Quantum Field Theory in Curved Spacetime

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Contents

I General info 2

II Lectures 3

1 Lecture I 3
  1.1 Rindler space 3
  1.2 The Unruh effect 3
  1.3 Euclidean time 4
  1.4 Euclidean time in Rindler space and the entanglement structure of the Minkowski vacuum 5

2 Lecture II 5
  2.1 Canonical technique 6
  2.2 Canonical technique in Rindler space 6
  2.3 Black holes 8

3 Lecture III 9
  3.1 Quantum state near black hole 9
  3.2 Hawking radiation 11
  3.3 Black hole information problem 11

4 Lecture IV 12

III Exercises 12

5 Exercise class I 12
  5.1 Temperature of a Black Hole 12
  5.2 Quantum fields in Rindler Space 14
    5.2.1 Preliminaries 14
    5.2.2 Bogolyubov Transformations 15
    5.2.3 Particle Density in the Modes 16

6 Exercise class II 17
Part I

General info

This module focuses on techniques and results from quantum field theory in curved space-times, such as the real world. Important applications we cover include

- Hawking radiation from black holes and the black hole information paradox.
- Quantum fluctuations during inflation that generate the temperature fluctuations in the CMB.

Exam. Problems 6.5, 7.2 and 8.1, to be handed in by the end of Friday 22/12/2017. Please email your solutions to both Joris and Ben (email addresses at the very beginning of this document).

You may use whatever references you can find in order to do the exam problems, but you must write up the solutions in a coherent manner. You must complete the exam problems individually. It is of course no problem if you have collaborated with fellow students on the exercises so far, but from this point onward you should not collaborate on the exam questions.

Note that problem 7.2 is mostly the same as what was already assigned in the exercise class, but we have added a couple of further questions at the end.

References.

- D. Harlow, Jerusalem Lectures on Black Holes and Quantum Information, [arXiv:1409.1231 [hep-th]].
- L. H. Ford, Quantum field theory in curved space-time, [gr-qc/9707062].
- N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space.
Notes. The lecture notes were written down by Ward Vleeshouwers, and he has kindly agreed to share them with the class. They are schematic, and should be seen as an overview of the topics covered in the course. The instructors have not edited the notes, which means we can’t guaranty their correctness, but we expect they will be quite useful for many students. See the references for more reliability.

Part II

Lectures

1 Lecture I

1.1 Rindler space

\[ ds^2 = -\rho^2 d\tau^2 + d\rho^2 + d\vec{y}^2 \]

A wedge of Minkowski space, accelerating observers: \( x = \ell \cosh s/\ell, \ t = \ell \sinh s/\ell. \ s = \) proper time, and acceleration is \( 1/\ell. \)

\( x_{\text{Mink}} = \rho \cosh \tau, \ t_{\text{Mink}} = \rho \sinh \tau. \) Rindler coordinates cover a single wedge. Rindler time is a boost parameter, which leaves Minkowski invariant: \( \tilde{c}_\tau \) is a Killing vector.

QFT of Rindler observers:

Take \( x^\phi \) for \( \phi \) free massless scalar in \( 3+1 \)-dimensional space.

\[ \langle \phi(x_1^\mu)\phi(x_2^\mu) \rangle = \frac{1}{(x_1 - x_2)^2 + i\epsilon} = \frac{1}{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (t_1 - t_2)^2 + i\epsilon} \]

Plug in \( x_1 = \rho_1 \cosh \tau_1 \) etc, gives:

\[ \langle \phi(x_1^\mu)\phi(x_2^\mu) \rangle = \frac{1}{(\Delta y)^2 + \rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2 \cosh(\tau_1 - \tau_2)} \]

1.2 The Unruh effect

Time dependence of \( \langle \phi\phi \rangle \):

In Minkowski we get power law decay. In Rindler, we get exponential decay: \( e^{-|\tau_1 - \tau_2|} \), this is reminiscent of thermal behavior. Rindler correlator is invariant under \( \tau \rightarrow \tau + 2\pi in, \ n \in \mathbb{Z} \) i.e. correl fnct is periodic in imaginary/Euclidean time, which is also indicative of thermal behavior.

To show this, consider some thermal expectation value \( \langle \mathcal{O} \rangle_\beta, \ \beta = 1/T. \)

\( \langle \mathcal{O} \rangle_\beta = tr (\hat{\rho}_\beta \mathcal{O}) = Z^{-1} tr \left( e^{-\beta H} \mathcal{O} \right) = Z^{-1} \sum_i \langle i | e^{-\beta H} \mathcal{O} | i \rangle, \) where \( |i\rangle \) is the basis where \( \mathcal{O} \) is diagonal.
Typically: $|\psi(t)\rangle = e^{-iHT}|\psi(0)\rangle$, $U(t) = e^{-iHT}$. Now: $e^{-tE}H$ is the Euclidean time evolution operator. Then $\sum_i |i\rangle e^{-\beta H}O|i\rangle$ tells us that we evolve $|i\rangle$ by $e^{-\beta H}$ after which it returns to its original state. Hence $|i\rangle$ is periodic in imaginary time with period $t_E = \beta$.

Rindler observer sees a thermal spectrum with $\beta = 1/T = 2\pi$ (dimensionless). Dimensionlessness comes from the fact that $\tau$ in Rindler metric is dimensionless. Physical temperature is then given in terms of proper time $\tau_{\text{prop}} = \rho \tau$, so that $T_\tau = \frac{1}{2\pi}; T_{\text{prop}} = \frac{1}{2\pi \rho}$, where $\rho$ is proper distance from horizon. This is called the Unruh effect: accelerating observer inside Minkowski vacuum sees thermal radiation with temperature $T = \frac{1}{2\pi \rho} = \frac{a}{2\pi}$, where $a$ is acceleration.

Is Rindler really thermal?

### 1.3 Euclidean time

**Here:** Euclidean path integral

**Exercises:** Canonical

Why talk about Euclidean time? Because $\langle \phi \phi \rangle Z^{-1} \int D[\phi] e^{iS}$ is not well-defined. Lorentzian QFT: many states possible.

Choose state: $|0\rangle_M \sim \lim_{t_E \rightarrow \infty} e^{-Ht_E} |\chi\rangle$. Calculate vacuum correlation function.

Take SHO: $S = \frac{1}{2} \int dt \left( \dot{\phi}^2 - \omega^2 \phi^2 \right)$. We take $\psi_{GS}(\phi_0) = \int_\phi^{(t_E=0)=\phi_0} [D\phi] e^{-S_{Euc}[\phi]}, S_{Euc} = \frac{1}{2} \int dt_E \left( \dot{\phi}^2 + \omega^2 \phi^2 \right)$

Use saddle point approx: $\psi_{GS}(\phi_0) \sim e^{-S_{Euc}[\phi_0]}$, with $\phi_{cl}(-\infty) = 0, \phi_{cl}(0) = \phi_0$

Calculate $\psi_{GS}(\phi_0)$ for SHO. $S_{Euc} = \omega^2 \phi_0^2 \int_{-\infty}^{0} dt \omega^2 = \frac{1}{2} \omega \phi_0^2$. Then: $\psi_{GS}(\phi_0) = \exp \left( \frac{-\omega^2}{2} \phi_0^2 \right)$

Generalizing:

$$\psi_{GS}(\phi(x)) = \langle \phi(x)|0\rangle = \frac{1}{\#} \int_\phi^{(x,t=0)=\phi_0(x)} D[\phi] e^{-S_{Euc}(\phi(x))} e^{-S_{Euc}(\phi_{cl})}$$

**Slogan:** Euclidean evolution selects state. This state is called the “Euclidean vacuum”.

**Correlators:**

$$\langle 0|\phi\phi|0\rangle = \langle \chi|e^{-HT_E} \phi \phi e^{-HT_E}|\chi\rangle.$$

Then:

$$\langle 0|\phi(x, t = 0)\phi(y, t = 0)|0\rangle = Z^{-1} \int D[\phi] e^{-S_{Euc}[\phi]} \phi(x, 0)\phi(y, 0)$$

At finite temperature:

$$Z = \text{tr} \rho = \sum_i \langle i| e^{-\beta H} |i\rangle = \int_{\phi(t_E+\beta,x)=\phi(t_E,x)} D[\phi] e^{-S_{Euc}(\phi)}$$

I.e. we integrate over field configurations that are periodic in Euclidean time. Hence:

$$\langle \phi(x, t = 0)\phi(y, t = 0) \rangle = Z^{-1} \int_{\text{periodic}} D[\phi] e^{-S_{Euc}(\phi)} \phi(x, 0)\phi(y, 0)$$
1.4 Euclidean time in Rindler space and the entanglement structure of the Minkowski vacuum

Rindler
d$s^2 = -\rho^2 d\tau^2 + d\rho^2 + d\vec{y}^2$

Euclidean: $\tau = i\tau_{Euc}$

d$s^2_E = \rho^2 d\tau^2_E + d\rho^2 + d\vec{y}^2$

Metric smooth at $\rho = 0$: $\tau$ has to have period $2\pi$. Quick and dirty way to see thermality is to check periodicity of Euclidean time.

A cleaner way to see this is from the fact that $|0\rangle_M$ is prepared by Euclidean time evolution. We have $\psi(\phi(x)) = \langle \phi(x)|e^{-HT_E}|\chi\rangle$

$\tau_E \to \tau_E + \pi$ takes us from right to left Rindler wedge. We have:

$$\psi_{GS}(\phi_L(x)\phi_R(x)) = \int_{\phi(x,t_E=\pi)=\phi_L(x)}^{\phi(x,t_E=0)=\phi_R(x)} e^{-S_{Euc}[\phi]} = \langle \phi_R(x)|e^{-\pi H_R}|\phi^*_L(x)\rangle$$

This is the Minkowski vacuum in Rindler basis. Go to energy basis of Rindler Hamiltonian:

$$\psi_{GS}(E_L, E_R) = \langle E_L | e^{-\pi H_R} | E_R \rangle$$

Hence $E_L = E_R$. We thus have:

$$|0\rangle_M \sum_{E \text{ eigenstates}} e^{-\pi E_i} |i\rangle^*_L \langle i|_R$$ (1)

This is the thermofield double state. It is pure now in the form of entangled states of left and right wedges. Trace over left wedge:

$$\rho_R = tr_L |0\rangle_M \langle 0|_M = \sum_i e^{-2\pi E_i} |i\rangle^*_R \langle i|_R = e^{-\beta H}$$

2 Lecture II

Last time: Rindler: $ds^2 = -\rho^2 d\tau^2 + d\rho^2$, gives finite $T$.

Let $\tau \to \beta \tau_E$: $ds^2 = d\rho^2 + \rho^2 d\tau^2_E$, $\tau_E \sim \tau_E + 2\pi$. Hence: $T_\tau = \frac{1}{2\pi}$, so that $T_{\text{proper}} = \frac{1}{2\pi \rho} = \frac{1}{2\pi}$

This derivation is a bit slick, Euclidean continuation is not always ‘available’ for general curved spaces.
2.1 Canonical technique

Canonical technique: Valid in general, but requires a lot of assumptions, e.g. free massless scalar in $1 + 1$ dimensions.

In flat space:

$$\tilde{\phi}(x, t) = \int \frac{dk}{2\pi^{d-1}} \left( \frac{1}{\sqrt{\omega_k}} a_k e^{i(kx-\omega t)} + h.c. \right), \quad \left[ a_k, a_k^\dagger \right] = \delta_{kk'}$$

I.e. we have one SHO for each $k$.

In curved space:

$$\tilde{\phi}(x^\mu) = \int d\lambda \left( u_\lambda(x^\mu) \hat{a}_\lambda + v_\lambda(x^\mu) \hat{a}_\lambda^\dagger \right), \quad \Box u_\lambda = 0 = \Box v_\lambda$$

Nice choice:

$$(u_\lambda, u_{\lambda'})_{KG} = \delta_{\lambda\lambda'} \quad , \quad (u, v) = 0$$

And

$$(u_\lambda, u_{\lambda'})_{KG} = -\delta_{\lambda\lambda'}$$

Where:

$$(f, g)_{KG} = i \int \sqrt{-g} n^\mu (f^* \partial^\mu g - (\partial f^*) g)$$

Here: $g$ is induced metric, $n^\mu$ is normal vector to spatial slice.

$u_\lambda$ are called positive frequency modes.

Flat: $|0\rangle_M$ defined by $\hat{a} |0\rangle_M = 0$.

For curved space: $\hat{a}_\lambda |0\rangle = 0$ for all $\lambda$: this state $|0\rangle$ is not unique in general. Choice of state is in the choice of which solutions we call positive frequency.

Time-like Killing vector $\leftrightarrow$ Hamiltonian, we can then define $|0\rangle$ by $H |0\rangle = 0$.

2.2 Canonical technique in Rindler space

For $1 + 1$ free massless scalar:

$$\frac{1}{\rho^2} \partial^\rho \partial^\rho \phi = \frac{1}{\rho} \partial^\rho (\rho \partial^\rho \phi)$$

$$u_p = \rho^p e^{-i\omega_p \tau} \quad R, \; u_p = 0 \quad L$$

With $\omega_p = |p|$

And $v_p^R = u^*_p$
\[
\hat{\phi} = \int dp \left( u_p^R \left( \hat{b}^R \right)^\dagger + v_p^R \hat{b}^R + u_p^L \hat{b}^L + v_p \left( \hat{b}^L \right)^\dagger \right)
\]

How does \( a_k |0\rangle_M = 0 \) look in terms of Rindler modes \( \hat{b} \)?

Define \( f_k = \frac{e^{i(kx-\omega t)}}{\sqrt{\omega_k}} \):

\[
\hat{a}_k = (f_k, \hat{\phi}) , \quad \hat{b}_p = \left( u_p, \hat{\phi} \right)_{KG}
\]

\[
\hat{b}_p = \left( u_p, \int dk \left( f_k \hat{a}_k + c.c. \right) \right) = \int dk \left( \alpha_{pk} \hat{a}_k + \beta_{pk} \hat{a}_k^\dagger \right)
\]

Where \( \alpha_{pk} = (u_p, f_k) \), \( \beta_{pk} = (u_p, f_k^* \).

i.e. they are the inner product of Minkowski \( f_k \) and Rindler \( u_p \) modes.

Calculate number of particles in each mode in Minkowski vacuum:

\[
\langle \hat{N} \rangle = \langle 0_M | \hat{b}^\dagger \hat{b} | 0_M \rangle = \langle 0_M | \int dk' \left( \alpha_{k'p}^* \alpha_{k'p} + \beta_{k'p}^* \beta_{k'p} \right) \int dk \left( \alpha_{kp} \hat{a}_k + \beta_{kp} \hat{a}_k^\dagger \right) | 0_M \rangle = \int dk |\beta_{kp}|^2
\]

**Trick:** Find combination of Rindler modes which is positive frequency from Minkowski point of view.

\( e^{i(kx-\omega \tau t)} \) is well-behaved in lower half of complex \( t \)-plane. Let \( t = i \tau \): \( e^{\omega \tau} \) blows up at infinity. If we take a function which is analytic (well-behaved) in lower half complex \( t \)-plane, this is some superposition of positive frequency modes. We combine Rindler modes in such a way that they are well-behaved in lower half \( t \)-plane.

\[
u_p^R = (\rho e^{-\tau})^ip = (x-t)^ip , \quad u_p^L = 0
\]

With \( x = \rho \cosh \tau , \quad t = \rho \sinh \tau \). \( u_p \) is not analytic since it jumps to zero when we cross from \( R \) to \( L \).

Write: \( u = (x-t)^ip = e^{i p (\ln(x-t))} \), gives branch cut at \( x-t = 0 \) in upper half \( t \)-plane. Hence one side \( x-t \) is positive and other side \( x-t \) is negative. Hence on right hand side we have \( e^{ip \ln(x-t)} \), on left hand side we have same function but with \( x-t < 0 \) i.e. \( \ln(x-t) = \ln(t-x) + \ln(-1) = \ln(t-x) \pm i\pi \), \( \pm \) depending on where we have branch cut.

Hence: \( u_R = e^{ip \ln(x-t)} , \quad u_L = e^{ip \ln(x-t)+i\pi} \). Hence \( u = u_p^R + e^{-\omega \pi} u_p^L \) has positive Minkowski frequency. Choosing branch cut gives unique analytic continuation. We thus get positive frequency on the right and negative on the left from Rindler point of view, which is positive frequency for Minkowski.

I.e. \( u_p^R + e^{-\omega \pi} u_p^L \sim \) positive in right wedge and negative in left wedge has positive Minkowski frequency.

Similarly: \( u_p^L + e^{-\omega \pi} u_p^R \) has pos freq.

Negative Minkowski frequency is given by \( u_p^R + e^{-\omega \pi} u_p^L , \quad u_p^L + e^{-\omega \pi} u_p^R \).
This gives an annihilation operator for Minkowski vacuum for each Rindler mode:

\[
\left( b_p^R + e^{-\omega_p \pi} \left( b_p^L \right)^\dagger \right) |0\rangle_M = 0
\]

*break*

We have:

\[
\left( \hat{b}^R_p - e^{-\pi \omega_p} \left( \hat{b}^L_p \right)^\dagger \right) |0\rangle_M = 0
\]

\[
\left( \hat{b}^L_p - e^{-\pi \omega_p} \left( \hat{b}^R_p \right)^\dagger \right) |0\rangle_M = 0
\]

Ex: Show that \( |0\rangle_M = \frac{1}{2} \prod_p \left( \sum_n e^{-\beta \omega_p n} |n\rangle_L |n\rangle_R \right) \), using \( b |n\rangle = \sqrt{n} |n - 1\rangle \), \( b^\dagger |n\rangle = |n + 1\rangle \).

Easy exercise. \( |0\rangle_M \) is unique!

### 2.3 Black holes

Schwarzschild black hole in 3 + 1 dimensions.

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_2^2
\]

Where \( f(r) = 1 - \frac{r_s}{r} \), \( r_s = 2GM \). Horizon a \( r = r_s \). Near horizon, change of coordinates to give:

\[
ds^2 = -\frac{\rho^2}{4r_s^2} dt^2 + d\rho^2 + r_s^2 d\Omega_2^2
\]

I.e. we get Rindler coordinates close to horizon (except for spherical part).

*conf diagram*

If you are much smaller than the Schwarzschild radius, you’re safe. Proper time to \( r = 0 \) \( \sim r_s \).

We look at eternal black holes \( \sim \) non-eternal black holes at late times, which behaves as Rindler.

Euclidean near-horizon geometry:

\[
ds^2 = r_s^2 d\Omega_2^2 + d\rho^2 + \rho^2 \left( \frac{t}{2r_s} \right)^2
\]

Hence: \( \frac{t_E}{2r_s} \sim \frac{t_E}{2r_s} + 2\pi \) i.e. \( t_E \sim t_E + 4\pi r_s \). Observers at \( r = \text{constant} \) see thermal radiation at temperature \( T = \frac{1}{4\pi r_s} \), such that the proper temperature is:
\[ T_{\text{proper}} = \frac{1}{4\pi r_s} \frac{1}{\sqrt{f(r)}} = \frac{1}{4\pi r_s} \frac{1}{\sqrt{(1 - \frac{r}{r_s})}} \]

Euclidean black hole:

\[ ds^2 = f(r)dt_E^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2 \]

I.e. \( t_E \sim t_E + 4\pi r_s \)

Physics:

An observer at infinity sees thermal radiation: observer in locally inertial frame sees observer at infinity emit radiation.

\( \langle T_{tr} \rangle \sim \) energy flux. Everything is static i.e. invariant under \( t \to -t \). However, energy flux is not static, hence it is equal to zero. I.e. things look thermal without energy flux, hence we have a black hole in thermal equilibrium with Hawking radiation.

State = Euclidean vacuum = Hartle-Hawking state.

Consider a free massless scalar in black hole background:

Exercises: Tortoise coordinate, spherical harmonics.

Modes:

\[ u^k_{\ell m} = \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} Y_{\ell m}(\Omega_k) \Psi_k^\ell(\tilde{r}) \]

Where \( \Psi(\tilde{r}) \) solves a Schrödinger type equation.

3 Lecture III

We look at real black hole i.e. collapsing matter. After relaxation time: black hole no hair theorem.

In Euclidean vacuum (Hartle-Hawking state) \( T = \frac{1}{4\pi r_s} \), respects symmetries such as time-like translation. This means that everything has to be static i.e. \( \langle T_{tr} \rangle \sim \) flux = 0. Hence, H.H. state describes black hole in equilibrium with radiation, which is an unstable equilibrium. This means that H.H. is not the appropriate state to describe real black holes in asymptotically flat space.

3.1 Quantum state near black hole

\[ \square \phi = 0 \]

\[ u^k_{\ell m} = \frac{1}{r\sqrt{\omega_k}} Y_{\ell m}(\Omega) \Psi^\ell_{k} (\tilde{r}) \quad , \quad \frac{dr}{f(r)} \equiv d\tilde{r} \]
Then, ‘effective Schrödinger equation’:

\[(\hat{\mathcal{E}}_\ell^2 + V_\ell(\vec{r})) \Psi_\ell^f(\vec{r}) = k^2 \Psi_\ell^f, \quad V_\ell = \frac{f(r)}{r^2} \left[ \ell(\ell + 1) + \frac{r_s^2}{r} \right] \]

Notice the mixed use of \( r \) and \( \vec{r} \), look at exercises for week 2 and picture on phone.

For large \( \ell \), we can ignore the \( r_s/r \) term. The maximum value of the angular momentum potential barrier is achieved at \( r = \frac{3}{2}r_s \), where the potential is roughly \( \frac{\ell(\ell+1)}{r_s^2} \).

These wave-functions make our problem a quantum mechanics problem. We consider ingoing modes (from the left) \( \sim e^{ik\vec{r}} \) with transmitted and reflected pieces:

\[ e^{ik\vec{r}} \rightarrow R(k)e^{-ik\vec{r}} + T(k)e^{ik\vec{r}} \]

Similar for ingoing modes from the right \( \sim e^{ik\vec{r}} \). Note that the horizon is at \( \vec{r} = -\infty \) and we are looking at scattering from angular momentum barrier outside horizon.

Quantum mechanics energy: \( E = k^2 = \omega^2 \). For \( \omega < \frac{\ell(\ell+1)}{r_s^2} \), \( T(k) \ll 1 \), i.e. low transmission for low energy, compare to 'tH.

We have \( T = \frac{1}{4\pi r_s} \), modes with \( \omega \gg T \sim \frac{1}{r_s} \) are not excited. We want \( \frac{\ell(\ell+1)}{r_s^2} \ll \omega^2 \) for the modes to get through the barrier, and \( \omega \lesssim \frac{1}{r_s^2} \), hence we want \( \frac{\ell(\ell+1)}{r_s^2} \ll \omega \lesssim \frac{1}{r_s^2} \).

Hence: high \( \ell \) is boring, \( \omega_k \) is the frequency seen by the asymptotic observer. Note that the frequencies that we are interested in are of order \( \omega \sim T \sim \frac{1}{r_s} \) for asymptotic observer. Focus on \( \ell = 0 \) “s-wave”. s-Wave has barrier for the modes to get through the barrier, and \( \omega \approx r \), hence outgoing modes are approximately plane waves. Near the horizon \( r = r_s, \vec{r} = r_s \log \left( \frac{r_s}{r} - 1 \right) \).

Hence:

\[ e^{ik\vec{r}} \rightarrow \text{near horizon} \left( \frac{r}{r_s} - 1 \right)^{ikr_s} \text{proper distance} \rho^{2ikr_s} \]

Recall: For Rindler metric \( ds^2 = dp^2 - \rho^2 d\tau^2 \), \( u \sim \rho^\beta e^{i\omega \rho} \)

I.e. Rindler modes close to horizon, Minkowski plane wave for \( r \rightarrow \infty \). If we take \( |B\rangle \) close to the horizon, it becomes the Rindler vacuum \( |0_R\rangle \), which is singular at the horizon. This state \( |B\rangle \) is called the Boulware vacuum. A quick way to see that it is singular at the horizon is that \( |B\rangle \) is a pure state outside the horizon i.e. \( |B\rangle \) is pure in ‘region I’. Hence there is no entanglement across horizon. We have a stress energy term \( (\hat{\mathcal{E}}_x \hat{\phi})^2 \), hence we get a singular stress tensor when there is no entanglement between fields since fields aren’t correlated.
3.2 Hawking radiation

Consider Unruh state. We put in-moving modes from the right in a vacuum state, and out-moving modes (in-moving modes from the left) in a thermal state. Tracing back Hawking particles to very close to the horizon, it looks like a Rindler wave packet. Proper wavelength redshifts as \( \lambda_{\text{prop}} \sim \sqrt{T(r)} \), \( \lambda_{\text{prop}} \rightarrow \infty \) as \( r \rightarrow r_s \), \( \lambda_{\text{prop}} \rightarrow 0 \).

We have three states \( |\text{HH}\rangle, |\text{B}\rangle, \) and \( |\text{U}\rangle \): \( |\text{U}\rangle \) is the relevant state, which has thermally populated outgoing modes and ingoing modes in the vacuum. \( |\text{U}\rangle \) is thus a mixed state. Look at outgoing mode \( H \) with \( \lambda \sim r_s \). There is a non-zero energy flux out to infinity i.e. black hole is evaporating via Hawking radiation. The energy out-flux goes as \( T^4 \sim \text{Area} \), \( T \sim \frac{1}{r^2} \). Recall: interesting modes have \( \omega \sim \frac{1}{r_s} \) and \( \ell \approx 0 \). These have occupation number \( \mathcal{O}(1) \) and spacing between Hawking photons of order \( r_S \).

We have:

\[
\langle \hat{N}_\omega \rangle = \frac{1}{e^{\beta \omega} - 1} \left( \frac{T(k)}{k} \right)^2 \text{greybody factor}
\]

Lifetime of black hole:

\[
\frac{dM}{dt} = \mathcal{O}(1) * \frac{-1}{r_s^2}, \quad \frac{1}{G} \frac{dr_s}{dt} = -\frac{1}{r_s}
\]

Hence

\[
t_{\text{life}} \approx r_s^3 / G_N
\]

3.3 Black hole information problem

Suppose we make a black hole in a pure state. It seems to evaporate to produce a mixed, thermal state. But unitarity demands that pure states evolve into pure states, hence we have a apparent violation of unitarity.

We analyzed near-horizon region + exterior. This is weakly curved, so we can use QFT in curved spaces. We are basically using an effective field theory. This gives thermal Hawking radiation, which is (of course) independent of thermal state. Estimate amount of Hawking particles: \( \omega \sim 1/r_s, \) \( M \sim r_s / G \). Hence \( N \sim \frac{M}{\omega} \sim \frac{r_s^2}{4G} = S_{BH} \), i.e. the number of Hawking particles is of the order of the entropy.

Schematically: black hole evaporation gives \( S \) photons, which, by unitarity, should be in a pure state. We (thus) expect that these photons are highly entangled. A cartoon picture is given by \( S \) qubits. Page: calculate entanglement entropy \( S_E \) as a function of the number of photons. Page then calculated that \( S_E \) roughly increases linearly with the number of qubits, since they are roughly randomly entangled. We thus get \( S_E \sim N_\gamma \) for \( N_\gamma < S/2 \), and \( S_E = S - N_\gamma \) for \( N_\gamma > S/2 \). Page: let half of a black hole evaporate. If outgoing Hawking photon is entangled with early radiation, we know that black hole was (likely) in a pure state.
Unitarity demands: $S_{HR} \approx 0$ (Hawking particle and previous Hawking radiation), $S_{HP} \approx 0$, $S_H \approx 1$.

Strong subadditivity:

$$S_{HR} + S_{HP} \geq S_H + S_{RH}$$

But this is in conflict with values for the entropy as demanded by unitarity.

pure $\rightarrow$ mixed via transmission/reflection

Partner particle $P$ goes to $r = 0$ in finite time.

$\Rightarrow \frac{3}{2} r_s = 3M$: location of potential maximum for Schwarzschild as in Carroll analysis.

4 Lecture IV

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Part III

Exercises

5 Exercise class I

5.1 Temperature of a Black Hole

Consider a metric of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_{d-2}^2,$$  \hspace{1cm} (2)

where, for example, $f(r) = 1 - \frac{2GM}{r}$ for a $d = 4$ Schwarzschild black hole.

a) Analytically continue to Euclidean signature by sending $t \rightarrow -i\tau$. Assume that $f(r)$ changes sign at a certain horizon radius $r_h$. Now zoom in at the near-horizon region by defining a coordinate $\xi = r - r_h \geq 0$, and show that the metric can be brought into the form

$$ds^2 = f'(r_h)\xi d\tau^2 + \frac{d\xi^2}{f'(r_h)\xi},$$  \hspace{1cm} (3)

where we’ve suppressed any angular dependence.

b) Figure out a coordinate transformation which brings the the metric into

$$ds^2 = \rho^2 \left( d \left[ \frac{f'(r_h)\tau}{2} \right] \right)^2 + d\rho^2.$$  \hspace{1cm} (4)
c) Does this metric look familiar? Explain why this metric makes sense.

d) In order for the metric to be well-behaved near the origin, what condition do you need to impose on the $\tau$ coordinate? Argue that this leads to a black hole temperature of

$$T = \frac{f'(r_h)}{4\pi}.$$ (5)

e) Compute the temperature of a Schwarzschild black hole in S.I. units. What can you say about the various constants that appear?
5.2 Quantum fields in Rindler Space

In this exercise you will do quantum field theory in Rindler space, and derive the spectrum of the particles that an accelerating observer detects in the Minkowski vacuum.

5.2.1 Preliminaries

Consider the action of a massless scalar field $\phi$ in $d$ dimensions,

$$S[\phi] = \int d^dx\sqrt{-g} \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi.$$  

(a) Show that this action is invariant under conformal transformations of the metric

$$g_{\mu\nu} \rightarrow \Omega^2(x) g_{\mu\nu}$$

when $d = 1 + 1$.

The Rindler metric is conformally flat, i.e. it can be written as

$$ds^2 = e^{2a\xi}(d\tau^2 + d\xi^2)$$

where $a$ is related to the acceleration of the observer. Since the action is invariant under conformal transformation of the metric, we know that the equation of motion in both coordinate systems can be written as

$$\frac{\partial^2 \phi}{\partial \tau^2} = \frac{\partial^2 \phi}{\partial \xi^2}.$$  

(7)

Remember that, using Minkowski coordinates, the free field operator can be expanded over the plane waves as

$$\hat{\phi}(t,x) = \int_{-\infty}^{+\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|k|}} \left[ \hat{a}_k e^{-ik(t+ix)} + \hat{a}_k^* e^{ik(t-ix)} \right],$$  

(8)

where standard commutation relations for $\hat{a}$ and $a^\dagger$ have been imposed. Since the field satisfies the same eom in the Rindler coordinates, writing down the mode expansion of the field $\hat{\phi}$ in the Rindler frame becomes trivial

$$\hat{\phi}(\tau,\xi) = \int_{-\infty}^{+\infty} \frac{dp}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|p|}} \left[ \hat{b}_p e^{-ip(\tau+i\xi)} + \hat{b}_p^* e^{ip(\tau-i\xi)} \right].$$  

(9)

Note that we now have expanded the field operator $\hat{\phi}$ into linear combinations of two different sets of mode functions with operator valued coefficients $\hat{a}_k$ and $\hat{b}_k$. Both sets of operators define vacuum and particle states, which are different quantum states of the same field. The Minkowski vacuum $|0_M\rangle$ is defined as the state for which $\hat{a}_k|0_M\rangle = 0$ for all $k$. Similarly, the Rindler vacuum is defined as the state for which $\hat{b}_p|0_R\rangle = 0$ for all $p$.

For further analysis, it is convenient to introduce lightcone coordinates

$$\bar{u} \equiv t - x \quad \bar{v} \equiv t + x \quad \text{and} \quad u \equiv \tau - \xi \quad v \equiv \tau + \xi.$$  

(10)

They satisfy

$$\begin{cases} \bar{u} &= -a^{-1}e^{-au} \\ \bar{v} &= a^{-1}e^{av} \end{cases}.$$  

(11)

1This mode expansion is only valid within the Rindler wedge $|x| > t$. The Rindler coordinates are not defined outside this domain, and in this case the two mode expansions cannot be compared with each other.
In these coordinates the metric becomes
\[ ds^2 = d\bar{u}d\bar{v} = e^{\alpha(u-v)}dudv. \] (12)

b) Substitute the lightcone coordinates in the obtained mode expansions while denoting \( \omega = |k| \) in the Minkowski frame and \( \Omega = |p| \) in the Rindler frame, and obtain the following lightcone mode expansions
\[
\hat{\phi}(\bar{u}, \bar{v}) = \int_0^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[ \hat{a}_\omega e^{-i\omega\bar{u}} + \hat{a}^\dagger_\omega e^{i\omega\bar{u}} + \hat{a}_- e^{-i\omega\bar{u}} + \hat{a}^\dagger_- e^{i\omega\bar{u}} \right],
\]
\[
\hat{\phi}(u, v) = \int_0^{+\infty} \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left[ \hat{b}_\Omega e^{-i\Omega u} + \hat{b}^\dagger_\Omega e^{i\Omega u} + \hat{b}_- e^{-i\Omega u} + \hat{b}^\dagger_- e^{i\Omega u} \right].
\] (13) (14)

Notice that the \( u/\bar{u} \) and \( v/\bar{v} \) coordinates are separated from each other within the mode expansions,
\[
\hat{\phi}(\bar{u}, \bar{v}) = \hat{A}(\bar{u}) + \hat{B}(\bar{v}),
\]
\[
\hat{\phi}(u, v) = \hat{P}(u) + \hat{Q}(v).
\] (15) (16)

According to the transformations (11), we can now extract two separate equations for the field expansion,
\[
\hat{A}(\bar{u}(u)) = \hat{P}(u), \quad \hat{B}(\bar{v}(v)) = \hat{P}(v).
\] (17)

Writing out the first relation yields
\[
\int_0^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[ \hat{a}_\omega e^{-i\omega\bar{u}(u)} + \hat{a}^\dagger_\omega e^{i\omega\bar{u}(u)} \right] = \int_0^{+\infty} \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left[ \hat{b}_\Omega e^{-i\Omega u} + \hat{b}^\dagger_\Omega e^{i\Omega u} \right],
\]
which shows that there is no mixing between operators with positive and negative momentum. This means we can do a separate analysis for the positive and negative momentum operators: in everything that follows, the results derived for positive modes will be equally valid for negative modes.

### 5.2.2 Bogolyubov Transformations

a) We now Fourier-transform eq. (18)\(^2\). We start with the RHS. Show that
\[
\int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} \hat{P}(u) = \frac{1}{\sqrt{2|\Omega|}} \left\{ \begin{array}{ll}
\hat{b}_\Omega & \text{if } \Omega > 0 \\
\hat{b}^\dagger_\Omega & \text{if } \Omega < 0
\end{array} \right.,
\] (19)

b) Do exactly the same thing to the LHS of (18), and find the following Bogolyubov transformation,
\[
\hat{b}_\Omega = \int_0^{+\infty} d\omega \left[ \alpha_{\omega\Omega} \hat{a}_\omega + \beta_{\omega\Omega} \hat{a}^\dagger_\omega \right] \quad \text{where} \quad \left\{ \begin{array}{l}
\alpha_{\omega\Omega} = \sqrt{\frac{i}{\Omega}} F(\omega, \Omega) \\
\beta_{\omega\Omega} = \sqrt{\frac{i}{\Omega}} F(-\omega, \Omega)
\end{array} \right. \quad \omega > 0, \Omega > 0,
\] (20)

\(^2\)As the result of this calculation clearly shows, the case \( \Omega = 0 \) is not covered, since the mode expansions (8) and (9) ignore the \( k = 0/p = 0 \) solution, also called the zero mode. In 1+1 dimensions, quantization of the zero mode is a technical issue, but it can be shown that it doesn’t contribute in four-dimensional theory, so we’ll ignore it.
with

\[ F(\omega, \Omega) = \int_{-\infty}^{+\infty} \frac{du}{2\pi} \exp \left[ i\Omega u + i\frac{\omega}{a}e^{-au} \right]. \quad (21) \]

This Bogolyubov transformation expresses the annihilation operator of the Rindler modes in terms of annihilation and creation operators of the Minkowski modes, with various \( \omega \). Hermitian conjugation of equation (20) gives a Bogolyubov transformation for the creation operator \( \hat{b}^\dagger \Omega \). The construction of the Bogolyubov transformations for the negative momentum modes is completely analogous.

### 5.2.3 Particle Density in the Modes

We are interested in computing what particle excitations an accelerated observer (using the Rindler frame) will see in the Minkowski vacuum. Of course,

\[ R(0|\bar{N}|0)_{R} = R(0|\hat{b}^\dagger \Omega \hat{b}|0)_{R} = 0. \quad (22) \]

a) Now show that

\[ M(0|\bar{N}|0)_M = \int d\omega |\beta_{\omega,\Omega}|^2. \quad (23) \]

This is the number of particles in the Minkowski vacuum the Rindler observer is expected to see. We will now compute this integral explicitly.

b) *Bonus question.* Prove the following essential property

\[ F(\omega, \Omega) = F(-\omega, \Omega) \exp \left( \frac{\pi\Omega}{a} \right). \quad (24) \]

Hint: extend \( u \) to the complex plane.

c) Compute the expected Rindler particle density in the Minkowski vacuum

\[ M(0|\bar{N}|0)_M = \left[ \exp \left( \frac{2\pi\Omega}{a} \right) - 1 \right]^{-1} \delta(0). \quad (25) \]

Hint: The commutation relation

\[ [\hat{b}_\Omega, \hat{b}^\dagger_{\Omega'}] = \delta(\Omega - \Omega') \quad (26) \]

imposes a normalization condition on the Bogolyubov coefficients. Write out this normalization condition and set \( \Omega = \Omega' \) in the end. Use the property of the function \( F \) above.

d) Interpret your result. Is there a non-trivial particle spectrum? What temperature can you associate to it? Why is the divergent factor \( \delta(0) \) there? Where do these particles come from, and who is providing them with the necessary energy?

e) Reinstate the correct factors of \( \hbar, c \) and \( k_B \), and compute the temperature you’d measure when accelerating at \( 6.5 \frac{m}{s^2} \) with your Aston Martin in the Minkowski vacuum.
6 Exercise class II

6.1 Euclidean Harmonic Oscillator

"The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.”
– Sidney Coleman

In this exercise, you will compute the partition function of the harmonic oscillator by taking a detour through euclidean time. This is very inefficient, but because you are already familiar with the answer you should get, you can focus more on the techniques. So, the goal is not to obtain the partition function, but rather to interpret the steps in between carefully.

a) From memory or with a small derivation, write down the partition function of the Harmonic Oscillator $Z = \text{tr}(e^{-\beta H})$ in closed form. (Without using path integrals.)

It is a quite trivial observation that

$$Z = \text{tr} e^{-\beta H} = \text{tr} e^{-itH}|_{t \to -i\beta}.$$

This result, however, is the reason that by going to Euclidean time, we go from unitary time-evolution to (thermal) statistics.

b) Without making the substitution, write out the trace in the last expression in the position basis, and write the propagator you find in your expression as a path integral. Keep the boundary conditions explicit.

c) Now make the substitution.

The substitution changes the contour of the integral over the Lagrangian ($\int_0^t dt' L \to \int_0^{-i\beta} dt' L$). Instead, change the integrand so that you get an expression involving the integral $\int_0^\beta dt_E L_E$, where $L_E$ is the Euclidean Lagrangian.

d) Can you see why ‘Euclidean time is cyclic’?

We could proceed by finding the Euclidean action and solving the subsequent path integral. However, to save some work, we’ll just take the non-Euclidean propagator for the Harmonic Oscillator from a book, and substitute to get the Euclidean propagator. In closed form, it reads

$$K(x_2, t_2; x_1, t_1) = \sqrt{\frac{m\omega}{2\pi i \sin[\omega(t_2 - t_1)]}} \times \exp \left\{ \frac{im\omega}{2\sin[\omega(t_2 - t_1)]} \left[ (x_1^2 + x_2^2) \cos[\omega(t_2 - t_1)] - 2x_1x_2 \right] \right\}.$$

e) Find the Euclidean propagator for the Harmonic oscillator.

f) Using what you’ve learned in questions b and e, compute the partition function of the Harmonic Oscillator. Compare your answer with a.

The ground state wave-function $\langle x | \Omega \rangle$ of a quantum mechanical system can be computed by

$$\langle x | \Omega \rangle \sim \lim_{\beta \to \infty} \langle x | e^{-\beta H} | \chi \rangle.$$

Here $| \chi \rangle$ can be any state that is not orthogonal to $| \Omega \rangle$. 

g) Compute the ground state of the Harmonic Oscillator using the Euclidean propagator you found in e. (If you already solved this in class, you can skip this exercise.)

6.2 Black Hole Thermodynamics

6.3 The Laws

Last session we showed that a $d = 3 + 1$ Schwarzschild black hole has a Hawking temperature $T = \frac{1}{8\pi GM}$.

a) Using the first law of thermodynamics, show that you can associate an entropy to the black hole

$$S_{BH} = \frac{A}{4G},$$

where $A$ is the surface area of the black hole.

b) What does the second law of thermodynamics say about $A$? For now, assume you can ignore evaporation. Your result is called the second law of black hole thermodynamics.

b) Compute the heat capacity of this black hole, and interpret. If you place a black hole in an infinite heat bath with $T \neq T_{BH}$, is a stable equilibrium possible? Why (not)?

6.4 Black Hole Scattering and the Consequences of the Area Law

Consider two well-separated Schwarzschild black holes at rest. Let their masses be $M_1$ and $M_2$. Assume that they coalesce to form a Schwarzschild black hole of mass $M_3$, with gravitational waves carrying off energy $E$ in the process.

a) Use conservation of energy and the second law of black hole thermodynamics (as derived in the previous exercise) to prove that the efficiency of the creation of gravitational waves obeys $\eta \leq 1 - \frac{1}{\sqrt{2}}$.

The radiated energy could be used to do work, hence the second law of black hole thermodynamics restricts the useful energy that can be extracted from merging black holes.

b) On 14 September 2015, LIGO detected the gravitational waves created by the coalescence of two black holes with $M_1 \approx 36 M_\odot$, $M_2 \approx 29 M_\odot$, and $E = 3 M_\odot$. For the sake of simplicity, assume the black holes 1 and 2 were once two well-separated Schwarzschild black holes at rest, and that black hole 3 is also a Schwarzschild black hole. Did the efficiency of this process come close to bound that was shown in a?

Now consider an initial Schwarzschild black hole of mass $M$ that evolves to form two well-separated approximately Schwarzschild black holes at rest.

c) Use conservation of energy and the second law of black hole thermodynamics to show that this cannot happen. (This is a special case of a completely general result that black holes cannot bifurcate.)

\[\text{This isn’t true, but according to LIGO, } M_3 \approx 62 M_\odot, \text{ so that indeed } M_1 + M_2 \approx M_3 + E.\]
6.5 The Schwarzschild-AdS Black Hole

The unique spherically-symmetric solution of Einstein’s field equations with a negative cosmological constant is known as the Schwarzschild-AdS metric. In \( d = 3 + 1 \) dimensions it is given by

\[
\mathrm{ds}^2 = -f(r)\mathrm{dt}^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \quad f(r) = 1 - \frac{2MG}{r} + \frac{r^2}{b^2}.
\]

(28)

Here \( b \) is related to the cosmological constant by \( \Lambda = -\frac{3}{b^2} < 0 \). The event horizon of this black hole is at \( r_+ \), the largest root of \( f(r) \).

a) The classical equation of motion of the massless scalar field reads

\[
\Box \phi = 0.
\]

Here the d’Alembertian (‘box’) is, of course, the generalized d’Alembertian for curved spacetimes. As an ansatz, take \( \phi \) to be a spherical harmonic,

\[
\phi = e^{-i\omega t} \frac{\psi_{\omega l}(r)}{r} Y_{lm}(\theta, \varphi).
\]

Show that this indeed solves the equation of motion if the radial part \( \psi_{\omega l} \) satisfies the effective Schrödinger equation

\[
-r_{s}^2 \psi_{\omega l}(r) + V(r)\psi_{\omega l}(r) = \omega^2 \psi_{\omega l}(r),
\]

\[
V(r) = \frac{f(r)}{r^2} [l(l + 1) + r\partial_r f(r)].
\]

Here \( r \) is always to be understood as \( r(r_*) \), where \( r_* \) is the coordinate that takes the metric (28) to the form

\[
\mathrm{ds}^2 = f(r)(-\mathrm{dt}^2 + \mathrm{dr_*}^2) + r^2 d\Omega_2^2.
\]

b) Plug in the Schwarzschild-AdS \( f(r) \), rewrite, and sketch the effective potential. Interpret your result. In particular, comment on the physical differences with the \( b \to \infty \) limit, which is the usual Schwarzschild effective potential discussed in class.

c) Finally, compute the temperature of the Schwarzschild-AdS black hole as a function of \( r_+ \) and \( b \) only. Plot the temperature as a function of its size and compare with the Schwarzschild case. Interpret carefully. Are there thermodynamically stable black holes in AdS?

7 Exercise class III

7.1 Klein-Gordon Inner Product

The space of solutions to the Klein-Gordon equation is a vector space, endowed with the Klein-Gordon inner product

\[
(\phi_1, \phi_2) = -i \int_{\Sigma} (\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1) n^\mu \sqrt{-\gamma} d^{d-1}\Sigma.
\]

(29)
Here $\Sigma$ is a spacelike hypersurface with unit normal $n^\mu$, and $\gamma_{\mu\nu}$ is the (induced) metric on $\Sigma$.

a) Show that the Klein-Gordon product does not depend on the choice of $\Sigma$.

*Hint.* You may use Stoke’s theorem, which relates the integral over a manifold $M$ to an integral over the boundary of that manifold, $\partial M$,

$$\int_M d^n x \sqrt{|g|} \nabla_\mu V^\mu = \int_{\partial M} d^{n-1} y \sqrt{|\gamma|} n_\mu V^\mu.$$  

7.2 Stress Energy Tensor in Rindler Space

The action for a free, massless scalar field in flat spacetime is given by

$$S = -\int d^d x \sqrt{-g} \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi.$$  

(30)

The stress-energy tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}.$$  

(31)

a) *Bonus.* Derive that

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}.$$  

b) Show that $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\sigma \phi \partial^\sigma \phi$.

For further analysis it is convenient to introduce light cone coordinates in $d = 1 + 1$ Minkowski space, $U = t - x$ and $V = t + x$. The right Rindler wedge is then given by $U < 0 < V$.

c) Write down the metric of flat, 1+1 dimensional spacetime in lightcone coordinates.

A complete basis for the right-moving modes (i.e. functions of $t - x = U$ that solve the eom.) is formed by

$$\phi_{\omega, R} = \Theta(-U) \frac{1}{\sqrt{4\pi \omega}} (-aU)^{-\frac{\omega}{4}}, \quad \phi_{\omega, L} = \Theta(U) \frac{1}{\sqrt{4\pi \omega}} (aU)^{-\frac{\omega}{4}}.$$  

We can now expand the field $\phi$ as

$$\phi(U) = \int d\omega \left( b_{\omega, R} \phi_{\omega, R} + b^\dagger_{\omega, R} \phi_{\omega, R}^* \right) + (L \leftrightarrow R),$$  

where we have neglected the left moving modes. (Later in the exercise, you will see why we do not have to consider the left-moving modes.)

d) Interpret the quantity $T_{UU}$ in (classical) GR.

e) Compute the expectation value of the $UU$ component of the stress-energy tensor in the Rindler vacuum.

*(Note: Your answer to the previous part should give $T_{UU} = \partial_U \phi \partial_U \phi_*.$)*

To get rid of a UV-divergence, the expectation value is defined as

$$\langle T_{UU} \rangle_R = \langle 0_L | T_{UU} | 0_L \rangle - \langle 0_M | T_{UU} | 0_M \rangle.$$  

20
Hint. First show that
\[ \langle 0 | T_{U \nu} | 0 \rangle = \int_0^\infty d\omega \frac{\omega}{4\pi a^2 U^2} (2n_\omega + 1), \tag{32} \]
where \( |0\rangle \) could be either the Minkowski vacuum or the Rindler vacuum, and where \( n_\omega = \langle b^\dagger_\omega b_\omega \rangle \) is the expected number density of particles in the respective state. Also, you can use that
\[ \int_0^\infty \frac{d\omega}{a} \frac{1}{e^{\frac{\omega}{\pi}} - 1} = \frac{1}{24}. \tag{33} \]
As discussed in class, for real black holes that form in a collapse, the outgoing modes are thermal while the ingoing modes are in the vacuum. So roughly, when we zoom in near the horizon and ignore the angular directions, it looks like \( 1+1 \) Rindler spacetime in the ‘Unruh vacuum’ state \( |0_U\rangle \) defined by putting the left-moving modes in the Minkowski vacuum and the right-moving modes in the Rindler vacuum.

f) Calculate all components of the stress tensor in the Unruh vacuum,
\[ \langle T_{\mu \nu} \rangle_U = \langle 0_U | T_{\mu \nu} | 0_U \rangle - \langle 0_M | T_{\mu \nu} | 0_M \rangle. \tag{34} \]
g) Calculate the energy flux \( \langle T_{tx} \rangle \) in the both the Unruh vacuum and the Rindler vacuum. As above, these quantities are defined relative to the Minkowski vacuum. Give a physical interpretation of your result.

7.3 Firewalls

We have seen that the Minkowski vacuum \( |0\rangle_M \) can be expressed as a sum of Rindler modes, which are entangled in very particular way across the surface \( x = 0 \). In this exercise we will show that this entanglement is crucial in order to have a smooth transition from the left or right Rindler wedge to the future interior.

Instead of putting system in the ground state \( |0\rangle_M \), let us put it in the mixed state \( \rho_L \otimes \rho_R \) where \( \rho_{L,R} \) is the thermal density matrix obtained by tracing out the \( R, L \)-wedge in the Minkowski vacuum. Note that this state is fully separable: there is no entanglement. However, for any observer who lives in the \( L,R \) wedge, the state is indistinguishable from the state \( |0\rangle_M \).

Given is the fact that the Minkowski vacuum \( |0\rangle_M \) is annihilated by the operator
\[ c_{\omega k} = \frac{1}{\sqrt{1 - e^{-2\pi\omega}}} \left( b_{R \omega k} - e^{-\pi\omega} b^\dagger_{L \omega(-k)} \right), \tag{35} \]
where the \( b_{L,R} \)'s are the Rindler creation and annihilation operators, and
\[ |0\rangle_M = \bigotimes_{\omega,k} \left[ \sqrt{1 - e^{-2\pi\omega}} \sum_n e^{-\pi\omega n} |n\rangle_{L \omega(-k)} \otimes |n\rangle_{R \omega k} \right]. \tag{36} \]
a) Compute the expectation value of the Minkowski number operator in this state and show that it equals
\[ \langle c^\dagger_{\omega k} c_{\omega k} \rangle_\rho = \frac{2}{(e^{\pi\omega} - e^{-\pi\omega})^2} \tag{37} \]
One can see that this will become a large number for \( \omega \ll 1 \). This means that the state will be quite singular: a Minkowski observer will observe very high energy quanta near the horizon.


8 Exercise class IV

8.1 Scalar field in de Sitter

Consider $d = 3 + 1$ de Sitter space in planar slicing

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2,$$

$$a(t) = H^{-1}e^{Ht}.$$  

(38)

Here $H$ is the Hubble parameter, a fixed constant.

a) Introduce a conformal time coordinate $\tau$ such that the metric becomes

$$ds^2 = a^2(t)(-d\tau^2 + d\vec{x}^2).$$

(39)

Find $a(\tau)$.

b) Show that the massless scalar field equation $\Box \phi = 0$ leads to

$$\phi'' + \frac{2a'}{a} \phi' - \nabla^2 \phi = 0,$$

(40)

where prime denotes differentiation with respect to conformal time. Make sure to use the definition of the d’Alembertian (‘box’) that is valid in curved spacetime.

c) Compare equation (40) to the equation of motion for a massless scalar field in Minkowski spacetime. In what do they differ, and how can this term be interpreted?

We are now going to solve the eom. Define a new field variable $\chi = a(t)\phi$. By definition of the Fourier transform $\chi_{\vec{k}}$, we can write

$$\chi = \int \frac{d^3k}{(2\pi)^3} \left( b_{\vec{k}}\chi_{\vec{k}}(\tau)e^{i\vec{k}\cdot\vec{x}} + b^*_{\vec{k}}\chi^*_{\vec{k}}(\tau)e^{-i\vec{k}\cdot\vec{x}} \right).$$

(41)

d) Show that this eliminates the term $2\frac{a'}{a} \phi'$ so that the eom now reads

$$\chi''_{\vec{k}} = \left( \frac{2}{\tau^2} - k^2 \right) \chi_{\vec{k}}, \quad k = |\vec{k}|.$$  

(42)

The solutions of the previous equation are given by

$$\chi_{\vec{k}}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left( 1 - \frac{i}{k\tau} \right) + \frac{e^{ik\tau}}{\sqrt{2k}} \left( 1 + \frac{i}{k\tau} \right).$$  

(43)

e) Typically one picks $\alpha = 1$ and $\beta = 0$. Explain why this is natural choice of mode function. The associated (de Sitter invariant) vacuum is often called the Bunch-Davies vacuum.

f) After quantizing the field, compute the spatial two-point function in the Bunch-Davies vacuum and bring it into the form

$$\langle 0|\hat{\phi}(\tau, \vec{x})\hat{\phi}(\tau, \vec{y})|0 \rangle = \int \frac{d^3k}{8\pi^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \sigma_k^2.$$  

(44)

g) An observationally relevant quantity is the power spectrum $P(k) \equiv \frac{k^3\sigma_k^2}{2\pi^2}$, which we can measure by looking at the sky. Compute the power spectrum, assuming that $k\tau \ll 1$.4

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4Considering distance scales of observational interest today, let’s say $\vec{x}$ and $\vec{y}$ are two points on the surface of last scattering which can be resolved by CMB observations. We know that the corresponding momenta $k$ were well inside this "frozen" regime at the end of inflation.