define a natural vacuum state, which we discussed in section 5.2.1. In the so-called adiabatic regime the frequency changes slowly and becomes approximately constant. This allowed us to define the adiabatic vacuum, which was not the true lowest energy vacuum, but to a very good approximation it was the lowest energy state in the adiabatic regime. We saw that in quasi de Sitter space the adiabatic regime was reached in the infinite asymptotic past, and in fact the adiabatic vacuum became the true vacuum in the limit $\eta \to -\infty$. In this adiabatic regime particle production due to the expansion of the universe is minimal, and we could define a natural vacuum state.

In section 5.2.2 we showed that we can actually solve the field equations in quasi de Sitter space exactly in terms of Hankel functions. An important requirement was the proportionality of the classical background field to the Hubble parameter during inflation, which we have both analytically and numerically verified in chapter 4. We could define a natural vacuum state, the so-called Bunch-Davies vacuum, which corresponds to zero particles in the infinite past. This allowed us to calculate the scalar propagator in section 5.2.3, which is the probability for the particle to travel from one space time point to another. For reasons of dimensional regularization and renormalization, which we will use in the coming chapter, we calculated the propagator in $D$ dimensions and made an expansion around $D - 4$.

In section 5.3 we generalized our results for the nonminimally coupled real scalar field to the two-Higgs doublet model. The difference is that instead of one real scalar field there are two complex scalar fields. In the quantization procedure for the fields this led to two changes: each scalar field was expanded in independent sets of creation and annihilation operators, and to incorporate the complexity of the fields we needed to introduce an additional set of creation and annihilation operators for each scalar field. These new operators created or annihilated antiparticles. We were able to write the expansion in one single matrix valued equation, with the fields as a vector and the mode functions as matrices. This allowed us to write the field equation for the two-Higgs doublet model in precisely the same form as the field equation for the real scalar field. The solutions were therefore also Hankel functions with an index $\nu$ which is a constant hermitean matrix. For the propagator, which is now a $2 \times 2$ matrix, in the Bunch-Davies vacuum we could simply copy the result from the propagator for the real scalar field. The result in Eq. (5.130) is the main result of this chapter. In the next chapter we will consider the fermion sector of the Standard Model action and use this propagator to calculate the effect of the nonminimally coupled inflaton field on the dynamics of massless fermions during inflation.
Chapter 6

The fermion sector

In the previous chapters we only considered the scalar section of the action. In chapter 4 we have seen that the scalar particle can be the inflaton in Linde’s chaotic inflationary model [8], where the scalar particle has a large initial value and slowly rolls down the potential hill. There are many scalar field models in which chaotic inflation is possible, and in this thesis we considered specifically the inflaton field with strong negative coupling to the Ricci scalar. Benefits of this model are the fact that inflation is successful for relatively small initial values of the scalar field and the quartic self coupling does not have to be very small in order to agree with the observed spectrum of density fluctuations from the CMBR. The latter allows the Higgs boson to be the inflaton as proposed by Bezrukov and Shaposhnikov [4]. The beautiful thing is that we do not need to introduce new scalar particles into the Standard Model, but we can simply use the existing particles to explain inflation by only introducing a nonminimal coupling to gravity.

Of course, the Standard Model of particles does not contain only the Higgs boson. It also contains the gauge bosons (the photon, $W^\pm$ and $Z$-bosons, the gluons) and the fermions (quarks, electrons, neutrinos). The masses of these particles are generated through the Higgs mechanism. The Higgs boson couples to these particles and because it has a nonzero expectation value at low energy, it effectively generates a mass for the gauge bosons and fermions. The coupling of the gauge bosons to the Higgs boson appears naturally in the Standard Model. By demanding $U(1)$, $SU(2)$ and $SU(3)$ symmetry of the action, one needs to introduce covariant derivatives that contain the gauge bosons. We will not consider the gauge bosons in this thesis, but focus on the fermions. For the fermions the coupling to the Higgs boson does not appear naturally and has to be constructed. This is the Yukawa sector of the action, and it contains the scalar-fermion interactions.

In this chapter we will construct the fermion sector of the Standard Model action, including the coupling of the fermions to the inflaton, i.e. the Higgs boson. For good literature on quantum field theory in general see [35]. We will consider the simplest supersymmetric extension of the Standard Model, the two-Higgs doublet model. The reason why we use this model is that it provides a mechanism for baryogenesis, see for example [36][37][29]. First we will first construct the fermion action in Minkowski space time in section 6.1. Then in section 6.2 we will generalize this to a curved space time. We will see that in a conformal FLRW universe we can write our action again as the Minkowski fermion action by performing a conformal rescaling of the fields. The only difference is that the mass of the fermions is multiplied by the scale factor. In section 6.2.2 we construct the field equation for the fermions (the Dirac equation) and we try to solve this equation classically. We will see that in case of the two-Higgs doublet model the Dirac equation is modified slightly, which could lead to an axial vector current, baryon number violation and baryogenesis.
In section 6.2.1 we calculate the propagator for the fermions, which for massless fermions turns out to be a simple conformal rescaling of the Minkowski propagator. In section 6.3 we use our main result from the previous chapter, the scalar propagator for the two-Higgs doublet model, and the massless fermion propagator to construct and renormalize the one-loop fermion self-energy. We will see that this one loop effect generates a mass for the fermions that is proportional to the Hubble parameter.

6.1 Fermion action in Minkowski space time

In a flat Minkowski space the general fermion action is

\[ S_f = \int d^D x \left\{ \frac{i}{2} \bar{\psi} \gamma^a \partial_a \psi - (\partial_a \bar{\psi}) \gamma^a \psi + \mathcal{L}_Y \right\}, \]  

(6.1)

where \( \psi \) is the fermion field doublet and \( \mathcal{L}_Y \) is the Yukawa Lagrangian. For generality we have taken a \( D \) dimensional Minkowski space, but one should remember that for the Standard Model \( D = 4 \). The anti-fermion \( \bar{\psi} \) is defined as

\[ \bar{\psi} = i \psi^\dagger \gamma^0, \]  

(6.2)

making sure that the action Eq. (6.1) is Lorentz invariant. The gamma matrices \( \gamma^a \) satisfy

\[ \{ \gamma^a, \gamma^b \} = -2 \eta^{ab} \quad (a, b = 0, 1, ..., D - 1). \]  

(6.3)

In the Weyl or chiral representation, that we will use throughout this thesis, the \( 4 \times 4 \) gamma matrices are in 4 dimensions given by

\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \]  

(6.4)

where \( \sigma^i \) are the \( 2 \times 2 \) Pauli matrices. In this chapter we are mostly interest in the influence of the inflaton (the Higgs particle) on the dynamics of the fermions (quarks, electrons, neutrinos). Our goal is to calculate the one-loop fermion self-energy, with a scalar in the loop. The coupling of the scalar to the fermions is contained in the Yukawa sector, and therefore we will give an explicit expression for this sector.

First of all, in the Standard Model we deal with chiral fermions which break the chiral symmetry. For example, there is only a left handed neutrino, whereas for quarks and electrons the symmetry is broken because of the mass of these particles. To be more precise, for the fermion fields we can project out two different chiralities,

\[ \psi = (P_L + P_R) \psi = \psi_L + \psi_R \]  

(6.5)

where the projection operators are defined as

\[ P_L = \frac{1 - \gamma^5}{2}, \quad P_R = \frac{1 + \gamma^5}{2}. \]  

(6.6)

The matrix \( \gamma^5 \) is defined through the other Dirac matrices, and in 4 dimensions it is

\[ \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3. \]  

(6.7)

In the chiral representation the explicit form in 4 dimensions is

\[ \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]  

(6.8)
In this representation the projection operators in 4 dimensions are
\[ P_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \] (6.9)

Note that in principle \( \psi_L \) and \( \psi_R \) are 4-spinors, but because of the \( \gamma^5 \) matrix it is easy to see that two components of these 4-spinors are zero. In the chiral representation we can thus write the 4-spinor \( \psi \) as two 2-spinors,
\[ \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \] (6.10)

It is also easy to see that \( (\gamma^5)^2 = 1 \) which allows us to derive the following useful relations
\[ (P_L)^2 = P_L \]
\[ (P_R)^2 = P_R \]
\[ P_R P_L = P_L P_R = 0. \] (6.11)

An important property of the matrix \( \gamma^5 \) is that it anticommutes with all the other Dirac matrices, i.e.
\[ \{ \gamma^5, \gamma^\mu \} = 0. \] (6.12)

This allows us to derive the following relation
\[ \bar{\psi} P_L = i \psi^+ \gamma^0 \left( \frac{1 - \gamma^5}{2} \right) \]
\[ = i \psi^+ \left( \frac{1 + \gamma^5}{2} \right) \gamma^0 \]
\[ = i (P_R \psi)^+ \gamma^0 \]
\[ = \bar{P_R} \psi, \] (6.13)

and the same relation when \( P_L \) and \( P_R \) are interchanged. Using this relation we can express the fermion action in terms of left- and right-handed fermions. For a kinetic term such as \( \bar{\psi} \gamma^a \partial_\alpha \psi \) this gives
\[ \bar{\psi} \gamma^a \partial_\alpha \psi = \bar{\psi} (P_L + P_R) \gamma^a \partial_\alpha (P_L + P_R) \psi \]
\[ = \bar{\psi} P_L \gamma^a \partial_\alpha (P_R \psi) + \bar{\psi} (P_R \psi) \gamma^a \partial_\alpha (P_L \psi) \]
\[ = \bar{P_R} \psi \gamma^a \partial_\alpha (P_R \psi) + \bar{P_L} \psi \gamma^a \partial_\alpha (P_L \psi) \]
\[ = \bar{\psi} \gamma^a \partial_\alpha \psi_R + \bar{\psi} \gamma^a \partial_\alpha \psi_L. \] (6.14)

The terms with \( P_L \gamma^a P_L \) vanish because of the anticommutation relation in Eq. (6.12) and the fact that \( P_L P_R = 0 \). Thus we see that for the kinetic terms there is no mixing of the right- and lefthanded fermions.

Now let us have a look at the Yukawa sector. In a very simple theory of one real scalar field that couples to the fermions we have a term \( -f \phi \bar{\psi} \psi \), where \( f \) is a coupling constant. We can therefore identify the time dependent mass of the fermions as \( m = f \phi \). Let us express this mass term again in terms of the right- and lefthanded fields, which gives
\[ -f \phi \bar{\psi} \psi = -f \phi (P_L + P_R)(P_L + P_R) \psi \]
\[ = -f \phi (\bar{\psi} (P_L)(P_L) \psi + \bar{\psi} (P_R)(P_R) \psi) \]
\[ = -f \phi \left( \bar{P_R} \psi (P_L) \psi + \bar{P_L} \psi (P_R) \psi \right) \]
\[ = -f \phi \left( \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R \right). \] (6.15)
Thus, the mass term of the fermion sector couples right- and lefthanded fermions. Now we want to find an explicit expression for the Yukawa sector of the fermion action in case of the two-Higgs doublet model. In this specific case, one of the Higgs doublets will couple only to down-type quarks and the other only to up-type quarks. The reason is to suppress flavor-changing neutral currents, see for example [29]. Define the two Higgs doublets as

\[ H_1 = \left( \begin{array}{c} \phi_1^0 \\ \phi_1^+ \end{array} \right), \quad H_2 = \left( \begin{array}{c} \phi_2^0 \\ \phi_2^- \end{array} \right), \] (6.16)

where the \( \phi^{+0} \) are charged and neutral complex scalar fields. We can fix a gauge such that

\[ H_1 = \left( \begin{array}{c} \phi_1^{+} \\ 0 \end{array} \right), \quad H_2 = \left( \begin{array}{c} \phi_2^{0} \\ 0 \end{array} \right), \] (6.17)

where \( \phi_1 \) and \( \phi_2 \) are complex scalar fields with expectation values \( \langle \phi_1 \rangle = v_1 \) and \( \langle \phi_2 \rangle = v_2 e^{i\theta} \). If we want the field \( \phi_2 \) to couple to the down-type quarks only, we have to define a new Higgs doublet \( \tilde{H}_2 \) as

\[ \tilde{H}_2 = i\sigma_2 H_2^* = \left( \begin{array}{c} -\phi_2^- \\ (\phi_2^0)^* \end{array} \right), \] (6.18)

which after fixing a gauge will be

\[ \tilde{H}_2 = \left( \begin{array}{c} 0 \\ \phi_2^* \end{array} \right). \] (6.19)

Now we are ready to write down the Yukawa Lagrangian. For the coupling of the quarks to the inflaton, it is

\[ \mathcal{L}_Y = -f_u (\overline{\psi}_L \cdot H_1) u_R - f_d (\overline{\psi}_L \cdot \tilde{H}_2) d_R + \text{h.c.}, \] (6.20)

where the index \( u \) corresponds to the up-type quarks \( (u, c, t) \) and \( d \) to the down-type quarks \( (d, s, b) \). The fermion field doublet \( \overline{\psi}_L \) is now explicitly

\[ \overline{\psi}_L = \left( \begin{array}{c} \overline{u}_L \\ \overline{d}_L \end{array} \right). \] (6.21)

We see that after fixing the gauge for the Higgs fields by using Eqs. (6.17) and (6.19), the Yukawa Lagrangian takes the simple form

\[ \mathcal{L}_Y = -f_u \overline{u}_L \phi_1 u_R - f_d \overline{d}_L \phi_2^* d_R - f_u^* \overline{u}_R \phi_1^* u_L - f_d^* \overline{d}_R \phi_2 d_L, \] (6.22)

where I have written down the Hermitean conjugate terms for completeness. Combining Eqs. (6.1) and (6.22) we find the fermion action in terms of the left- and righthanded fermion fields,

\[ S = \int d^4x \left\{ \frac{i}{2} [\overline{\psi}_L \gamma^a \partial_a \psi_L - (\partial_a \overline{\psi}_L) \gamma^a \psi_L] + \frac{i}{2} [\overline{\psi}_R \gamma^a \partial_a \psi_R - (\partial_a \overline{\psi}_R) \gamma^a \psi_R] \\
- f_u \overline{u}_L \phi_1 u_R - f_d \overline{d}_L \phi_2^* d_R - f_u^* \overline{u}_R \phi_1^* u_L - f_d^* \overline{d}_R \phi_2 d_L \right\}. \] (6.23)

Next we use that we can write

\[ f_u \overline{u}_L \phi_1 u_R = f_u \phi_1 \overline{P}_L u P_R u = f_u \phi_1 \overline{P}_R^2 u = f_u \phi_1 \overline{u} \frac{1}{2} (1 + \gamma^5) u \] (6.24)

and similarly

\[ f_d \overline{d}_L \phi_2^* d_R = f_d \phi_2^* \overline{d} \frac{1}{2} (1 + \gamma^5) d \\
f_u^* \overline{u}_R \phi_1^* u_L = f_u^* \phi_1^* \overline{u} \frac{1}{2} (1 - \gamma^5) u \\
f_d^* \overline{d}_R \phi_2 d_L = f_d^* \phi_2 \overline{d} \frac{1}{2} (1 - \gamma^5) d. \]
These relations allow us to rewrite the action in terms of the general fermion fields as

\[ S = \int d^Dx \left\{ \frac{i}{2} \left[ \overline{\psi} \gamma^a \partial_a \psi - (\partial_a \overline{\psi}) \gamma^a \psi \right] 
- f_u \phi_1 \overline{u} \frac{1}{2} (1 + \gamma^5) u
- f_d \phi_2 \overline{d} \frac{1}{2} (1 + \gamma^5) d \right\}. \]  

(6.25)

This action will become useful when we calculate the one-loop fermion self-energy and we need the Feynman rules for the scalar-fermion vertices in section 6.3.

### 6.2 Fermion action in curved space times

The action (6.23) describes the fermions in a flat Minkowski space time, but in the end we want to describe fermions in a flat FLRW space time. Thus we need to rewrite the fermion action to a curved space time. Curved space time can be described by the metric \( g_{\mu\nu} \), that we can transform to a local Minkowski frame by the following transformation

\[ g_{\mu\nu}(x) = e^a_{\mu}(x) e^b_{\nu}(x) \eta_{ab}, \]  

(6.26)

where \( e^a_{\mu} \) is the so-called vierbein field. In curved space times the action (6.23) takes the form

\[ S = \int d^Dx \sqrt{-G} \left\{ \frac{i}{2} \left[ \overline{\psi}_L \gamma^\mu \nabla_\mu \psi_L - (\nabla_\mu \overline{\psi}_L) \gamma^\mu \psi_L \right] + \frac{i}{2} \left[ \overline{\psi}_R \gamma^\mu \nabla_\mu \psi_R - (\nabla_\mu \overline{\psi}_R) \gamma^\mu \psi_R \right] 
- f_u \phi_1 \overline{u}_R \frac{1}{2} (1 + \gamma^5) u_R
- f_d \phi_2 \overline{d}_R \frac{1}{2} (1 + \gamma^5) d_R \right\}, \]  

(6.27)

where we have added the invariant measure \( \sqrt{-G} \) and replaced the ordinary derivative by the covariant derivative for fermions

\[ \partial_a \rightarrow e^a_{\mu} \nabla_\mu = e^a_{\mu} (\partial_\mu - \Gamma^\mu_{ab} \gamma^b), \]  

(6.28)

where the spin connection \( \Gamma^\mu \) is defined by

\[ \Gamma^\mu = -\frac{1}{8} e^c_\rho e^d_\sigma (\partial_\rho e^\sigma_{\mu} - \Gamma^\sigma_{\rho\mu} e^\sigma_{\nu}) \left[ \gamma^c, \gamma^d \right]. \]  

(6.29)

with \( \Gamma^\rho_{\mu\nu} \) the Christoffel connection. Actually the covariant derivative for a particle with spin in Eq. (6.28) should be the familiar covariant derivative for spin-0 particles plus the spin connection \( \Gamma_\mu \), so \( \nabla_\mu = \Gamma^0_\mu - \Gamma_\mu \) with \( \Gamma^0_\mu \) the familiar covariant derivative defined in Eq. (B.2). However, since the fermion field is not a vector field (it does not have an index) and because in the equation of motion for \( \Psi \) there is only one derivative, the familiar covariant derivative reduces to the partial derivative. The covariant derivative is derived by demanding that \( \nabla_\mu \gamma^\nu = 0 \). Notice also that for spinless particles the spin connection vanishes, since \( \left[ \gamma^c, \gamma^d \right] = 0 \).

We finally that we can again write the action in the form of Eq. (6.25), i.e. in terms of the fermion fields without making the distinction between left- and righthanded fields. The scalar-fermion interaction terms will now have an additional scale factor \( \sqrt{-G} = a^D \), such that the Feynman rules for the vertices will include this scale factor. We will come back to this when calculating the one-loop fermion self-energy in section 6.3.

Let us continue with the action (6.27). The Dirac matrices \( \gamma^\mu \) are defined by

\[ \gamma^\mu = e^a_{\mu} \gamma^a \]  

(6.30)
and satisfy therefore
\[ \{ \gamma^\mu, \gamma^\nu \} = e^\mu_a e^\nu_b \{ \gamma^a, \gamma^b \} = -2g^{\mu \nu}. \]  
(6.31)

Now we use the conformal FLRW metric as the metric to describe our expanding universe
\( g_{\mu \nu} = a^2(\eta)\eta_{\mu \nu}. \) A comparison to Eq. (6.26) shows that
\[ e^\mu_a = a(\eta)\delta^\mu_a, \]
\[ e^\nu_a = \frac{1}{a(\eta)}\delta^\nu_a, \]
\[ e_{\mu a} = \eta_{ka}e^k_\mu = a(\eta)\eta_{\mu a}. \]  
(6.32)

Using the form of the Christoffel symbols with the metric (B.20) and the expressions above we find that
\[ \Gamma_\mu = \frac{1}{4a} \eta_{\mu b} \left[ \gamma^0, \gamma^b \right]. \]  
(6.33)

We can now write
\[ i\gamma^\mu \nabla_\mu \psi = \frac{i}{a} i\gamma^c \partial_c \psi + i\frac{D - 1}{2} \frac{a'}{a^2} \gamma_0 \psi \]
\[ = a^{-\frac{D+1}{2}} i\gamma^c \partial_c (a^{-\frac{D-1}{2}} \psi). \]  
(6.34)

Therefore we can rewrite the action from Eq. (6.27) as
\[ S = \int d^Dx a^\frac{D}{2} \left\{ \frac{i}{2} [\bar{\chi}_L a^{-\frac{D-1}{2}} \gamma^c \partial_c (a^{-\frac{D-1}{2}} \chi_L) - \partial_c (a^{-\frac{D-1}{2}} \bar{\chi}_L) a^{-\frac{D-1}{2}} \gamma^c \chi_L] + \frac{i}{2} [\bar{\psi}_R a^{-\frac{D-1}{2}} \gamma^c \partial_c (a^{-\frac{D-1}{2}} \psi_R) - \partial_c (a^{-\frac{D-1}{2}} \bar{\psi}_R) a^{-\frac{D-1}{2}} \gamma^c \psi_R] \right. 
- f_u \bar{\psi}_L \phi^0 u_R - f_d \bar{\psi}_L \phi^0 d_R - f_u^* \bar{\psi}_R \phi^0^* u_L - f_d^* \bar{\psi}_R \phi^0^* d_L \}
\[ - f_u \bar{\psi}_L \phi^0 u_R - f_d \bar{\psi}_L \phi^0 d_R - f_u^* \bar{\psi}_R \phi^0^* u_L - f_d^* \bar{\psi}_R \phi^0^* d_L \right\} \]  
(6.35)

We now define a new conformally rescaled fermion field
\[ \chi = a^{-\frac{D-1}{2}} \psi, \]  
(6.37)

which allows us to write the action as
\[ S = \int d^Dx a^\frac{D}{2} \left\{ \frac{i}{2} [\bar{\chi}_L a^{-\frac{D-1}{2}} \gamma^a \partial_a \chi_L - (\partial_a \bar{\chi}_L) a^{-\frac{D-1}{2}} \gamma^a \chi_L] + \frac{i}{2} [\bar{\chi}_R a^{-\frac{D-1}{2}} \gamma^a \partial_a \chi_R - (\partial_a \bar{\chi}_R) a^{-\frac{D-1}{2}} \gamma^a \chi_R] \right. 
- a f_u \bar{\psi}_L \phi^0_1 u_R - a f_d \bar{\psi}_L \phi^0_2 d_R - a f_u^* \bar{\psi}_R \phi^0_1^* u_L - a f_d^* \bar{\psi}_R \phi^0_2^* d_L \right\}. \]  
(6.38)

We stress that the \( u \) and \( d \) quarks have also been conformally rescaled, \( a^{-\frac{D-1}{2}} u \rightarrow u \). Thus we see that by performing a conformal rescaling of the fermion field, the action is almost the same as the flat Minkowski action from Eq. (6.23). The only difference is that the mass term (the coupling of the fermions to the scalar field) is multiplied by the scale factor. In section 4.2 we made a statement that a massless fermion field is invariant under a conformal transformation, and this should be clear from the action (6.38). For a massless fermion the action in a \( D \) dimensional FLRW universe is precisely the same as the action in Minkowski space after a conformal rescaling of the fermion field.
We now want to calculate the Feynman propagator for the fermions. The time ordered Feynman propagator is given by

\[
iS_{F}^{ab}(x,x') = \langle 0|T[\psi_a(x)\bar{\psi}_b(x')]|0 \rangle = \theta(x-x')\langle 0|\psi_a(x)\bar{\psi}_b(x')|0 \rangle - \theta(x'-x)\langle 0|\bar{\psi}_b(x')\psi_a(x)|0 \rangle,
\]

where \(a\) and \(b\) are the spinor indices and there is a minus sign in front of the second term because the fermions anticommute. Let us consider a simplified fermion action with fermion fields \(\psi\) and mass \(m\). The Feynman propagator satisfies

\[
\sqrt{-g}(i\gamma^\mu \nabla_\mu + m)iS_F(x,x') = i\delta^D(x-x'),
\]

where the index \(x\) in \(\nabla_x\) denotes a covariant derivative with respect to the coordinate \(x\). In a FLRW space time we can write Eq. (6.40) as

\[
\left( a^{D-1}i\gamma^\mu \partial_\mu + a^D m \right) iS_F(x,x') = i\delta^D(x-x').
\]

We now write the equation for the propagator equation as

\[
(i\gamma^\mu \partial_\mu + am) i\tilde{S}_F(x,x') = i\delta^D(x-x'),
\]

where \(\tilde{S}_F(x,x') = a^{D-1}S_F(x,x')\) is the conformally rescaled propagator. The mass term is multiplied with the scale factor \(a\) with respect to the propagator equation in a flat Minkowski. This prevents us from simply copying the Minkowski space result for the fermion propagator. However, for massless fermions the above equation is quite easily solved. We define special de Sitter distance functions,

\[
\Delta S_{++}(x,x') = \frac{1}{2}[(\eta - \eta')^2 + |\vec{x} - \vec{x}'|^2]
\]

\[
\Delta S_{+-}(x,x') = -(\eta - \eta' + i\epsilon)^2 + |\vec{x} - \vec{x}'|^2
\]

\[
\Delta S_{-+}(x,x') = -(\eta - \eta' - i\epsilon)^2 + |\vec{x} - \vec{x}'|^2
\]

\[
\Delta S_{--}(x,x') = -|\eta - \eta'|^2 + |\vec{x} - \vec{x}'|^2.
\]

We can compare this to the different pole prescriptions in the momentum space integral for the propagator, where choosing a different position for the two poles (upper/lower half of complex plane) defines a different contour integration and a different propagator. These four different propagators can then be described as the same function of the appropriate distance function, i.e \(S_{lm} = F(x_{lm})\), \(l, m = +, -, \). The Feynman propagator is now a function of \(\Delta S_{++}\), whereas the anti time ordered (anti-Feynman) propagator is \(S_{--}\) and the Wightman propagators are \(S_{+-}\) and \(S_{-+}\). We are mostly interested in the time ordered Feynman propagator, and we therefore define

\[
\Delta S = \Delta S_{++}.
\]

We now use the following identities for the different distance functions

\[
\frac{1}{[\Delta S_{\pm \pm}(x,x')]^{D-1}} = \pm \frac{4\pi^D}{\Gamma(\frac{D}{2} - 1)} i\delta(x-x')
\]

\[
\frac{1}{[\Delta S_{\pm \mp}(x,x')]^{D-1}} = 0,
\]
to obtain the solution for the Feynman propagator
\[
iS_F(x, x') = \frac{1}{(aa')^{D/2}} i\bar{S}_F(x, x')
\]
\[
= \frac{1}{(aa')^{D/2}} i\gamma^c \frac{\Gamma(\frac{D}{2} - 1)}{4\pi^{D/2}} \frac{1}{[\Delta x^2(x, x')]^{D/2 - 1}}.
\] (6.44)

This is simply a conformal rescaling of the massless fermion propagator in flat Minkowski space.

For the massive fermions the calculation for propagator is much more difficult. The propagator for massive fermions in expanding space times with constant \( \epsilon = -\dot{H}/H^2 \) was solved by Koksma and Prokopec in [38]. They explicitly solved the field equation for the conformally rescaled fermion fields \( \chi \) in quasi de Sitter space. The solutions were again Hankel functions, but compared to the scalar propagator the indices \( \nu \) are complex. With these explicit solutions the massive fermion propagator could be calculated from the primary definition (6.39). Considering the infrared behavior of the fermions, the propagator is now found to be finite in the infrared, instead of divergent for the scalar case. Precisely the complexity of the index \( \nu \) of the Hankel functions makes sure that the propagator is IR finite. Physically, the Pauli exclusion principle forbids an accumulation of fermions in the infrared.

In [38] the propagator was solved for fermions with a real mass \( m \). This happens for example when the mass is generated by Standard Model Higgs particle. In case of a single Higgs doublet, we can fix a gauge such that the scalar field is real. In that case the mass of the fermions is \( m = f\phi \) and is a real quantity. In our two-Higgs doublet model however, the scalar fields are complex and the masses of the fermions are therefore complex as well. This will change the field equations for the fermions and therefore also the massive fermion propagator. In the next section we show that the complex fermion mass could generate an axial vector current and could be a possible source for baryogenesis.

6.2.2 Solving the chiral Dirac equation

In this thesis we will not redo the complete calculation done in [38] for a complex mass. Instead, we will outline the calculation and see where the differences pop up. Let us first give the field equations for the left- and righthanded conformally rescaled fermions from the action (6.38). I will call these field equations with an asymmetry in the mass term the chiral Dirac equations. For the conformally rescaled \( u \) quarks these equations are
\[
i\gamma^c \partial_c u_L - am_u u_R = 0
\]
\[
i\gamma^c \partial_c u_R - am_u^* u_L = 0,
\]
where I have defined \( m_u = f_u \phi_1 \). For the field equations for the \( d \) quarks we would find the same, but instead the mass would be \( m_d = f_d^2 \phi_2 \). For now I will simply disregard the fact that we have different fermions with different masses, but instead consider just a general conformally rescaled fermion field \( \chi \) with complex mass \( m \). So the field equations are
\[
i\gamma^c \partial_c \chi_L - am\chi_R = 0
\]
\[
i\gamma^c \partial_c \chi_R - am\chi_L = 0.
\] (6.45)

In Ref. [38] the conformally rescaled field \( \chi(x) \) is quantized by expanding it in creation and annihilation operators as we did for the scalar field in chapter 5. These operators are of different helicity \( h \), which is the spin in the direction of motion and can be either +1 or
The mode functions in the expansion are then \( \chi^{(h)}(k, \eta) \) and are Fourier transforms of the fields \( \chi(x) \). The explicit expansion we do not need here so we will refer to [38] to see the explicit form. Important for us is the fact that we can decompose the mode functions \( \chi^{(h)}(k, \eta) \) into a direct product of chirality and helicity 2-spinors,

\[
\chi(k, \eta) = \sum_h \chi^{(h)}(k, \eta) = \sum_h \left( \begin{array}{c} \chi_{L,h}(k, \eta) \\ \chi_{R,h}(k, \eta) \end{array} \right) \otimes \xi^{(h)},
\]

(6.46)

The decomposition in different chiralities we have already seen and done. But also the left- and righthanded 2-spinors can again be decomposed in the two helicity eigenvectors \( \xi^{(h)} \). Thus we see that we can rewrite each of the Eqs. (6.45), that are 2-spinor equations, into two equations for each \( h \). To actually do this we use that the action of the helicity operator \( \hat{h} \xi_h \) is

\[
\hat{h} \xi_h = \frac{k}{\hat{k}} \sigma \xi_h = h \xi_h,
\]

(6.47)

where \( \sigma \) are the three Pauli matrices and \( \hat{b} \cdot \hat{c} = \frac{\hat{k}}{\hat{k}} \) is the direction of propagation. The term \( \gamma_i \partial_i \) contains precisely the term \( k \cdot \sigma \) because \( \sigma_i \) appears in the matrices \( \gamma_i \) and the spatial partial derivative brings down a factor \( ik_i \) from the Fourier transformed field \( \chi(x) \). The decomposition (6.46), together with the explicit form of the \( \gamma \) matrices from Eq. (6.4), allows us to write the field equations (6.45) as

\[
i \chi_{L,h} + h k \chi_{L,h} - am \chi_{R,h} = 0
\]
\[
i \chi_{R,h} - h k \chi_{R,h} - am^* \chi_{L,h} = 0,
\]

(6.48)

where \( k = |k| \) and \( h = \pm 1 \). We see that we have obtained a set of four different equations that we now want to solve. In the end we want to find a solution for the 4-spinor field \( \chi(x) \).

In Ref. [38] the field equations (6.48) are solved for a real mass \( m \). In that case one can take a linear combination of the equations, insert the scale factor \( a \) in quasi de Sitter space and decouple the equations by taking a second derivative to \( \eta \). One then obtains Bessel’s equation with the usual Hankel functions as solutions, now with a complex index \( \nu \).

In this case the mass is complex and it is not so easy to solve the field equations. There are a few approaches to solve Eqs. (6.48). The first and most straightforward way is to decouple the field equations. We insert the scale factor \( a \) from Eq. (2.23) and write the field equations as

\[
i \chi_{L,h} + h k \chi_{L,h} - \frac{\zeta}{\eta} \chi_{R,h} = 0
\]
\[
i \chi_{R,h} - h k \chi_{R,h} + \frac{\zeta^*}{\eta} \chi_{L,h} = 0,
\]

(6.49)

where

\[
\zeta = \frac{m}{H(1 - \epsilon)}.
\]

\( \zeta \) is a constant since the Hubble parameter \( H \) is proportional to \( m = f \phi \) during inflation in the strong negative nonminimal coupling regime. We decouple the two equations by taking another derivative to \( \eta \) and do various substitutions of one equation into another. This leads to the equations

\[
\chi_{\eta \eta,h} + \frac{1}{\eta} \chi_{\eta,h} \chi_{\eta,h} + \left( \frac{\zeta^*}{\eta^2} + \frac{h k}{i \eta} \right) \chi_{\eta,h} = 0.
\]

(6.50)
We now again rescale the field $\chi_{\nu R, \eta}^{i}$ to a new field $\tilde{\chi}_{\nu R, \eta}^{i} = \sqrt{-\eta} \chi_{\nu R, \eta}^{i}$. This removes the linear term $\frac{1}{\eta} \chi_{\nu R, \eta}^{i}$ in the field equation, and it becomes

$$\tilde{\chi}_{\nu R, \eta}'' + \left( \frac{1}{\eta^2} + \frac{k^2}{\eta^2} \right) \tilde{\chi}_{\nu R, \eta} = 0. \tag{6.51}$$

The solutions to this equation are the so-called Whittaker functions $M_{\nu, \mu}(z)$ and $W_{\nu, \mu}(z)$. The general solution to Eq. (6.51) we can write as

$$\tilde{\chi}_{\nu R, \eta}^{i} = a M_{\nu, \mu}(z) + b W_{\nu, \mu}(z), \tag{6.52}$$

$$\nu = \mp \frac{h}{2}, \quad \mu = \mp i|\zeta|, \quad z = 2i k \eta.$$

This result was obtained in the last stages of the research for this thesis. A first study about Whittaker functions shows that for the cases $h = \pm 1$ the Whittaker functions might be rewritten in terms of Hankel functions, such that the solutions for $\chi_{\nu R, \eta}^{i}$ become

$$\chi_{\nu R, \eta}^{i} \simeq a \sqrt{-k \eta} H^{(1)}_{\nu}(k \eta) + b \sqrt{-k \eta} H^{(2)}_{\nu}(k \eta) \tag{6.53}$$

$$\nu_{\pm} = \frac{1}{2} \mp i |\zeta|.$$}

These are the same solutions obtained by Koksma and Prokopec in [38], which allows one to continue in the same way as in their paper and obtain the massive fermion propagator. However, we stress that these results are very preliminary and a careful study still has to be done. This might reveal some interesting differences compared to the case of the real fermion mass. In future work we hope to do a careful analysis and thoroughly solve the fermion propagator in case of a complex mass.

There are more ways to solve the chiral Dirac equations (6.48). We return to the fermion action (6.38). Suppose now we write our complex fields $\phi_{1}$ and $\phi_{2}$ as

$$\phi_{1}(\eta) = |\phi(\eta)| e^{i \alpha(\eta)}, \quad \phi_{2}(\eta) = |\phi(\eta)| e^{i \beta(\eta)} \tag{6.54}$$

At zero temperature the fields take their vacuum expectation values $|\phi_{1}| \rightarrow v_{1}, \ |\phi_{2}| \rightarrow v_{2}, \ \alpha \rightarrow 0$ and $\beta \rightarrow 0$. Remember that the only physical degrees of freedom are the real scalar fields and the phase difference between these fields. This means we can basically make one of the two fields $\phi_{1}$ or $\phi_{2}$ real and keep the other complex. The phase of this complex field is time dependent and can therefore not be removed by a field redefinition. Let us now consider a toy model where we have a complex scalar field $\phi$ with a time dependent phase $\theta(\eta)$,

$$\phi(\eta) = |\phi(\eta)| e^{i \theta(\eta)}. \tag{6.55}$$

The fermion action in this toy model is

$$S = \int d^{D}x \left\{ i\overline{\chi}_{L}\gamma^{a} \partial_{a} \chi_{L} + i\overline{\chi}_{R}\gamma^{a} \partial_{a} \chi_{R} - af \overline{\chi}_{L} \phi \chi_{R} - af \overline{\chi}_{R} \phi^{\dagger} \chi_{L} \right\},$$

where $f$ is a real Yukawa coupling and $\chi_{\nu R}$ are the conformally rescaled fermion fields. These Yukawa interactions lead to the field equations (6.48) with a complex mass $m = f \phi$. We can however remove the phase from the mass term by redefining the fermion fields,

$$e^{i \frac{\theta(\eta)}{2}} \chi_{L} \rightarrow \chi_{L}, \quad e^{i \frac{\theta(\eta)}{2}} \chi_{R} \rightarrow \chi_{R}, \tag{6.56}$$

The Yukawa interaction terms now have a real mass $m = f |\phi|$, but the additional time dependent phase in the kinetic term introduces an extra term in the fermion action,
This term is precisely the derivative of $\theta$ times the zeroth component of the axial vector current $J^{\delta \mu}$,

$$J^{\delta \mu} = \bar{\chi} \gamma^{\mu} \gamma^{5} \chi = \bar{\chi}_L \gamma^{\mu} \gamma^{5} \chi_L - \bar{\chi}_R \gamma^{\mu} \gamma^{5} \chi_R .$$  \hspace{1cm} (6.57)

This term explicitly violates $C$ and $CP$ (it is odd under $CP$). If we now derive the field equations from the fermion action with the additional term (6.56), we find the field equations

\begin{align*}
i \chi'_{L,h} + h \bar{k}_h \chi_{L,h} - a m \chi_{R,h} &= 0 \\
i \chi'_{R,h} - h \bar{k}_h \chi_{R,h} - a m \chi_{L,h} &= 0,
\end{align*}

\hspace{1cm} (6.58)

where the mass $m = f |\phi|$ and where

$$k_h = k - h \frac{\dot{\theta}}{2} .$$  \hspace{1cm} (6.59)

If $\dot{\theta}$ is constant, we find that we have obtained the field equations (6.48) with the difference that the mass is now real and the momentum $k$ is shifted by $\dot{\theta}$. These are precisely the field equations that are solved by Kokosma and Prokopec in [38]. We can therefore simply copy their results and replace all the momenta $k$ by $\bar{k}$. In calculating the propagator one has to insert the solutions in the definition of the propagator Eq. (6.39). Roughly this means one has to integrate the mode functions $\chi(k, \eta)$ over the momenta $k$, analogous to the calculation of the scalar propagator in Eq. (5.92). We can write this as an integration over $k$, but remember that the Hankel functions are functions of $\bar{k}$. One can then try to solve this by shifting the integration variable $k$ to $\bar{k}$, which means we have to do an extra integration over the mode functions from $-h \frac{\dot{\theta}}{2}$ to 0. Another way to solve this is by considering the $h \frac{\dot{\theta}}{2}$ as a small parameter, such that we can write $\bar{k} = k + \delta k$ and do a Taylor expansion for $\delta k$ small.

The study on the calculation of the massive fermion propagator with a shifted momentum $\bar{k}$ was only started towards the end of this thesis. Therefore we cannot present any rigorous results here, but only give an outline for further study. We hope that the shifted momentum will give an extra term in the propagator that is similar to the axial vector current and is $CP$ violating. This axial vector current can then be converted by sphaleron processes into a baryon asymmetry and could source baryogenesis. If this is the case, then the two-Higgs doublet model would not only be a good model for inflation, but could also be used to explain the baryon asymmetry of the universe. We hope to do a rigorous study on the fermion propagator in future work.

This concludes our study on the massive fermion propagator derived from a fermion action where the mass is generated by a complex scalar field. The calculations in this section where all focused on solving the Dirac equation for the fermion fields at tree-level. In other words, we tried to calculate the fermion propagator, but did not yet include any loop effects. In the next section we will in fact calculate the one-loop fermion self-energy.

### 6.3 One-loop fermion self-energy

In quantum field theory, an interaction is seen as a small perturbation to the quadratic part of the action. By expanding the path integral in terms of the small perturbation, one can derive the Feynman rules and Feynman diagrams for the interactions. Instead of taking only the “classical” tree-level propagator, the quantum propagator is constructed by taking into account all loop corrections of arbitrary order. Since each loop is proportional to a factor $h$, the biggest loop contribution to the propagator will be the one-loop effect. In Fig.
6.1 we show the one-loop diagram that we want to calculate. The fermion propagates, then splits into a scalar (the Higgs boson) and a fermion and recombines again into a fermion. These loop effects can be reexpressed as an energy or mass term in the fermion action. This is what we call the fermion self-energy, and in this chapter we specifically calculate the one-loop self-energy. The one-loop self-energy and its effect on the dynamics of fermions was already calculated by Garbrecht and Prokopec in [39]. We will redo their calculation here, with the difference that we will work now in quasi de Sitter space, instead of exact de Sitter.

In order to calculate the one-loop self-energy, we calculate the diagram in Fig. 6.1 by multiplying the scalar-fermion vertices with the internal scalar and fermion propagator. For the scalar propagator we use the Feynman propagator for the two-Higgs doublet model defined in Eq. (5.129). We found the explicit expression for this propagator in quasi de Sitter space in Eq. (5.130), which we list here again as a matter of convenience,

$$i \Delta (x, x') = \frac{[(1 - \epsilon)^2 H H']^\frac{D-1}{2}}{(4\pi)^\frac{D}{2}} \frac{D - 1}{\Gamma(D - 1)} \left(\frac{v}{4}\right)^{1 - \frac{D}{2}} + \frac{(1 - \epsilon)^2 H H'}{(4\pi)^2} \Psi_0 \left(1 - \frac{1}{4} - v^2\right) + \mathcal{O}(s^3 + \epsilon).$$

(6.60)

For the fermion propagator we should use the massive fermion propagator derived in [38]. However, for matters of simplicity we use the massless fermion propagator from Eq. (6.44), which is

$$i S_F(x, x') = \frac{1}{(a a')^{\frac{D-1}{2}}} i \gamma^\mu \partial_\mu \left(\frac{\Gamma(D - 1)}{4\pi^D} \frac{1}{[\Delta x^2(x, x')^{\frac{D-1}{2}}} \right).$$

In principle the only fermions we can describe with this massless propagator are neutrinos. Of course we also want to describe one-loop effects for the quark propagator, which are much larger because the Yukawa couplings are much larger (remember that the mass of the fermions is generated through the Yukawa interactions, so the largest mass has the largest Yukawa coupling). Therefore by taking the massless fermion propagator, we actually make an incorrect assumption. We will leave the calculation of the one-loop self-energy for the massive fermion propagator to future work. For now we will calculate the fermion one-loop self-energy for the simple case of a massless fermion propagator, and we keep in mind that in fact we only describe neutrinos with this calculation.

We continue by writing down the the Feynman rules for the vertices from the action (6.36). The results are shown in Table 6.1.

As we will see, if we simply multiply these vertices and propagators we will find that the self-energy is divergent for $D = 4$. In fact we can isolate the singularity to a local term $\delta(x - x')$. Looking at the diagram in Fig. 6.1 we see that the diagram is divergent if the point $x$ where the fermion splits into a scalar and a fermion is equal to the point $x'$ where
the scalar and fermion recombine again in a fermion. This local divergency however is non-physical and can be removed by adding a suitable counterterm to the one-loop fermion self-energy, see for example [35]. This counterterm, the so-called fermion field strength renormalization is

\[ i\delta Z_2(aa')D^\perp(x-x'). \]  

(6.61)

In section 6.3.1 we will actually isolate the divergency in the diagram of Fig. 6.1 to a local term and cancel this by a suitable choice of the parameter \( \delta Z_2 \).

Combining the expressions above we can calculate the one-loop fermion self-energy by simply multiplying vertices and propagators and adding the renormalization counterterm. Let us for simplicity focus on the one-loop self-energy for the \( u \) propagator. This diagram only involves the scalar propagator with the field \( \phi_1 \), since only this field couples to up-type quarks. The one-loop fermion self-energy is then

\[ -i[\Sigma](x,x') = (-if_u a^D_1(1+\gamma^5)i\{S\}(x,x')(-if_u^* a^D_1(1-\gamma^5)i\Delta(x,x')_{11} + i\delta Z_2(aa')D^\perp_{ij} \gamma^\mu_{ij} \delta \mu \delta^D(x-x'), \]  

(6.62)

where we have chosen the part \( \Delta(x,x')_{11} \) of the \( 2 \times 2 \)-matrix propagator \( \Delta(x,x') \) from Eq. (??), which is the propagator \( \langle \phi_1\phi_1^{a\ast} \rangle \). For the down-type quarks, we would find exactly the same expression, but only the couplings \( f_u \) are replaced by \( f_d \) and we take the part \( \Delta(x,x')_{22} \) of the scalar propagator. From now on we will simply omit the indices \( u/d \) in the Yukawa couplings and the indices 11 or 22 in the scalar propagator, and remember that we have to pick a specific part if we look at up- or down-type quarks. If we now insert the explicit expressions for the scalar and fermion propagators, and use that

\[ \gamma = \frac{\Delta x^2}{\eta^\prime}(1-c)^2HH'aa'\Delta x^2, \]  

(6.63)

we find

\[ -i[\Sigma](x,x') = |f|^2(aa')^2\frac{2}{8\pi^2D\Gamma(D-1)}\frac{i\gamma^\mu \Delta x_{\mu}}{(\Delta x^2)^D-1} + |f|^2(1-c)^2HH'(aa')^2\frac{8}{32\pi^2}\frac{i\gamma^\mu \Delta x_{\mu}}{(\Delta x^2)^D-1} + i\delta Z_2(aa')^{D\perp}_{ij} \gamma^\mu_{ij} \delta \mu \delta^D(x-x'). \]  

(6.64)
Note that the fact that the scalar propagator is a $2 \times 2$ matrix is still hidden in the $\nu^2$ term, since this is a $2 \times 2$ matrix. Since we have to integrate the self-energy over $\int d^4x$, we find that there is a singularity in the first term for $D = 4$. The other terms are integrable, and they are therefore finite. The first term requires renormalization, which we will do in the next section.

### 6.3.1 Renormalization of the one-loop self-energy

In the previous section we saw that the first term of the self-energy in Eq. (6.64) is singular for $D = 4$. Our goal is to isolate the singularity in the self-energy and write it as a local term (i.e. with a $\delta(x-x')$) such that we can remove it by a suitable choice of $\delta Z_2$. Let us first give the singular term again

$$\left| f \right|^2 \frac{1}{2} \frac{(a')^2}{8\pi^D} \Gamma\left(\frac{D}{2}\right) \Gamma\left(\frac{D}{2} - 1\right) \frac{i\gamma^c \Delta x_c}{\left[\Delta x^2\right]^{D-1}}.$$  \hspace{2cm} (6.65)

We now express the last part as a derivative, i.e.

$$\frac{\Delta x_c}{\left[\Delta x^2\right]^{D-1}} = \frac{-1}{2(D - 2)} \partial_c \frac{1}{\left[\Delta x^2\right]^{D-2}}.$$  \hspace{2cm} (6.66)

Then we calculate

$$\partial^2 \frac{1}{\left[\Delta x^2\right]^{a}} = \eta^{\mu \nu} \partial_\mu \left[ -2a \frac{\Delta x_\nu}{\left[\Delta x^2\right]^{a+1}} \right]$$

$$= -2a \left[ \frac{\eta^{\mu \nu} \delta_\mu}{\left[\Delta x^2\right]^{a+1}} - 2(\alpha + 1) \frac{\eta^{\mu \nu} \Delta x_\mu \Delta x_\nu}{\left[\Delta x^2\right]^{a+2}} \right]$$

$$= 2a \left[ \frac{2a + 2 - D}{\left[\Delta x^2\right]^{a+1}} \right].$$  \hspace{2cm} (6.67)

If we now compare Eqs. (6.65) and (6.67) and identify $D - 2 = \alpha + 1$ (and therefore also $2\alpha = 2(D - 3)$ and $2\alpha + 2 - D = D - 4$), we see that we can write Eq. (6.65) as

$$\frac{\Delta x_c}{\left[\Delta x^2\right]^{D-1}} = \frac{-1}{2(D - 2)} \partial_c \frac{1}{2(D - 3)(D - 4)} \partial^2 \frac{1}{\left[\Delta x^2\right]^{D-3}}.$$  \hspace{2cm} (6.68)

We recognize the singularity for $D = 4$ immediately. Now we insert a so-called 0 into this equation by using the identity (6.43). This gives

$$\frac{\Delta x_c}{\left[\Delta x^2\right]^{D-1}} = \frac{-1}{2(D - 2)} \partial_c \frac{1}{2(D - 3)(D - 4)} \left( \partial^2 \frac{1}{\left[\Delta x^2\right]^{D-3}} - \partial^2 \frac{\mu^{D-4}}{\left[\Delta x^2\right]^{D-3}} - \frac{\mu^{D-4}}{\left[\Delta x^2\right]^{D-3}} \right) \frac{4\pi \frac{D}{2} \Gamma\left(\frac{D}{2} - 1\right)}{\Gamma\left(\frac{D}{2} - 1\right) i\delta(x-x')}.$$  \hspace{2cm} (6.69)

Note the appearance of the renormalization scale $\mu$! This scale appears to make the dimensionality of the terms equal. We now focus our attention on the two terms with derivatives $\partial^2$. We can write

$$\frac{1}{\left[\Delta x^2\right]^{D-3}} - \frac{\mu^{D-4}}{\left[\Delta x^2\right]^{D-3}} = \mu^{2D-6} \left( \frac{1}{\left[\mu^2 \Delta x^2\right]^{D-3}} - \frac{1}{\left[\mu^2 \Delta x^2\right]^{D-3}} \right).$$  \hspace{2cm} (6.70)

Now we perform an expansion around $D = 4$ and we obtain

$$\mu^{2D-6} \left( \frac{1}{\left[\mu^2 \Delta x^2\right]^{D-3}} - \frac{1}{\left[\mu^2 \Delta x^2\right]^{D-3}} \right) = \mu^{2D-6} \left( \frac{1}{\left[\mu^2 \Delta x^2\right]^{D-3}} - \frac{1}{\left[\mu^2 \Delta x^2\right]^{D-3}} \left( 1 - (D - 4)\ln\left(\mu^2 \Delta x^2\right) - 1 + \frac{1}{2}(D - 4)\ln\left(\mu^2 \Delta x^2\right) \right) \right)$$

$$= -\mu^{2D-6} \frac{1}{\left[\mu^2 \Delta x^2\right]} \left( (D - 4)\ln\left(\mu^2 \Delta x^2\right) \right).$$  \hspace{2cm} (6.71)
Note that this term is proportional to \((D - 4)\)!. When we plug this expression back into Eq. (6.69), we find the following

\[
\frac{\Delta x_c}{[\Delta x^2]^{D-1}} = \frac{-1}{2(D - 2)} \partial_c \frac{1}{2(D - 3)(D - 4)} \left( -\partial^2 \frac{\mu^{2D-6}}{[\mu^2 \Delta x^2]} \frac{1}{2}(D - 4) \ln(\mu^2 \Delta x^2) + \frac{4\pi^2}{\Gamma(\frac{D}{2} - 1)} \right)
\]

\[
= \frac{1}{8(D - 2)(D - 3)} \partial_c \partial^2 \frac{\mu^{2D-6}}{[\mu^2 \Delta x^2]} \ln(\mu^2 \Delta x^2)
\]

\[
- \frac{1}{(D - 2)(D - 3)(D - 4)} \partial_c \frac{\pi^2 \mu^{D-4}}{\Gamma(\frac{D}{2} - 1)} i\delta(x - x')
\]

\[
= \frac{1}{16} \partial_c \partial^2 \frac{\ln(\mu^2 \Delta x^2)}{[\Delta x^2]} - \frac{1}{(D - 2)(D - 3)(D - 4)} \partial_c \frac{\pi^2 \mu^{D-4}}{\Gamma(\frac{D}{2} - 1)} i\delta(x - x').
\]

(6.72)

We have succeeded in isolating the singularity at \(D = 4\)! Note that in the third line we have substituted \(D = 4\) for the first term, since this term does not contain a singularity at \(D = 4\). We now substitute our result (6.72) into the self energy Eq. (6.64) and we expand the renormalization counterterm containing \(\delta Z_2\) up to first order in \(D - 4\). The self-energy now reads

\[
\Sigma(x, x') = -|f|^2 \frac{(aa')^\frac{3}{2}}{2^3\pi^D} \Gamma(\frac{D}{2}) \Gamma(\frac{D}{2} - 1) \gamma^c \partial_c \partial^2 \frac{\ln(\mu^2 \Delta x^2)}{[\Delta x^2]}
\]

\[
+ |f|^2 \frac{(aa')^\frac{3}{2}}{8\pi^2} \mu^{D-4} \Gamma(\frac{D}{2}) \frac{(D - 2)(D - 3)(D - 4)}{i\gamma^c \partial_c (x - x')}
\]

\[
- |f|^2 \frac{(1 - e^2)^2 HH'(aa')^\frac{3}{2}}{32\pi^4} \Psi_0 \left( \frac{1}{4} - \nu^2 \right) \frac{\gamma^\mu \Delta x_\mu}{(\Delta x^2)^2}
\]

\[
- \delta Z_2(aa')^\frac{3}{2} \gamma^\mu \partial_\mu \delta(x - x') - \frac{1}{2} \delta Z_2(aa')^\frac{3}{2} \ln(aa') \frac{1}{2} (D - 4) i\gamma^c \partial_c \partial^D (x - x')
\]

(6.73)

We have used that \((aa')^{\frac{D-1}{2}} = (aa')^\frac{3}{2} + \frac{1}{2} \ln(aa')(D - 4)\). Now we use the local renormalization counterterm to cancel the singularity at \(D = 4\) in the second term. We therefore choose

\[
\delta Z_2 = |f|^2 \frac{\mu^{D-4}}{16\pi^2} \frac{\Gamma(\frac{D}{2}) - 1}{(D - 3)(D - 4)},
\]

(6.74)

which precisely cancels the singularity if we recognize that \(2\Gamma(\frac{D}{2}) = (D - 2)\Gamma(\frac{D}{2} - 1)\). Now we also see that the last term in the self-energy Eq. (6.73) is proportional to \((D - 4)^0\), so we can safely set \(D = 4\) in this last term. Thus, after canceling the singularity we find the renormalized self-energy

\[
\Sigma^{\text{ren}}(x, x') = -|f|^2 \frac{(aa')^\frac{3}{2}}{2^3\pi^D} \gamma^c \partial_c \partial^2 \frac{\ln(\mu^2 \Delta x^2)}{[\Delta x^2]}
\]

\[
- |f|^2 \frac{(aa')^\frac{3}{2}}{2^5\pi^2} \ln(aa') i\gamma^c \partial_c (x - x')
\]

\[
- |f|^2 \frac{(1 - e^2)^2 HH'(aa')^\frac{3}{2}}{2^5\pi^4} \Psi_0 \left( \frac{1}{4} - \nu^2 \right) \frac{\gamma^\mu \Delta x_\mu}{(\Delta x^2)^2}
\]

(6.75)
The next step is to get rid of the $\Delta x^2$ terms in the denominators. Therefore we use the following relations valid in 4 dimensions

$$
\partial_\mu \left( \frac{\ln(a\Delta x^2) + 1}{\Delta x^2} \right) = \frac{\Delta x_\mu}{(\Delta x^2)^2}
$$

$$
\partial^2 \ln^2 (\mu^2 \Delta x^2) = 8 \frac{\ln(\mu^2 \Delta x^2)}{\Delta x^2}
$$

$$
\partial^2 \ln (\mu^2 \Delta x^2) = \frac{4}{\Delta x^2},
$$

which we can combine to get

$$
\frac{\ln(\mu^2 \Delta x^2)}{[\Delta x^2]} = \frac{1}{2^3} \partial^2 \left[ \ln^2 (\mu^2 \Delta x^2) - 2\ln(\mu^2 \Delta x^2) \right].
$$

Furthermore we rewrite the $\Psi_0$ term in the following way

$$
\Psi_0 = \ln(y\mathcal{K}), \quad \mathcal{K} = \frac{1}{4} \exp \left\{ \frac{3}{2} + \nu \left( \frac{3}{2} - \nu \right) + 2\gamma_E - 1 \right\}
$$

where we use the expression for $y$ from Eq. (6.63). Then we split up the term in Eq. (6.78) and write

$$
\ln(y\mathcal{K}) = \ln(aa') + \ln(KHH' (1 - c)^2 \Delta x^2).
$$

Now we use the relations (6.76) to find that

$$
\ln(y\mathcal{K}) \frac{\Delta x_\mu}{(\Delta x^2)^2} = -\frac{1}{2^4} \{ \partial_\mu \partial^2 \ln(\mu^2 \Delta x^2) + 2\ln(aa') \partial_\mu \partial^2 \ln(\Delta x^2) \}. \tag{6.79}
$$

By using Eqs. (6.77) and (6.79) and the definition $\gamma^\mu \partial_\mu \equiv \partial$ we find the final renormalized form of the self-energy

$$
\Sigma_{\text{ren}}(x, x') = -|f|^2 \frac{(aa')^2}{2^4 \pi^4} \partial^4 \left[ \ln^2 (\mu^2 \Delta x^2) - 2\ln(\mu^2 \Delta x^2) \right]
$$

$$
-|f|^2 \frac{(aa')^2}{2^5 \pi^2} \ln(aa')i\partial \delta(x - x')
$$

$$
-|f|^2 \frac{(1 - c)^2 HHH'(aa')^2}{2^9 \pi^4} \left( \nu^2 - \frac{1}{4} \right)
$$

$$
\times \{ \partial^2 \ln(\mu^2 \Delta x^2) + 2\ln(aa') \partial^2 \ln(\Delta x^2) \}. \tag{6.80}
$$

6.3.2 Influence on fermion dynamics

The one-loop self-energy will have an effect on the dynamics of fermions. To be precise, the Dirac equation will be modified in the following way

$$
a^2 i \partial a^2 \psi(x) - \int d^4 x' \Sigma_{\text{ren}}(x; x') \psi(x') = 0, \tag{6.81}
$$

where $\Sigma_{\text{ren}}(x; x')$ is the retarded self-energy,

$$
\Sigma_{\text{ren}}(x; x') = \Sigma^{++} + \Sigma^{+-}. \tag{6.82}
$$

Note that we have calculated $\Sigma^{++} \equiv \Sigma_{\text{ren}}(x_+; x'_+)$ in the previous section since we have been working with $\Delta x^2 \equiv \Delta x^2_{++}$ all the time. We now also want to find the self-energy $\Sigma^{+-} \equiv$
\[ \Sigma^{\text{ren}}(x_+; x_-) \], which corresponds to the distance function \( \Delta x^2_{+\pm} \) defined in Eq. (6.41). There will be only two changes: first of all, there will be an overall minus sign because the vertices contain a sign. For the ++ case, the sign will be the same, but for the +− and −+ case the sign will be different. Secondly, remember that in the renormalization procedure we used the identity (6.43) for \( \Delta x^2_{+\pm} \) to extract the singularity for \( D = 4 \) to a local term (see Eq. (6.69)). For the \( \Delta x^2_{0\pm} \) term the local term does not appear in the identity (6.43), which means that the singularity disappears completely and that the counterterm \( \delta Z_2 = 0 \). This gives as a result for the self-energies \( \Sigma^{++} \) and \( \Sigma^{+-} \):

\[
\Sigma^{++}(x, x') = -f^2 (aa') \frac{\bar{\delta}}{2^{1/4}} \left[ \ln^2 (\mu^2 \Delta x^2_{++}) - 2 \ln (\mu^2 \Delta x^2_{++}) \right]
- f^2 \frac{(aa')^2}{2^{5/2}} \ln (aa') i \delta(x - x')
- f^2 \frac{(1 - e^2)^2 H H'(aa')^2}{2^{5/2}} \left( \nu^2 - 1 \right)
\times \left\{ \delta \theta^2 \ln^2 (K HH'(1 - e^2) \Delta x^2_{++}) + 2 \ln (aa') \delta \theta^2 \ln (\Delta x^2_{++}) \right\}.
\]

(6.83)

\[
\Sigma^{+-}(x, x') = +f^2 (aa') \frac{\bar{\delta}}{2^{1/4}} \left[ \ln^2 (\mu^2 \Delta x^2_{+-}) - 2 \ln (\mu^2 \Delta x^2_{+-}) \right]
+ f^2 \frac{(1 - e^2)^2 H H'(aa')^2}{2^{5/2}} \left( \nu^2 - 1 \right)
\times \left\{ \delta \theta^2 \ln^2 (K HH'(1 - e^2) \Delta x^2_{+-}) + 2 \ln (aa') \delta \theta^2 \ln (\Delta x^2_{+-}) \right\}.
\]

(6.84)

The retarded self-energy is simply the sum of these. Since the ++ and +− self-energies have a difference in sign, the only contributions to the retarded self-energy come from branch cuts and singularities in \( \Delta x^2_{+\pm} \) and \( \Delta x^2_{0\pm} \) and the local term in the ++ self-energy. We use the following relations to extract these branch cuts and singularities:

\[
\ln (\alpha \Delta x^2_{+\pm}) - \ln (\alpha \Delta x^2_{0\pm}) = \ln \left( \frac{\nu_{+\pm}}{4} \right) - \ln \left( \frac{\nu_{0\pm}}{4} \right) = 2 \pi i (\Delta \eta^2 - r^2) \Theta (\Delta \eta)
\]

\[
\ln^2 (\alpha \Delta x^2_{+\pm}) - \ln^2 (\alpha \Delta x^2_{0\pm}) = 4 \pi i \ln (\alpha (\Delta \eta^2 - r^2)) \Theta (\Delta \eta^2 - r^2) \Theta (\Delta \eta),
\]

(6.85)

where \( \Delta \eta = \eta - \eta' \), \( r \equiv |x - x'|^2 \) and \( \Theta \) is the Heaviside step function. Furthermore, \( \Theta (\Delta \eta^2 - r^2) = \Theta (\Delta \eta - r) \), since we are considering timelike distances. The retarded self-energy in Eq. (6.82) then becomes:

\[
\Sigma^{\text{ret}}(x, x') = -f^2 (aa') \frac{\bar{\delta}}{2^{5/2}} \left[ \Theta (\Delta \eta - r) \Theta (\Delta \eta) \ln |\mu^2 (\Delta \eta^2 - r^2)| - 1 \right]
- f^2 \frac{(aa')^2}{2^{1/2}} \ln (aa') i \delta(x - x')
- f^2 \frac{(1 - e^2)^2 H H'(aa')^2}{2^{1/2}} \left( \nu^2 - 1 \right)
\times \left\{ i \delta \theta^2 [\Theta (\Delta \eta - r) \Theta (\Delta \eta) \ln |\alpha (\Delta \eta^2 - r^2)|] + \ln (aa') i \delta \theta^2 [\Theta (\Delta \eta - r) \Theta (\Delta \eta) ] \right\}.
\]

(6.86)

where I have defined \( a = K HH'(1 - e^2) \). Now we want to use this self-energy in the modified Dirac equation (6.81). We first define the conformally rescaled function:

\[
\chi (x) = a^\frac{\bar{z}}{2} \psi (x).
\]

(6.87)

Because the background is homogeneous and isotropic, we seek plane wave solutions of the form:

\[
\chi (x) = e^{ik \cdot x} \chi (\eta),
\]

(6.88)
such that we can find
\[ \chi(x') = e^{i\vec{k} \cdot \vec{x'}} \chi(\eta) = e^{i\vec{k} \cdot \vec{x}} e^{-i\vec{k} \cdot \Delta \vec{x}} \chi(\eta'), \]
where $\Delta \vec{x} = \vec{x} - \vec{x'}$. Furthermore we write the integral in the modified Dirac equation in polar coordinates
\[ \int d^4x' = \int \eta' \int d^3\vec{x} = \int \eta' \int_0^{2\pi} \int_0^\pi \int_0^\infty dr d\theta d\phi r^2 \sin \theta, \]
and perform the integrations over the angles $\theta$ and $\phi$,
\[ \int_0^{2\pi} \int_0^\pi d\theta d\phi \ r^2 \sin \theta e^{-i\vec{k} \cdot \Delta \vec{x}} = 2\pi \int_0^{\pi} d\theta r^2 \sin \theta e^{-i k r \cos \theta} = 2\pi \int_{-1}^{1} dy r^2 e^{-ik y} \]
\[ = 2\pi r^2 \frac{1}{i k r} (e^{-ik r} - e^{ik r}) = 4\pi \frac{r}{k} \sin kr, \]
where $k = |\vec{k}|$. Of course for the local term in the self-energy it does not make sense to switch to polar coordinates, because we can perform the integration over the 4-dimensional $\delta$-function straight away. Furthermore we use
\[ \int d\eta' \int_0^{\infty} dr \Theta(\Delta \eta - r) \Theta(\Delta \eta) = \int_0^{\eta} d\eta' \int_0^{\Delta \eta} d\eta'' \]
With the assumption for the wave function in Eqs. (6.88) and (6.89) and the angular integration in Eq. (6.90) we find the modified Dirac equation
\[ 0 = i \delta \chi(x) + \frac{|f|^2}{2\pi^2} i \delta \partial^4 e^{i\vec{k} \cdot \vec{x}} \int_0^{\eta} d\eta' \int_0^{\Delta \eta} dr r \sin (kr) \ln |\mu^2 (\Delta \eta^2 - r^2)| - 1 \chi(\eta') \]
\[ + \frac{|f|^2}{2\pi^2} [\ln(a) i \delta \chi(x) + i \delta (\ln(a) \chi(x)) ] \]
\[ + |f|^2 (1 - e) H' a \left( \frac{1}{4} \right) \]
\[ \times \left\{ i \delta \partial^2 e^{i\vec{k} \cdot \vec{x}} k \int_0^{\eta} d\eta' a' \int_0^{\Delta \eta} drr \sin (kr) \ln [a (\Delta \eta^2 - r^2)] \chi(\eta') \]
\[ + \ln(a) i \delta \partial^0 e^{i\vec{k} \cdot \vec{x}} k \int_0^{\eta} a' d\eta' \int_0^{\Delta \eta} drr \sin (kr) \chi(\eta') \]
\[ + i \delta \partial^2 e^{i\vec{k} \cdot \vec{x}} k \int_0^{\eta} d\eta' a' \ln(a') \int_0^{\Delta \eta} drr \sin (kr) \chi(\eta') \right\}. \]
In the second line I have used that $\delta$ contains only derivatives with respect to $x$, such that I can write $\ln(a a' \delta (x - x') \chi(x')) = \ln(a) \delta (x - x') \chi(x') + \delta (a a') \delta (x - x') \chi(x')$. Now we perform the radial integration by using the following integrals
\[ \int_0^{\Delta \eta} drr \sin (kr) = \frac{1}{k^2} [\sin(k \Delta \eta) - k \Delta \eta \cos(k \Delta \eta)] \]
and we define
\[ \zeta(x) = \int_0^1 dxx \sin (xx) \ln (1 - x^2). \]
We can now rewrite \( \ln|\mu^2(\Delta \eta^2 - r^2)| = \ln(\mu^2 \Delta \eta^2) + \ln(1 - \left(\frac{r}{\Delta \eta}\right)^2) \) since \( r \leq \Delta \eta \), because of the \( \Theta \)-function. If we now recognize \( x = \frac{r}{\Delta \eta} \) and \( z = k \Delta \eta \), we can write

\[
0 = i \dot{\chi}(x) + \frac{|f|^2}{2\pi^2} \left( \frac{e^{i \vec{k} \cdot \vec{x}}}{k^3} \right) \int_0^\eta \frac{d\eta'}{k^3} \left\{ \left[ \ln(\mu^2 \Delta \eta^2) - 1 \right] \left[ \sin(k \Delta \eta) - k \Delta \eta \cos(k \Delta \eta) \right] + (k \Delta \eta)^2 \zeta(k \Delta \eta) \right\} \chi(\eta')
\]

\[
+ \frac{|f|^2}{2\pi^2} \left[ \ln(a) i \dot{\chi}(x) + i \dot{\chi}(\ln(a) \chi(x)) \right]
\]

\[
+ \frac{|f|^2}{2\pi^2} \left[ 1 - \frac{1}{2} \right] \left[ H^2 \right] \left( \frac{v^2 - 1}{4} \right)
\]

\[
\times \left\{ i \dot{\theta} \frac{e^{i \vec{k} \cdot \vec{x}}}{k^3} \int_0^\eta \frac{d\eta'}{k^3} \left[ \ln[\alpha \Delta \eta^2] \left[ \sin(k \Delta \eta) - k \Delta \eta \cos(k \Delta \eta) \right] + (k \Delta \eta)^2 \zeta(k \Delta \eta) \right] \chi(\eta')
\]

\[
+ \ln(a) i \dot{\theta} \frac{e^{i \vec{k} \cdot \vec{x}}}{k^3} \int_0^\eta \frac{d\eta'}{k^3} \left[ \sin(k \Delta \eta) - k \Delta \eta \cos(k \Delta \eta) \right] \chi(\eta')
\]

\[
\times \frac{i}{k^3} \left( \ln(a) \right) \left[ \sin(k \Delta \eta) - k \Delta \eta \cos(k \Delta \eta) \right] \chi(\eta') \right\}.
\]

The integral in Eq. (6.94) can be performed and gives

\[
z^2 \zeta(z) = 2 \sin(z) - [\cos(z) + z \sin(z)] [\text{Si}(2z)] + [\sin(z) - z \cos(z)] \left[ \text{Ci}(2z) - \gamma_E - \ln \left( \frac{z}{2} \right) \right]
\]

where

\[
\text{Si}(z) = \int_0^z \frac{\sin(t)}{t} dt = -\int_z^\infty \frac{\sin(t)}{t} dt + \frac{\pi}{2}
\]

\[
\text{Ci}(z) = -\int_z^\infty \frac{\cos(t)}{t} dt = \int_0^z \frac{\cos(t) - 1}{t} dt + \gamma_E + \ln(z).
\]

We now want to act with the derivative operators on the integrals. Since the derivatives are with respect to \( \eta \) which appears in both the integral and the integrands, we have to act with the derivative on both. Since at the upper limit where \( \eta' = \eta \) the integrands vanish at least as \( (\Delta \eta)^3 \ln(\Delta \eta) \), the derivative of the integral gives zero and we are allowed to take the derivatives of the integrand. Another way to see this is to write the integral again with the Heaviside step function \( \theta \), i.e. \( \int_0^\eta \frac{d\eta'}{k^3} \left[ \ln\theta(\Delta \eta) \right] d\eta' = \int \frac{d\theta}{k^3} \left[ \ln\theta(\Delta \eta) \right] \), and use that \( \partial \theta(\Delta \eta) / \partial \eta = 0 \). We also note that the derivative operator \( \partial^2 = -\partial_0^2 + \partial_1^2 \) can be written as \( -\partial_0^2 + k^2 \) because the only spatial dependence is contained in the plane wave exponential. We use the following relations

\[
(\partial_0^2 + k^2)[\sin(z) - z \cos(z)] = 2k^2 \sin(z)
\]

\[
(\partial_0^2 + k^2)[\cos(z) - z \sin(z)] = -2k^2 \cos(z)
\]

\[
\frac{d}{dz} \text{Si}(2z) = \frac{\sin(2z)}{2z}
\]

\[
\frac{d}{dz} \text{Ci}(2z) = \frac{\cos(2z)}{2z}
\]

To find the other relations

\[
(\partial_0^2 + k^2) \left[ \ln(\alpha z^2) (\sin(z) - z \cos(z)) \right] = 2k^2 \left[ (\ln(\alpha z^2) + 1) \sin(z) + \frac{d}{dz} \frac{\sin(z) - z \cos(z)}{z} \right]
\]

\[
(\partial_0^2 + k^2) z^2 \zeta(z) = \left[ -\cos(z) [\text{Si}(2z)] + \sin(z) [\text{Ci}(2z) - \gamma_E - \ln \left( \frac{z}{2} \right) - \frac{d}{dz} \frac{\sin(z) - z \cos(z)}{z} \right].
\]
where in all the equations we can take the derivatives, and we find

\begin{align}
0 &= i\partial \chi(x) + \frac{|f|^2}{2^5\pi^2} i\partial (\partial_0^2 + k^2) e^{-\bar{\kappa} \cdot x} \int_{\eta'}^\eta d\eta' \{ \ln(\mu^2 \Delta \eta^2) \sin(k \Delta \eta) - \cos(k \Delta \eta) \sin(k \Delta \eta) \} \\
&\quad + \sin(k \Delta \eta) \left[ \text{Ci}(2k \Delta \eta) - \gamma_E - \ln\left(\frac{k \Delta \eta}{2}\right) \right] \chi(\eta') \\
&\quad + \frac{|f|^2}{2^5\pi^2} \left[ \ln(a) i\partial \chi(x) + i\partial \ln(a) \chi(x) \right] \\
&\quad - |f|^2 \left( 1 - e^2 \right) \frac{HH' a}{2^4\pi^2} \left( \nu^2 - \frac{1}{4} \right) \\
&\quad \times \left\{ i\partial e^{i\bar{\kappa} \cdot x} \int_{\eta'}^\eta d\eta' a' \left\{ \ln(a \Delta \eta^2) + 1 \right\} \sin(k \Delta \eta) - \cos(k \Delta \eta) \sin(k \Delta \eta) \right\} \\
&\quad + \sin(k \Delta \eta) \left[ \text{Ci}(2k \Delta \eta) - \gamma_E - \ln\left(\frac{k \Delta \eta}{2}\right) \right] \chi(\eta') \\
&\quad + \ln(a) i\partial e^{i\bar{\kappa} \cdot x} \int_{\eta'}^\eta d\eta' a' \sin(k \Delta \eta) \chi(\eta') + i\partial e^{i\bar{\kappa} \cdot x} \int_{\eta'}^\eta d\eta' a' \ln(a') \sin(k \Delta \eta) \chi(\eta') \right\} \tag{6.96}
\end{align}

We now make a distinction between the terms. During inflation the scale factor \( a \) grows exponentially, and therefore the second term (the conformal anomaly) grows as \( \ln(a) \) and the third term as \( aa' \) and \( aa' \ln(aa') \). The first term is the conformal vacuum contribution and does not depend on the scale factor. Thus during inflation, the first term does not grow and is not relevant. Note that there are also higher order terms but these scale as \( s \ll 1 \) and are therefore suppressed (see the discussion below Eq. (5.104). To summarize, we keep only the conformal anomaly term and the third term. Now we write

\begin{equation}
(i\gamma^0 \partial_0 - i\gamma^\nu \partial_\nu) e^{i\bar{\kappa} \cdot x} \chi(k, \eta) = (i\gamma^0 \partial_0 - \tilde{\gamma} \cdot \bar{k}) e^{i\bar{\kappa} \cdot x} \chi(k, \eta),
\end{equation}

such that the modified Dirac equation simplifies to

\begin{align}
0 &= (i\gamma^0 \partial_0 - \tilde{\gamma} \cdot \bar{k}) \chi(k, \eta) \\
&\quad + |f|^2 \left( 1 - e^2 \right) \frac{HH' a}{2^4\pi^2} \left( \nu^2 - \frac{1}{4} \right) \\
&\quad \times \left\{ (i\gamma^0 \partial_0 - \tilde{\gamma} \cdot \bar{k}) \frac{1}{k} \int_{\eta'}^\eta d\eta' a' \left\{ \ln(a \Delta \eta^2) + 1 \right\} \sin(k \Delta \eta) - \cos(k \Delta \eta) \sin(k \Delta \eta) \right\} \\
&\quad + \sin(k \Delta \eta) \left[ \text{Ci}(2k \Delta \eta) - \gamma_E - \ln\left(\frac{k \Delta \eta}{2}\right) \right] \chi(\eta') \\
&\quad + \ln(a) (i\gamma^0 \partial_0 - \tilde{\gamma} \cdot \bar{k}) \frac{1}{k} \int_{\eta'}^\eta d\eta' a' \sin(k \Delta \eta) \chi(\eta') \\
&\quad + (i\gamma^0 \partial_0 - \tilde{\gamma} \cdot \bar{k}) \frac{1}{k} \int_{\eta'}^\eta d\eta' a' \ln(a') \sin(k \Delta \eta) \chi(\eta') \right\}. \tag{6.97}
\end{align}

We can simplify the above equation by noticing that

\begin{equation}
-(i\gamma^0 - \tilde{\gamma} \cdot \bar{k}) \frac{1}{k} \Theta(\Delta \eta) \sin(k \Delta \eta)
\end{equation}
is the retarded Green function of the operator \((i\gamma^0 - \vec{\gamma} \cdot \vec{k})\). Explicitly

\[
-(i\gamma^0 - \vec{\gamma} \cdot \vec{k})(i\gamma^0 - \vec{\gamma} \cdot \vec{k})^{12} \Theta(\Delta\eta)\sin(k\Delta\eta) = (\partial_0^2 + k^2)^{1} \Theta(\Delta\eta)\sin(k\Delta\eta)
\]

\[
= 2\cos(k\Delta\eta)\delta(\Delta\eta) + \frac{\sin(k\Delta\eta)}{k}\delta(\Delta\eta)\]

\[
= \cos(k\Delta\eta)\delta(\Delta\eta)
\]

where in the first line I have used that \(\delta^2 = \partial^2 = -(\partial_0^2 + k^2)\) and in the third line I have partially integrated the expression (note that the Green function always acts under an integral). We now guess the operator that makes the Dirac equation local, which is

\[
-a(i\gamma^0 \partial_0 - \vec{\gamma} \cdot \vec{k})^{12} ,
\]

(6.98)

and act with this operator on the Dirac equation (6.99). The result is

\[
0 = (a\partial_0 \frac{1}{a} \partial_0 + k^2 + ia\left(\gamma^0 \partial_0 \frac{1}{a} \vec{y} \cdot \vec{k}\right)\chi(k,\eta)
\]

\[
+ \frac{|f|^2}{2^5 \pi^2} \left\{ a\partial_0 \ln(a) \frac{1}{a} \partial_0 + \ln(a)k^2 + ia\left(\gamma^0 \partial_0 \ln(a) \frac{1}{a} \vec{y} \cdot \vec{k}\right)\right\} \chi(k,\eta)
\]

\[
+ \frac{1}{a} \partial_0 \int^n \eta a' \ln(H^2(1-e^2)\Delta\eta^2)\chi(\eta') + \frac{2}{a} \int^n \eta a' \cos(\Delta\eta) - \frac{1}{\Delta\eta} \chi(\eta')
\]

\[
+ \frac{1}{a} (\partial_0 \ln(a)) \gamma^0 \frac{\vec{y} \cdot \vec{k}}{k} \int^n \eta a' \sin(\Delta\eta)\chi(\eta')\right\}.
\]

(6.99)

We can combine the first three lines, containing the tree level operator and the conformal anomaly, which gives

\[
\left[ 1 + \frac{|f|^2}{16\pi^2} \ln(a) \right](\partial_0^2 + k^2)
\]

\[
-aH \left[ 1 + \frac{|f|^2}{16\pi^2} \ln(a) - \frac{3}{2} \right] \partial_0 \chi(k,\eta)
\]

\[
-iaH \gamma^0 \vec{y} \cdot \vec{k} \left[ 1 + \frac{|f|^2}{16\pi^2} \ln(a) - \frac{3}{2} \right] \chi(k,\eta),
\]

(6.100)

where I have used that \(H = \frac{\Delta}{\sigma} = \partial_0 \frac{1}{a^2}\). We see that we can neglect the conformal anomaly when \(\ln(a) \ll 16\pi^2/|f|^2\). Since the Yukawa couplings \(f\) are small and \(\ln(a)\) (the number of e-folds) grows to approximately 70 during inflation, we can neglect the conformal anomaly term.

Moreover, we can extract the leading order term from Eq. (6.99), which is the term of \(\mathcal{O}(s^{-1})\). Using \(\nu^2 - \frac{1}{4} = 2 + \mathcal{O}(s)\) and the expansion of \(\psi(\frac{3}{2} - \nu)\) in Eq. (5.107), we find

\[
(a\partial_0 \frac{1}{a} \partial_0 + k^2 + ia\left(\gamma^0 \partial_0 \frac{1}{a} \vec{y} \cdot \vec{k}\right)\chi(k,\eta) + a^2|f|^2 \frac{(1-e^2)H^2}{16\pi^2} \left| \frac{H^2}{\Xi R + M^2 + 3H^2\nu} \right| \chi(k,\eta) = 0.
\]

(6.101)
We can now compare this equation to the Dirac equation for a massive conformally rescaled fermion (see section 6.2.1),

$$(i\partial + am_\psi)\chi(x) = 0,$$

where $m_\psi$ is the mass of the fermions. If we now perform a Fourier transform of the conformally rescaled field $\chi(x)$ and act with the operator $-a(i\partial - am_\psi)\frac{1}{a}$ on this equation, we find

$$0 = -a(i\partial - am_\psi)\frac{1}{a}(i\partial + am_\psi)\chi(x)$$

$$= (a\partial_0 - \partial_0 + k^2 + ia\left(\gamma^0\partial_0 - \frac{1}{a}\right)\vec{\gamma}\cdot\vec{k})\chi(x) + a^2m_\psi^2\chi(x). \quad (6.102)$$

If we compare Eqs. (6.101) and (6.102) we see that the quantum one-loop effect of the massless fermion with the nonminimally coupled scalar inflaton in the loop generates an effective mass of the massless fermions of

$$m_\psi^2 = |f|^2\frac{(1-c)^2H^2}{16\pi^2}\left[\frac{H^2}{\Xi R + M^2 + 3H^2c}\right]. \quad (6.103)$$

Since the term $\Xi R + M^2$ is proportional to $H^2$ ($R = 6H^2(2-c)$ and $M^2$ contains the classical inflaton field that is proportional to $H$ during inflation), we find that the mass $m_\psi \propto H$. The Hubble parameter decreases linearly in time during inflation, see section 4.4.2, and so does the fermion mass.

The same analysis has been done for the photon interacting with the inflaton, and it is shown in [40] [41] [42] [43] that a mass is generated for the photon as well. Thus it seems that a dynamical mass generation is a generic feature of (massless) particles interacting with the inflaton.

We could now proceed as in section 6.2.2 and [38] and solve the modified Dirac equation (6.101) for the field $\chi(k, \eta)$. This is also done in [39] and it is found that there is no charge generation during inflation, but there is particle number generation, along the lines of [44]. It is found that towards the end of inflation, a particle distribution is generated which is similar to a Fermi-Dirac distribution with temperature $T = H/2\pi$ and energy $E = m_\psi$. For light particles $m_\psi \ll H$ the particle density is $\frac{1}{2}$, whereas for heavy particles $m_\psi \gg H$ the particle number is exponentially suppressed. This agrees with the adiabatic analysis.

To summarize this section, we found that during inflation the interaction of a massless fermion with the scalar inflaton field, i.e. the Higgs boson, generates an effective mass for the fermions of (6.103). This is the main result of this section. We stress that we have only calculated this effect for massless fermions, which in the Standard Model are effectively only the neutrinos. If we want to describe the dynamical effects on massive fermions (quarks, electrons), we would have to calculate the one-loop self-energy for massive fermions. This means we have to use the complicated massive fermion propagator as calculated in [38]. This remains to be done in future work.

### 6.4 Summary

In this chapter we considered the fermion sector of the Standard Model action. The masses of the fermions are generated through the scalar-fermion interaction terms, the so-called Yukawa interactions. If the scalar field, the Standard Model Higgs boson, is the inflaton as well, interesting effects can occur. In section 6.1 we first constructed the fermion sector of the action in Minkowski space. We saw that in case of the two-Higgs doublet model one
of the two Higgs bosons can be coupled exclusively to up-type quarks, and the other only to down-type quarks. Furthermore we have seen that we can decompose the fermions into left- and righthanded fermions. The mass term mixes these two chiralities.

In section 6.2 we constructed the fermion action in a conformal FLRW universe. We have seen that by performing a conformal rescaling of the fermion fields we could write the action as almost the same action in Minkowski space. The only difference is that the mass term is multiplied by the scale factor. For massless fermions the propagator was therefore easily calculated in section 6.2.1 and was simply a conformal rescaling of the flat space result. On the other hand, the massive fermion propagator requires much more work and was calculated in [38]. In that case the mass of the fermions was real because the Higgs field was real. This allowed the fields to be solved and the propagator to be constructed.

In this thesis however we considered the two-Higgs doublet model where the scalar fields are complex. The masses of the fermions generated by these fields are therefore also complex, and this complicates the solution of the Dirac equation. In section 6.2.2 we tried to solve the chiral Dirac equations in two different ways. The first was to simply try to decouple the equations and write it in a simpler form. In a recent and very brief study we found that we might be able to write the solutions in terms of the same solutions as for the fermions with a real mass, and would thus give the same massive propagator. Another approach is to remove the complex phase from the scalar fields by a redefinition of the fermion fields. This generates an axial vector current in the fermion action that violates $C$ and $\mathbf{CP}$ and could be a source for baryogenesis. The change in the Dirac equations was a shift in momentum and this momentum shift alters the calculation of the propagator. The study on the Dirac equation with a complex mass is very recent and remains to be thoroughly investigated in future work.

In section 6.3 we combined all our knowledge from the previous chapters. We calculated the one-loop fermion self-energy by using the scalar propagator from chapter 5 and the massless fermion propagator from section 6.2.1. This calculation unfortunately only describes the (effectively massless) neutrinos, but might give us an idea for quarks. After quite a tedious procedure where we renormalized the self-energy in section 6.3.1, calculated the effect of the self-energy on the dynamics of fermions in section 6.3.2 and extracted the leading order behavior, we find the main result of this chapter in Eq. (6.103). The one-loop effect generates a mass for the fermions that is proportional to the Hubble parameter $H$.

We stress that this effective mass is dynamically generated for massless fermions. If we want to one-loop effect on massive fermions, we would have to use the massive fermion propagator from [38] to calculate the one-loop self-energy. We hope to do this calculation in future work.
Chapter 7

Summary and outlook

Cosmology is a very active field of research that tries to describe the dynamics and evolution of the universe. The main principles of cosmology are the homogeneity and isotropy of the universe on the largest scales, which allow us to describe the entire universe as a perfect fluid with an energy density \( \rho \) and pressure \( p \). Since effectively the only interaction on the largest scales is the gravitational interaction, we can describe our universe with the Einstein equations that follow from Einstein's theory of general relativity. The foundation of general relativity is the metric \( g_{\mu\nu} \), and our expanding universe is on the largest scales most successfully described by the flat Friedman-Lemaître-Robertson-Walker metric. This FLRW metric is similar to the Minkowski metric, but has an additional time dependent scale factor \( a \) that multiplies the spatial part of the metric. The scale factor basically scales up space, and determines the size of our universe. Two important quantities in this respect are the Hubble parameter \( H \) that contains the derivative of the scale factor and is therefore also called the expansion rate. The other is the deceleration parameter \( \epsilon \) that contains the derivative of the Hubble parameter and therefore tells us if the expansion of space is accelerating or decelerating.

In chapter 2 we discussed some basic cosmology. We introduced the general Einstein equations and derived the specific Einstein equations for the flat FLRW metric. These equations contain the Hubble parameter \( H \) and deceleration parameter \( \epsilon \). We have seen that we could solve these equations for a perfect fluid, and we found that \( \epsilon \) is a constant. In section 2.3 we showed that we can derive the Einstein equations from an action principle. The gravitational interaction can be derived from the Einstein-Hilbert action, whereas the stress-energy tensor that contains the sources for the gravitational fields is derived from the matter Lagrangian.

In this thesis we considered the evolution of the very early universe. After the Planck era, where general relativity and quantum mechanics become equally important and a theory of quantumgravity is essential, the universe is suspected to have undergone a stage of inflation. During this inflationary era the expansion of space accelerates and the universe grows to a size many orders of magnitude larger than our visible universe. In chapter 3 we gave a short introduction to cosmological inflation. We have seen that inflation can solve many of the cosmological puzzles, such as the homogeneity, horizon, flatness and cosmic relics puzzle. During inflation the universe accelerates, and we showed that we need some special field with negative pressure to make this happen. An essential ingredient in inflationary models is a scalar field that can undergo the Higgs mechanism. The scalar field is trapped in a false vacuum, but wants to move to the true vacuum below some temperature and this is what causes the negative pressure. When the scalar field finally moves to the true vacuum, the latent heat is released and inflation takes place. The scalar field that
drives inflation is called the inflaton.

The first inflationary models suffered from some problems, such as the graceful exit and fine-tuning problems. The most accepted scenario is the chaotic inflationary model, where a scalar field initially has a very large value and then slowly rolls down the potential well to its minimum. We can define the slow-roll conditions that must generically be satisfied in order for successful inflation to take place. These slow-roll conditions are determined by the potential of the scalar field. So far there have not been any direct measurements that confirm whether or not inflation took place in the early universe, and we therefore also do not know the form of the scalar potential. This is also a hard nut to crack, since inflation concerns the expansion of space and the only thing we can observe are photons and particles, which have in principle not much to do with inflation. However, cosmological inflation does make some predictions about our universe. One of the most important is that inflation predicts a nearly scale invariant spectrum of density perturbations. This has actually been measured from the CMBR and many cosmologists consider this to be the confirmation of the inflationary hypothesis. The deviations from a scale invariant spectrum can actually be expressed in terms of the slow-roll parameters, and since these parameters are derived from the inflationary potential, measurements of the spectrum constrain the form of the inflaton potential. Inflation also predicts primordial gravitational waves that leave their imprint on the CMBR, and cosmologists hope to see this very soon with the new Planck satellite that was launched in May 2009. The question remains what the correct inflationary model really is. Chaotic inflation is possible in most scalar field models, and this is on the one hand nice because the most general models are allowed, but on the other hand it leaves room for many exotic models.

In this thesis we considered a specific type of inflationary models, the nonminimal inflationary models. In these models the scalar inflaton field is coupled to gravity, which in the action formalism corresponds to the Ricci scalar $R$ in the Einstein-Hilbert action. The term nonminimal coupling term $\xi R \phi^2$ effectively changes the gravitational constant $G_N$ such that it can become much smaller. On the other hand it acts as an additional mass term in the potential. We considered specifically nonminimal inflationary models with strong negative coupling, $\xi \ll -1$. In section 4.1 we considered the simplest model of a real scalar field $\phi$ with a quartic potential with self-coupling $\lambda$ that is nonminimally coupled to $R$. We derived the dynamical equations for $H$ and for the field $\phi$ and solved these analytically in the slow-roll approximation and in the strong negative coupling limit. We saw that the field rolls down slowly to its minimum and that the large negative $\xi$ reduces the initial value of $\phi$ necessary for successful inflation, i.e. to solve the cosmological puzzles. The Hubble parameter $H$ has the special property that during inflation it is proportional to the field $\phi$. Numerically we have solved the complete field equations and found that our analytical results are verified to great accuracy.

Besides providing a successful inflationary model, the nonminimally coupled inflaton field has the additional benefit that it lowers the constraint on the quartic self-coupling $\lambda$. As mentioned, the measurements of the spectrum of density perturbations put constraints on the slow-roll parameters, and since these are derived from the inflaton potential they therefore constrain the form of the potential. For minimally coupled models, the spectrum of density perturbation is proportional to $\sqrt{\lambda}$ and the value of the self-coupling must therefore be $\lambda \approx 10^{-12}$. However, if we include a nonminimal coupling term, we find that the spectrum is proportional to $\sqrt{\lambda \xi^2}$. In order to derive this, we first needed to perform a conformal transformation of the metric that removes the nonminimal coupling term. We introduced the general conformal transformation in section 4.2 and performed the specific conformal transformation that removes the $\xi R \phi^2$ term in section ??.
resulting action as the action of a minimally coupled scalar field with a modified potential. This new action after the conformal transformation is called the Einstein frame. In the Einstein frame we could easily derive the slow-roll parameters from the potential and found the proportionality of the spectrum of density perturbations to $\sqrt{\lambda/\xi^2}$. A strong negative $\xi$ therefore reduces the constraint on $\lambda$, and in fact for large enough $\xi$ we found that $\lambda$ can have a reasonable value of $\sim 10^{-3}$.

The fact that $\lambda$ does not have to be extraordinary small allows the Higgs boson to be the inflaton. The great advantage is that we do not need to introduce any additional exotic scalar particles that drive inflation, but can use the Higgs boson that is already an essential part of the Standard Model of particles. The only thing we have to add is an extra term in the action with the nonminimal coupling of the Higgs boson to $R$. We showed that in the Einstein frame the potential reduces to the ordinary Higgs potential for small values of the Higgs inflaton field $\phi$. However, for large field values the potential becomes asymptotically flat and this makes sure that inflation is successful.

In section 4.4 we introduced the two-Higgs doublet model. This is the simplest supersymmetric extension of the Standard Model and features a second Higgs doublet. In the unitary gauge there is in addition to a second real scalar field also a phase between the two fields. This phase can lead to $\text{CP}$ violation through an axial vector current that could then be converted in a baryonic current that is responsible for baryogenesis. We derived again the field equations and solved these numerically for real Higgs fields. We found that one linear combination of the two real scalar fields serves as the inflaton, whereas the other is an oscillating mode that quickly decays. Furthermore because there are more nonminimal couplings and quartic self-coupling the initial value of the fields can be lowered even further.

In chapter 5 we quantized our nonminimally coupled inflaton field. It is well known that we cannot simply combine quantum field theory and general relativity in a single theory of quantum gravity because the theory is not renormalizable. However, for energies well below the Planck scale of $M_P = 2.4 \times 10^{18}$ GeV we expect quantum gravitational effects not to play a role and we have no problem to do quantum field theory in a curved background. First we gave a useful derivation of the quantization of a scalar field in Minkowski space in section 5.1. Although we showed that there is are infinite ways to quantize our field and therefore an infinite amount of choices for the vacuum, we found that there is only one unique vacuum that is also the zero particle and lowest energy state. The reason is that the Hamiltonian is constant in time. In an expanding universe the Hamiltonian is not constant because of the time dependent scale factor in the metric and therefore the choice of vacuum is not unique. By considering a simple example of a harmonic oscillator with a varying frequency with asymptotically constant regions we showed that the zero particle vacuum at one point in time is not the zero particle vacuum at a later time. Physically, particles are created by the expansion of the universe.

Since it is very difficult, if not impossible to do quantum field theory if the vacuum and therefore also the notion of a particle is not well-defined, we needed to find a natural choice for the vacuum. First we quantized the nonminimally coupled inflaton field in section 5.2 by considering the inflaton field as the classical inflaton field that drives inflation plus a small quantum fluctuation. In an expanding universe described by the FLRW metric we saw that we could write the field equation for the fluctuations as a harmonic oscillator with a time dependent frequency. In the special case of quasi de Sitter space however there are regions where the frequency of the harmonic oscillator is approximately constant and changes slowly. In these adiabatic regions we can define a natural vacuum state known as the Bunch-Davies vacuum that corresponds to zero particles and energy in the infinite
asymptotic past.

In the Bunch-Davies vacuum we could actually solve the field equations exactly and make a natural choice to quantize the nonminimally coupled inflaton field. This allowed us to calculate the scalar propagator in $D$-dimensions. The propagator was seemingly divergent in $D = 4$, but by making an expansion around $D = 4$ we found that the divergency disappears. In section 5.3 we extended our findings and quantized the two-Higgs doublet model. The complexity of the fields lead to slightly different quantization of the fields, but in the end we found that we could write the field equations for the mode functions in precisely the same form as for mode functions of the single real scalar field. The difference is that the field equation was now a matrix equation with matrix valued operators. In the end we found precisely the same form of the propagator for the two-Higgs doublet model as for the usual Higgs model, with the difference that the propagator is $2 \times 2$ matrix.

In chapter 6 we focused on a different part of the Standard Model action. Instead of the Higgs sector containing only scalar fields, we now considered the fermion sector with the Standard Model fermions and the coupling of the Higgs bosons to the fermions in the Yukawa sector. We first constructed the fermion sector in a flat Minkowski space in section 6.1 and then extended this to an expanding FLRW universe in section 6.2. We found that we could again write our action as the Minkowski action by performing a conformal rescaling of the fields, with the only difference that the mass term is now multiplied by the scale factor. The propagator for massless fermions is then simply a conformal rescaling of the propagator in Minkowski space. The massive propagator requires a lot more work but can be constructed by explicitly solving the Dirac equation for the fermion fields with a real mass.

The mass of the fermions is in fact generated through the coupling of the Higgs boson to the fermions. For the single Higgs model, the scalar Higgs boson is a real field and gives therefore a real mass to the fermions. In case of the two-Higgs doublet model the scalar fields are complex and so is the mass of the fermions. This makes it more difficult to solve the Dirac equations. Recently we did a study on solving the Dirac equations for fermions with a complex mass. First we tried to solve these equations by decoupling the fermion fields in left- and righthanded fermions and rewriting the equations. A first study suggests that we might be able to write the solutions in terms of the solutions for the real mass. Another approach is to remove the phase $\theta$ from the complex scalar fields by a field redefinition of the fermion fields. If the phase is time dependent, we find that we can make the mass term in the fermion sector real, at the expense of creating an extra term in the action that is proportional to the axial vector current. This axial vector current violates $\text{CP}$ and could be converted through sphaleron processes into a baryon asymmetry. If the phase derivative is constant, $\dot{\theta} = \text{constant}$, we find that the only change in the fermion field solutions is a shift in the momentum. In calculating the massive propagator this leads to additional parts that might be $\text{CP}$ violating.

In section 6.3 we combined our knowledge from all the previous chapters and calculate the one-loop self-energy for the fermions. We use the scalar propagator for the two-Higgs doublet model in quasi de Sitter space and the massless fermion propagator. The massless fermion propagator strictly speaking only describes neutrinos and is therefore an incorrect way to calculate the effect of the scalar inflaton field on the dynamics of fermions during inflation. However, we expect this to give an indication of the one-loop self-energy for the massive fermion propagator. After a long calculation where we calculate and renormalize the self-energy we study its effect on the dynamics of massless fermions during inflation. The final result is that the one-loop self-energy effectively generates a mass for the massless fermions that is proportional to the Hubble parameter $H$. 
In future work we hope to study some of the issues that we have not been able to investigate in this thesis. First of all we have numerically solved the field equations for real fields $\phi_1$ and $\phi_2$ in the two-Higgs doublet model. However, there is a third degree of freedom, which is the phase between the two fields. We would like to introduce also this phase in the field equations and see if the phase is changing in time. The time dependent phase leads to CP violation.

Moreover we showed that in the nonminimally coupled two-Higgs doublet model there is one linear combination of the two Higgses that is the actual inflaton, the other linear combination is an oscillating mode that quickly decays. In this thesis the focus was mostly on the inflaton mode. In future work we hope to study the dynamics and effects of the oscillating field. This field could decay into lighter particles and could lead to reheating of the universe. If the decay happens in a CP violating way, this could lead to baryogenesis.

We have seen that CP violation is also present in the fermion sector of the Standard Model. We want to continue the study on solving the Dirac equations for fermions with a complex mass. A thorough investigation is needed to see if we can really write the solutions as the same solutions for fermion fields with a real mass. Furthermore we want to do a proper study on baryogenesis in the fermion sector of the two-Higgs doublet model. As mentioned, the time dependent phase in this model generates an axial vector current in the fermion action. This axial vector can then be converted by sphalerons into a baryonic current that could generate the baryon asymmetry of the universe. We hope to more research on this in future work.

Finally we want to calculate the one-loop fermion self-energy for massive fermions. The massive fermion propagator was calculated recently in [38] and we can use the result to calculate the effect of the inflaton field on the dynamics of quarks in quasi de Sitter space. We hope that this future work will give us some interesting consequences for fermion dynamics in the early universe.
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Appendix A

Conventions

• Throughout this thesis we will use that \( h = c = 1 \).
• The signature of the metric is \( \text{sign}[g_{\mu\nu}] = (-, +, +, +) \).
• Greek indices \( \alpha, \beta, \gamma... \) are space time indices that run from 0 to \( D - 1 \). The 0 corresponds to the time index and 1, 2, ..., \( D - 1 \) are the space indices.
• Roman indices \( a, b, c \) run from 1 to \( D - 1 \) and correspond to the space indices alone.
• Greek indices are raised and lowered by the metric tensor \( g_{\mu\nu} \).
• We will frequently use the Einstein convention, which means that when there is a contraction of two indices, we have to sum over these indices. For example, \( a_\mu b^\mu = \sum_{\mu=0}^{D-1} a_\mu b^\mu \).
• An overdot denotes a derivative with respect to \( t \), i.e. \( \dot{a} = \frac{da}{dt} \), whereas a prime denotes a derivative with respect to conformal time \( \eta \), \( a' = \frac{da}{d\eta} \).
Appendix B

General Relativity in an expanding universe

B.1 General Relativity

In general relativity the infinitesimal line element is described by

\[ ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \]  

(B.1)

where \( g_{\mu\nu}(x) \) is the metric tensor. The covariant derivative acting on a vector field \( V^\mu \) is defined as

\[ \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \]  

(B.2)

where \( \Gamma^\nu_{\mu\lambda} \) is chosen such that the combination \( \nabla_\mu V^\nu \) transforms as a tensor. Now, by demanding that the connection is torsion free (symmetric in lower indices) and metric compatible (\( \nabla_\rho g_{\mu\nu} = 0 \)), we find the unique connection known as the Christoffel connection

\[ \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \]  

(B.3)

On a flat Minkowski background, the metric is given by \( g_{\mu\nu}(x) = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) and we can immediately see that the connection vanishes. Therefore, the covariant derivative reduces to the ordinary derivative on flat spaces. The curvature of space can be found by parallel transporting a vector around an infinitesimal closed loop. The commutator between two covariant derivatives acting on a vector \( V^\mu \) measures the differences between parallel transporting a vector one way around the loop and the other way. Thus

\[ [\nabla_\sigma, \nabla_\beta] V^\rho = R^\rho_{\alpha\beta\sigma} V^\alpha, \]  

(B.4)

where \( R^\rho_{\alpha\beta\sigma} \) is the Riemann curvature tensor

\[ R^\rho_{\alpha\beta\sigma} = \partial_\sigma \Gamma^\rho_{\alpha\beta} - \partial_\beta \Gamma^\rho_{\alpha\sigma} + \Gamma^\rho_{\alpha\lambda} \Gamma^\lambda_{\beta\sigma} - \Gamma^\rho_{\beta\lambda} \Gamma^\lambda_{\alpha\sigma}. \]  

(B.5)

Of course, in a flat Minkowski spacetime the curvature tensor vanishes. A contraction of two indices gives us the so called Ricci tensor \( R_{a\beta} \).

\[ R_{a\beta} = R^\rho_{a\rho\beta} = R^\mu_{a\mu\beta} = \partial_\rho \Gamma^\rho_{a\beta} - \partial_\beta \Gamma^\rho_{a\rho} + \Gamma^\rho_{a\lambda} \Gamma^\lambda_{a\beta} - \Gamma^\rho_{a\lambda} \Gamma^\lambda_{a\beta}. \]  

(B.6)

Another contraction gives us the Ricci scalar \( R \),

\[ R = R^a_{a} = g^{a\beta} R_{a\beta}. \]  

(B.7)
Finally, we use the Bianchi identity
\[ \nabla_\lambda R_{\rho\alpha\beta} + \nabla_\rho R_{\alpha\lambda\beta} + \nabla_\alpha R_{\lambda\rho\beta} = 0, \]  
(B.8)
and contract this whole expression twice (i.e. we multiply with \( g^{\beta\alpha} g^{\sigma\lambda} \)) to find that
\[ \nabla^\sigma (R_{\sigma\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}) = 0, \]  
(B.9)
which allows us to define the Einstein tensor
\[ G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}. \]  
(B.10)

### B.2 General Relativity in a flat FLRW universe

In cosmology the universe is best described by an expanding Minkowski spacetime. The metric that corresponds to such an expanding universe is the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric,
\[ g_{\mu\nu}(x) = \text{diag}(-1, a^2, a^2, a^2), \quad a = a(t), \]  
(B.11)
where \( a(t) \) is the scale factor. The line element then takes the form
\[ ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = -dt^2 + a^2(t)d\vec{x} \cdot d\vec{x}. \]  
(B.12)
We see that the spatial part of the line element scales with the scale factor \( a(t) \). This means that if \( a(t) \) grows in time, the universe expands. Using the FLRW metric we find that the nonvanishing elements of the Christoffel connection defined in Eq. (B.3) are
\[ \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j = H\delta^i_j \]  
(B.13)
\[ \Gamma^0_{ij} = \frac{\dot{a}}{a}g_{ij} = Ha^2\delta_{ij}, \]  
(B.14)
where we have defined the Hubble parameter
\[ H \equiv \frac{\dot{a}}{a}. \]  
(B.15)
We can now also calculate the nonvanishing components of the Ricci tensor from Eq. (B.6), and in 4 dimensions they are
\[ R_{00} = -3\frac{\ddot{a}}{a} = -3(H^2 + \dot{H}) \]  
(B.16)
The Ricci scalar \( R \) in Eq. (B.7) is then
\[ R = g^{\mu\nu}R_{\mu\nu} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = 6(H + 2H^2). \]  
(B.17)
The components of the Einstein curvature tensor are finally
\[ G_{00} = 3\left(\frac{\ddot{a}}{a}\right)^2 = 3H^2 \]  
\[ G_{ij} = -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]g_{ij} = -(2H + 3H^2)a^2. \]  
(B.18)
General Relativity in a conformal FLRW universe

All the above equations are valid in a regular FLRW spacetime. However, we can simplify and generalize many of our equations by performing the conformal transformation

\[ dt = a d\eta. \]  

(B.19)

The metric now takes the simple form

\[ g_{\mu\nu} = a^2(\eta) \text{diag}(-1, 1, 1, 1) = a^2(\eta) \eta_{\mu\nu}, \]  

(B.20)

where \( \eta_{\mu\nu} \) is the Minkowski metric and \( \eta \) is conformal time. The Christoffel connection takes the simple form

\[ \Gamma^a_{\mu\nu} = \frac{a'}{a} \left( \delta^a_\mu \delta^0_\nu + \delta^a_\nu \delta^0_\mu - \delta^a_0 \eta_{\mu\nu} \right), \]  

(B.21)

where \( a' = \frac{da}{d\eta}, \) or explicitly,

\[ \Gamma^0_{00} = \frac{a'}{a}, \quad \Gamma^i_{j0} = \frac{a'}{a} \delta^i_j, \quad \Gamma^0_{ij} = -\frac{a'}{a} \eta_{ij} = \frac{a'}{a} \delta_{ij}. \]

For convenience, we now list certain relations

\[ \dot{a} = \frac{da}{dt} = \frac{1}{a} \frac{da}{d\eta} = \frac{a'}{a}, \quad \ddot{a} = \frac{d\dot{a}}{dt} = \frac{1}{a} \frac{d(a'/a)}{d\eta} = \frac{a''}{a^2} - \frac{(a')^2}{a^3}, \quad H = \frac{\dot{a}}{a} = a'^2, \quad \dot{H} = \frac{1}{a} \frac{d}{d\eta} \left( \frac{a'}{a^2} \right) = \frac{a''}{a^2} - 2 \frac{a'}{a^3}. \]  

(B.22)

Note that the above connections are valid in \( D \) space time dimensions. The non-vanishing components of the Ricci tensor can now easily be derived, and in \( D \) dimensions they are

\[ R_{a\beta} = a^2 \left( H^2(D - 1) \eta_{a\beta} + \dot{H}(\eta_{a\beta} - (D - 2)\delta^0_a \delta^0_\beta) \right). \]  

(B.23)

Note that \( \dot{H} \) is

\[ \dot{H} = \frac{d}{dt} H = \frac{1}{a} \frac{d}{d\eta} H = \frac{1}{a} H'. \]  

(B.24)

The Ricci scalar \( R \) in Eq. (B.7) is then

\[ R = H^2(D + 2H(D - 1)). \]  

(B.25)

In 4 dimensions we again recover the Ricci scalar from Eq. (B.17). As a reference, we now calculate the components of the Ricci tensor and Ricci scalar in terms of the scale factor \( a \) and derivatives of the scale factor with respect to conformal time,

\[ R_{00} = 3 \left[ \frac{a''}{a} - \left( \frac{a'}{a} \right)^2 \right], \quad R_{ij} = -\left[ \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 \right] \eta_{ij} = -\left[ \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 \right] \delta_{ij}. \]  

(B.26)
Note that we could also have calculated these components by rewriting Eq. (B.16) and using Eq. (B.22). The results for the components of the Ricci tensor are the same in both coordinate systems, except the $R_{00}$-components are related by $R_{00}(\eta) = a^2 R_{00}(t)$, where I have used the $\eta$ and $t$ to distinguish between the calculation in conformal and spacetime coordinates, respectively. Of course this is due to the fact that $g_{00}(\eta) = a^2 g_{00}(t)$. The Ricci scalar in conformal coordinates is

$$R = g^{\mu\nu} R_{\mu\nu} = 6 \frac{a''}{a^3}. \quad (B.27)$$

After rewriting the Ricci scalar in terms of $a(t)$ using Eq. (B.22), we get exactly the same result as Eq. (B.17). Thus, the Ricci scalar is invariant under conformal transformations, which should be the case. The Einstein curvature tensor finally is

$$G_{00} = 3 \left( \frac{a'}{a} \right)^2$$

$$G_{ij} = - \left[ -2 \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 \right] \eta_{ij} = \left[ -2 \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 \right] \delta_{ij}, \quad (B.28)$$

where again the 00-component is related to $G_{00}$ in Eq. (B.18) coordinates by $G_{00}(\eta) = a^2 G_{00}(t)$. 
Bibliography


[34] T. M. Janssen and T. Prokopec, (2009), 0906.0666.


