THE COOLING OF NEUTRON STARS

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Abstract

The cooling of neutron stars is investigated. To this end we finally arrive at a numerical algorithm treating neutrino and photon radiation away from the star and heat conduction within. We consider the theory required for such a calculation in a general context but determine the parameters specifically for a star consisting purely of a Fermi gas of neutrons. This particular stellar model allows for only one particle species and thus strongly limits the possible cooling processes taking place in the neutron star.

For this model the equation of state is deduced and the general relativistic stellar density profile is found. On this background, which is assumed to be static, we write down the equations governing the energy flow within the star in a general relativistic manner. Three parameters play a role: the heat capacity, the thermal conductivity and the neutrino emission rate. The effect of these parameters on neutron star cooling is described and each is determined for the case of a star consisting of only neutrons.

For accuracy we use a four-point Runge-Kutta method to solve the temperature profiles in time. However, calculating numerical diffusion is unstable for too large a time step, putting significant strain on computer time. To overcome this problem we ignore internal diffusion after the temperature profile obtains an ‘isothermal’ shape and treat the star as a single object with an energy loss, a temperature and heat capacity.

With only bremsstrahlung as a neutrino producing process and photon radiation at the stellar surface, we determine the surface temperature of the star, as measured by an observer at infinity, as a function of time. For the model of a neutron star consisting of only neutrons, we find that for the cooling of the star the neutrino emission is negligible compared to the photon radiation.
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Neutron stars are fascinating, exotic objects. With masses of about a solar mass but with radii of only 8 to 12 kilometres, they are the densest stars in the known universe \cite{1}. Due to their incredibly dense interiors, neutron stars are unique astrophysical laboratories for the exploration of matter under extreme densities. The equation of state, i.e., the relation between pressure and density, is unknown at these densities, which can reach up to several times the nuclear density, \( n_0 \simeq 0.16 \text{ fm}^{-3} \). Under these conditions several novel states of matter, containing new, high-energy degrees of freedom, may be favoured \cite{2}. Hadronic phases, which contain, in addition to nucleons, a Bose condensate of pions or kaons, and/or hyperons, have been proposed and studied in some detail. There is also the exciting possibility that the neutron star core may contain deconfined quark matter.

Neutron stars are created in the aftermath of the gravitational collapse of the core of a massive star of 8 M\(_\odot\) (solar mass) or heavier. At the end of it’s life the star explodes in a Type II supernova, expelling the outer mantle, and the early neutron star remains. An enthralling picture of such a supernova remnant and the neutron star born from it is shown in Figure 1.1. Roughly 50 seconds after the initial shockwaves, in which neutrinos play a big role, the neutrino mean free path length becomes larger than the radius of the neutron star and all produced neutrinos leave the star unhindered, thereby cooling the neutron star.

Starting at the stellar boundary and working inward, a neutron star consists of an atmosphere, an envelope, a crust and a core. The atmosphere, some millimetres to several centimetres thick, is composed of hydrogen and possibly some heavy elements. The composition of the atmosphere determines the spectrum of emitted radiation which is of key importance to measuring the surface temperature. Some neutron stars have extremely high magnetic fields in the atmosphere. In 1992 Duncan and Thompson \cite{3} discussed the formation of so called ‘magnetars’, neutron stars with magnetic fields up to \( 10^{15} \text{ G} \). The magnetic fields of this magnitude can change the radiation emitted significantly.

Beneath the atmosphere lies an envelope of matter not yet completely degenerate. This layer of some ten’s of meters deep acts as a thermal insulator between the hot interior and the atmosphere.

Further down are the crust and the core. In this regime density increases to above the nuclear density. The nucleons are very degenerate and the state of matter changes from nuclei immersed in a quantum liquid of neutrons to nuclei completely dissolved into their constituents, neutrons and protons. It is assumed that the matter is in a superfluid state in this region, forming vortices in the higher layers. In the core itself many unknowns
CHAPTER 1. INTRODUCTION

Figure 1.1: A supernova remnant with surviving neutron star, IC 443. Seen in this false-colour composite image from NASA, the supernova remnant is still glowing across the spectrum, from radio (blue) to optical (red) to x-ray (green) energies. The wide view of IC 443, also known as the Jellyfish nebula, spans about 65 light-years at the supernova remnant’s estimated distance of 5,000 light-years.

arise. As mentioned, exotic condensates might be present and possibly the core consists of deconfined quark matter. According to the work of Diederix in 2007 [4], who used a baryon model with mesons as the particles that are exchanged in an interaction, a phase transition towards a deconfined quark phase occurs within the star.

Observationally, neutron stars pose several problems. The radiation, emitted by the star, is used to deduce the surface temperature. Magnetic fields change the isotropic nature of the radiation however, making measurements troublesome. Furthermore, the composition of the atmosphere, which determines the emitted spectrum, is mainly unknown since narrow spectral lines are not observed. The spectrum of radiation emergent from a neutron star atmosphere can be very different from a blackbody spectrum. This makes fitting the measurements to an atmospheric model to yield the temperature difficult [5].

Neutron star masses, mainly obtained from binary systems, lie mostly within a range of 1 to 1.7 $M_{\odot}$. The upper bound of this narrow range has recently been stretched to 2 $M_{\odot}$ using Shapiro delay as the measurement technique [6]. Besides the temperature and the mass, the age of the neutron star can be measured. For instance by looking at the supernova remnants from which the neutron star is born, but also by the spin-down rate of x-ray pulsars.

From a theoretical point of view, finding the equation of state is one of the main difficulties. For the atmosphere, the envelope and the crust, models are fairly well developed, but despite the current developments in heavy-ion colliders, the equation of state for matter at such densities as arise within a neutron star core is still unresolved. The
equation of state, combined with general relativity, determines the stellar structure such as the mass and density profiles, and thus the total mass and radius of the star.

The equation of state is determined by the model used to describe the matter inside the neutron star. In a particular model we choose which particles we allow for and what interactions act between them. The most simple model, for instance, is a star composed purely of neutrons. In this case no other particles are allowed and thus all particle transforming interactions are forbidden. Adding protons and electrons to our model automatically introduces beta and inverse-beta decay, leading already to a much more physical neutron star model where proton rich regions are gradually converted into neutron rich layers. Adding hyperons and finally deconfined quarks concludes the range of possible particles but choices have to be made in the way the effects of the interactions are described, especially in the more involved particle models.

One link connecting the theoretical work and the experimental findings is a cooling model. Using the stellar structure resulting from a chosen stellar model and consequently the equation of state, the neutrino emission rate, the thermal conductivity and the heat capacity at each point in the star can be computed. These are the parameters describing the diffusion of energy within the star and they determine, combined with photon emission at the stellar surface, the energy radiated away. From supernova theory an initial temperature for the neutron star can be deduced and the surface temperature as a function of the age of the star can be calculated with a numerical model, incorporating the above physics. The results can be compared to the measurements of temperature versus age of several neutron stars to see whether models are favoured or ruled out by observations. Finding a suitable model gives insight in ultra dense matter theory.

It is the cooling model that is investigated in this thesis. For the building and understanding of the numerical model we look at the most simple neutron star model there is: a pure neutron gas star. No other particles are allowed, there is no superfluidity, the star does not rotate nor does it have any magnetic field. No envelope is attached to the interior and we assume a blackbody radiation at the stellar surface. Each of these assumptions is a simplification from actual neutron stars. We believe however, that the main features of the cooling trajectory are determined by the model of the stellar core interior. Even in the most advanced models, more than 99% of the stars mass resides in the core. Although envelopes and superfluidity play significant roles in neutron star cooling, this simple model is a good starting point for cooling calculations.

We describe the theory and calculations required to find the surface temperature as a function of the age for a neutron star in a general manner and do all the explicit calculations for the case of a star consisting of only neutrons. In this model the entire star is made up of a degenerate Fermi gas of neutrons, of which the equation of state and many other properties are well known.

We compute the stellar profiles for the mass and the density in chapter 2 by sketching the general relativistic equations and the Fermi theory that yields the equation of state. This will form the background for interactions that create neutrinos and the processes that transport energy throughout the star. The equations governing this and the parameters involved will be discussed in chapters 3 and 4. With the theory established we will then turn in chapter 5 to the numerical algorithm to compute the cooling of the star. In chapter 6 we will discuss the obtained results and the methods and assumptions used. Finally, conclusions and an outlook to further development are given in chapter 7. Since the calculations in this thesis are carried out for the simple model of a neutron star consisting only of neutrons, no comparison to observations will be made here.
Chapter 2

Stellar model

To determine the cooling trajectory of a neutron star, we first need to find a reasonable model of our star. The total mass and radius, as well as the internal structure of the star, such as the mass and density profile need to be known. To find these, we need to decide what physics to incorporate, which particles to allow for and what many-body physics we introduce. These choices influence the way the star looks (mass, radius, etc.) and the way the relevant parameters behave (heat capacity, neutrino emission, etc.). To compute the relevant profiles of the star, we require two things: the Tolman-Oppenheimer-Volkoff (TOV) equations, the differential equations for the pressure and density as a function of radius in a curved space-time. And secondly, the equation of state (EOS), the relation between density and pressure from microscopic theory. The TOV equations and the equation of state form a closed set of equations which can be solved. From a chosen central density, which is a free parameter in the theory, the mass and density profiles as a function of radius are found, and thus the total stellar mass and radius. These profiles are the necessary starting point for further calculations of the cooling rates.

2.1 TOV equations

In 1939 Tolman, Oppenheimer and Volkoff, [7, 8] showed how to use general relativity to describe the inside of a star. The resulting Tolman-Oppenheimer-Volkoff (TOV) equations can be derived from Einstein’s equations and only need the metric and the form of the stress-energy tensor as physical input. As is often done in stellar computations, we take a perfect fluid stress-energy tensor, implying that the gas the star is made of is approximated to be non-viscous and stress free.

As we need to treat our star relativistically, it is Einstein’s equations that relate space-time curvature to the matter and energy of the star. We choose a spherically symmetric, static space-time, for which the most general case is described by the line element signature [9, 10],

\[
\frac{ds^2}{c^2} = -e^{2\phi(r)}(c dt)^2 + e^{2\lambda(r)} dr^2 + r^2 d\Omega^2. \tag{2.1}
\]

The functions \( \phi(r) \) and \( \lambda(r) \) are radial functions, to be connected to the gravitational potential, and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the metric on a unit sphere. In this space-time we can write down the metric tensor \( g_{\mu\nu} \), the Ricci tensor \( R_{\mu\nu} \) and the Ricci scalar \( R \),
CHAPTER 2. STELLAR MODEL

to find the Einstein tensor,

\[ G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}. \tag{2.2} \]

In addition to the Einstein tensor we need the stress energy tensor \( T \), describing a perfect fluid,

\[ T_{\mu\nu} = \rho u_\mu u_\nu + P \left( g_{\mu\nu} + \frac{u_\mu u_\nu}{c^2} \right). \tag{2.3} \]

Here \( u_\mu \) is the particle four velocity, which, for a fluid in its rest frame, is \((1, 0, 0, 0)\). The pressure and the energy density are denoted by \( P \) and \( \rho \) respectively. The Einstein equations then relate the two tensors by

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.4} \]

with \( G \) Newton’s gravitational constant. This relation yields four equations since both tensors are diagonal \( 4 \times 4 \) matrices. Two of these equations are identical and we are left with

\[ \frac{8\pi G}{c^4} \rho(r) = e^{-2\lambda(r)} \left( \frac{2\lambda'(r)}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}, \tag{2.5a} \]

\[ \frac{8\pi G}{c^4} P(r) = e^{-2\lambda(r)} \left( \frac{2\phi'(r)}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}, \tag{2.5b} \]

\[ \frac{8\pi G}{c^4} P(r) = e^{-2\lambda(r)} \left( \frac{\phi'(r) - \lambda'(r)}{r} - \lambda'(r)\phi'(r) + \phi'(r)^2 + \phi''(r) \right). \tag{2.5c} \]

We know that in the vacuum outside the star the metric \((2.1)\) is equal to the Schwarzschild solution,

\[ ds^2 = - \left( 1 - \frac{2GM}{c^2r} \right) (c dt)^2 + \left( 1 - \frac{2GM}{c^2r} \right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.6} \]

in which \( M \) denotes the total mass of the star. This suggests that within the star we can define

\[ e^{\lambda} = \left( 1 - \frac{2Gm(r)}{c^2r} \right)^{-\frac{1}{2}}, \tag{2.7} \]

which reduces equation \((2.5a)\) into the more familiar form,

\[ \frac{dm(r)}{dr} = \frac{4\pi}{c^2} \rho(r) r^2, \tag{2.8} \]

where \( m(r) \) is the mass contained inside the sphere of radius \( r \). The vacuum solution also fixes the integration constant of \( m(r) \) in equation \((2.8)\) at zero by connecting the interior solution to the exterior solution at the stellar boundary, \( r = R \). We see that the mass \( M \) in the Schwarzschild solution is to be identified with \( m(R) \) where \( R \) is the radius of the star,

\[ M = m(R) = \int_0^R 4\pi \rho(r) r^2 dr. \tag{2.9} \]
Note that $M$ is the total gravitating mass. This is the mass as seen by an outside observer. The mass-energy of the star without gravitational energy, the *proper* mass, is the integral of $\rho(r)$ times the proper volume element, $d\tilde{V} = 4\pi r^2 d\tilde{r} = 4\pi r^2 e^{\lambda} dr$. When interested in other integrals, for instance to compute the number of particles, we should use the proper volume element as well. For future reference, the proper time element is $d\tau = e^\phi dt$.

Now, using the definition of equation (2.7) we find for equation (2.5b),

$$
\frac{d\phi}{dr} = \frac{Gm(r) + \frac{4\pi}{3} Gr^3 P(r)}{r(c^2r - 2Gm(r))}.
$$

(2.10)

The third relation, equation (2.5c), is more difficult but we can use the energy conservation principle instead. Since the comoving derivative of the stress energy tensor equals zero,

$$
\nabla_\mu T^{\mu\nu} = 0,
$$

(2.11)

this holds for the Einstein tensor as well. The only non-trivial relation, given by $\nu = r$, yields

$$
\frac{dP(r)}{dr} = -(P(r) + \rho(r)) \frac{d\phi(r)}{dr},
$$

(2.12)

and, by inserting equation (2.10), we obtain

$$
\frac{dP(r)}{dr} = -(P(r) + \rho(r)) \left( \frac{Gm(r) + \frac{4\pi}{3} Gr^3 P(r)}{r(c^2r - 2Gm(r))} \right).
$$

(2.13)

Equation (2.13), together with equation (2.8), are known as the TOV equations. As a check it can be shown that they satisfy equation (2.5c).

### 2.2 Equation of state

To solve the TOV equations and find, for example, the density profile, we need the relation between pressure and energy density, i.e., the equation of state (EOS). The equation of state is determined by the model used to describe the matter inside the neutron star.

In this thesis we choose the model of a Fermi gas of neutrons. No other particles are allowed and we treat the gas as ideal. When we look at the neutrino emission rate of our star, later on, we do introduce an interaction between the neutrons where neutrinos are created, but we assume that this interaction does not influence the equation of state. Another simplification is to treat the star at zero temperature. Since the kinetic energy of the particles in the Fermi gas is much higher than the temperature of the star, this assumption can safely be made to calculate the EOS. When computing the relevant parameters in the star, such as the heat capacity, we use a low temperature expansion. Although the star loses energy due to neutrino and photon emission, we do not acknowledge this loss in the structure of the star. We compute the stellar profiles once and use these static mass and density profiles to calculate cooling. This is consistent with treating the star at zero temperature. The thermal energy is negligible to the total energy of the star and the reduction in energy due to cooling does not influence the equation of state significantly.
CHAPTER 2. STELLAR MODEL

Also, since our entire star is made up of a neutron gas, we do not introduce regions such as a core or a crust. Using only one model for the whole system makes calculations much more manageable, but on the boundary where densities are relatively low, the assumption of a degenerate Fermi gas is not entirely accurate.

Using all the above assumptions and simplifications makes finding the EOS for our star very tractable, since the theory of a degenerate Fermi gas is well known. The key concept is the Fermi distribution function $f(\epsilon)$, giving the average number of particles with energy $\epsilon$,

$$f(\epsilon) = \frac{1}{e^{(\epsilon - \mu)/k_B T} + 1}.$$  \hspace{1cm} (2.14)

At lower and lower temperatures $T$, this function becomes more and more like a step function at chemical potential $\mu$. We can use this to relate the Fermi energy $\epsilon_F$, the Fermi momentum $k_F$, and the particle density $n$, of the Fermi gas. The particle density is the integral of the Fermi distribution over the momenta and for the zero temperature limit we can write,

$$n(r) = 2\int_0^\infty \frac{d^3k}{(2\pi)^3} \theta(k-k_F(r)) = 2\int_0^{k_F(r)} \frac{dk}{(2\pi)^3} 4\pi k^2 = \frac{k_F(r)^3}{3\pi^2},$$  \hspace{1cm} (2.15)

and hence,

$$k_F(r) = (3\pi^2 n(r))^{1/3}.$$  \hspace{1cm} (2.16)

The derivative of the Fermi momentum to particle density is then

$$\frac{\partial k_F}{\partial n} = \frac{1}{(3\pi^2 n(r))^{1/3}} = \frac{1}{3} \frac{(3\pi^2)^{1/3}}{n^{2/3}} = \frac{\pi^2}{k_F^2}.$$  \hspace{1cm} (2.17)

The pressure now, is written as minus the derivative of energy with respect to volume, and we find,

$$P = -\frac{\partial E}{\partial V} = -\frac{\partial}{\partial V}(\rho V) = -V \frac{\partial n}{\partial V} \frac{\partial \rho}{\partial n} - \rho = n \frac{\partial \rho}{\partial n} - \rho.$$  \hspace{1cm} (2.18)

The derivative of the energy density to particle density, for a Fermi gas is,

$$\frac{\partial \rho}{\partial n} = \frac{\partial}{\partial n} \left( \int_0^{k_F} \frac{k^2}{\pi^2} \epsilon(k)dk \right) = \frac{\partial k_F}{\partial n} \frac{\partial}{\partial k_F} \left( \int_0^{k_F} \frac{k^2}{\pi^2} \epsilon(k)dk \right)$$

$$= \frac{\partial k_F}{\partial n} \frac{k_F^2}{\pi^2} \epsilon(k_F) = \epsilon(k_F).$$  \hspace{1cm} (2.19)

We define the energy at the Fermi momentum to be the Fermi energy, $\epsilon(k_F) \equiv \epsilon_F$, which yields for equation (2.18),

$$P = n\epsilon_F - \rho.$$  \hspace{1cm} (2.20)

By writing the Fermi energy and the energy density as a function of particle density we can bring everything together. For a relativistic Fermi gas the Fermi energy is

$$\epsilon_F = \sqrt{c^2 h^2 k_F^2 + m^2 c^4} = mc^2 \sqrt{an^{2/3} + 1}, \quad \text{with} \quad a = \frac{(3\pi^2)^{2/3} h^2}{m^2 c^2},$$  \hspace{1cm} (2.21)
and \( m \) denotes the rest mass of the neutrons. Using this, the energy density is

\[
\rho = \int_0^{k_F} \frac{k^2}{\pi^2} \varepsilon(k) \, dk = \int_0^{k_F} \frac{k^2}{\pi^2} \sqrt{c^2 k^2 + m^2 c^4} \, dk
\]

\[
= \frac{3}{8} mc^2 \left[ \sqrt{an^{2/3} + 1} \left( \frac{n^{1/3}}{a} + 2n \right) - \frac{\arcsinh \sqrt{an^{1/3}}}{a^{3/2}} \right]. \tag{2.22}
\]

We now have an expression for the pressure in terms of the Fermi energy and the energy density, all of which can be expressed in terms of the particle density. With this result we can solve the TOV equations and solve for the particle density.

### 2.3 Stellar profiles

Our main objective is to find an equation for the particle density and to this end we rewrite the differential equation for the pressure, equation (2.12),

\[
\frac{dP}{dr} = \frac{n \, d\epsilon_F}{dr} = n \, \frac{d\epsilon}{dn} \frac{dn}{dr} - \frac{dp}{dn} \frac{dn}{dr} = n \, \frac{d\epsilon_F}{dn} \frac{dn}{dr}, \tag{2.23}
\]

From equation (2.19) we know that \( \frac{dp}{dn} = \epsilon_F \) and thus,

\[
\frac{dP}{dr} = n \, \frac{d\epsilon_F}{dn} \frac{dn}{dr} = \frac{3 \epsilon_F}{am^2 c^4 n^{2/3}} \frac{dP}{dr} \tag{2.24}
\]

which we can rewrite into

\[
\frac{dn}{dr} = \frac{1}{n} \left( \frac{d\epsilon_F}{dn} \right)^{-1} \frac{dP}{dr} = \frac{3 \epsilon_F}{am^2 c^4} \frac{dP}{dr} = -\frac{3n^{1/3} \epsilon_F}{am^2 c^4} \left[ \frac{Gm(r) + \frac{4 \pi}{3} Gr^3 (n \epsilon_F - \rho)}{r(c^2 r - 2Gm(r))} \right]. \tag{2.25}
\]

This is the equation we have to solve to find the particle density \( n \) as a function of radius. This can be done numerically and the only input necessary is a particle density in the centre of the star. From the density we find the mass, energy and energy density at each radial step and, moving outwards, the entire stellar structure is found. This yields a certain total mass and radius for the star, depending on the central particle density chosen. The relation between the resulting mass and radius as a function of the central particle density is plotted in Figure 2.1. In this figure the central particle density starts at \( 10^{38} \) cm\(^{-3} \) (upper left), and increases along the curve toward \( 10^{40} \) cm\(^{-3} \) (lowest point). It can be seen that although the central particle density is continuously increased, a maximum mass exists. For higher central particle densities the total mass of the star decreases again.

The maximum mass thus found is the mass above which the star would collapse. The Fermi pressure is too small to withstand the gravitational pull for those masses. Solutions for equation (2.25) exist for central particle densities higher than that yielding the maximum mass star, as can be seen by the points curving back in Figure 2.1. These solutions however, are unstable since \( \frac{dM}{dn_c} < 0 \) in this region.

In all the further calculations and results of this thesis we used the structure found for the maximum mass star, which has \( M = 0.767 \, M_\odot \), \( R = 8.77 \) km and the central particle density \( n_c = 2.92 \times 10^{39} \) cm\(^{-3} \). The particle density profile and the mass profile of this star are given in Figures 2.2 and 2.3 respectively.
Figure 2.1: The total mass and radius of the star as a function of central particle density. The central particle density ranges from $10^{38}$ cm$^{-3}$ (upper left) to $10^{40}$ cm$^{-3}$ (lowest point). The mass is maximal for $n_c = 2.92 \times 10^{39}$ cm$^{-3}$ and is $M = 0.767 M_\odot$, with a radius of $R = 8.77$ km.

For the calculation of these profiles an accurate grid has to be used in order to avoid numerical problems at the stellar boundary. From equation (2.25), the particle density should approach zero asymptotically. However, the radial step size has to approach zero to find this. For a non-zero step size, the particle density at some point becomes negative. The last positive value found is what we define to be the stellar boundary $R$. For later calculations where computer time becomes more and more of a problem the grid thus found is thinned out. The remaining grid points are displayed as points on the curves in Figures 2.2 and 2.3.

For a more advanced model containing several particle species and elaborate many-body physics, finding the structure profiles of the star is not as easy as it is for the neutron Fermi gas, described above. However, we note that all the curvature and gravity is incorporated in the TOV equations and not in the equation of state. With it’s origin in microscopic theory, the EOS has no relation with gravity at all. This is illustrated by the fact that we can always do a local Lorentz transformation such that we have a locally flat metric. Adding to this that we are effectively applying a local-density approximation, we can use the EOS of a homogeneous gas. Hereby, finding the stellar structure for a complex model of interacting particles in a gravitational field is reduced to finding the equation of state of this system of particles. As mentioned before, this is by no means an easy task but a numerical relation will do. With the energy densities expressed in terms of the particle densities, we can use the (numerical) equation of state to find the pressure in terms of the particle densities and insert everything into the TOV equation. The total particle density profile found will then be composed of all the particle density profiles of all the particles in the stellar model.
Figure 2.2: The particle density $n(r)$ as a function of radius for the maximum mass star. At $r = R = 8.77$ km the density becomes zero. The solid line is the curve found from solving equation (2.25) numerically and the dots represent the thinned out grid, used for computations later in this text.

Figure 2.3: The mass $m(r)$ as function of radius for the maximum mass star. The mass $m(r)$ depicted is the mass within the sphere of radius $r$. Again, the solid line is the solution from solving equation (2.25) together with equation (2.8). The dots represent the thinned out grid, used for computations later in this text.
Chapter 3

Energy equations

The cooling of our star is due to two mechanisms. Firstly, in particle reactions, neutrinos are produced which leave the star and therefore take energy away from it. Secondly, the star emits photons at its boundary, also resulting in an energy loss. To describe the cooling of our star we thus need to understand these two mechanisms and determine how the temperature throughout the star is affected by them.

The physics that describes the temperature throughout the star as a reaction to neutrino and photon emission is diffusion with a loss term. Classically the equation is

\[ C_V \frac{dT}{dt} = \kappa \nabla^2 T - q, \]

(3.1)

where \( T \) is the temperature, \( C_V \) the heat capacity per unit volume, \( \kappa \) the thermal conductivity and \( q \) an added loss term, which in our case describes the neutrino losses.

It can be seen that the cooling, the \( \frac{dT}{dt} \) term times the heat capacity, is directly proportional to the neutrino emissions. When the other term, the Laplacian of the temperature field, is non-zero, heat conduction is induced with the thermal conductivity as a proportionality constant.

Usually, equation (3.1) is split in two equations using the heat current \( J \),

\[ \nabla \cdot J = -C_V \frac{dT}{dt} - q, \]

(3.2a)

\[ \kappa^2 \nabla T = -J. \]

(3.2b)

This makes the system easier to handle numerically since it contains only first order derivatives. It is also a little more transparent how the physics of the diffusion works in further calculations.

3.1 Relativistic diffusion

The energy diffusion equations (3.1) and (3.2) are insufficient for our star since they do not describe the energy flow in a relativistic, curved space-time. Deriving the relativistic form of these equations in the metric of equation (2.1) can be done in several ways. In 1966 Thorne [11] derived the equations completely based on energy conservation arguments, in which he incorporated redshift factors \( e^\phi \), and proper radial distances and proper volume elements, using \( e^\lambda \).
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The resulting equations are, written as to correspond with the classical equations (3.2),

\[ \frac{dL e^{2\phi}}{dr} = -4\pi r^2 e^\lambda \left\{ C_V \frac{dT e^\phi}{dt} + e^{2\phi} q \right\}, \]  
(3.3a)

\[ \frac{dT e^\phi}{dr} = -\frac{L e^\phi}{4\pi r^2} e^\lambda. \]  
(3.3b)

The gradient is replaced by its form in spherical coordinates and the luminosity \( L \) is defined through

\[ J = \frac{L}{4\pi r^2}, \]  
(3.4)

scaling out the surface of the energy flux. In our spherically symmetric system, only the radial component of \( J \) is non-zero and we thus have only a non-zero radial luminosity, \( L \), yielding the equations above.

Another way of deriving the same equations is from an action principle of diffusion, in the general relativistic form. The motivation for this is a more intuitive derivation, since redshift and proper radial distances, \( e^\phi \) and \( e^\lambda \), do not have to be inserted by hand, but follow from theory through the proper four-volume element \( d^4 x \sqrt{-g} \), where \( g \) is the determinant of the metric. In this approach, we must use Langevin's theory for dissipative systems. It is the dissipative term, \( dT/dt \) in the diffusion equation that causes trouble when using a normal action. It is however, instructive to show what would happen with such an action, also to introduce concepts and notation.

Naively, the action to describe diffusion of a scalar field \( \phi \) with a sink term \( Q \) is

\[ S = \int d^4 x \sqrt{-g} \frac{1}{2} \left\{ \phi C_V n^\mu \partial_\mu \phi + \kappa \gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + Q \phi \right\}. \]  
(3.5)

In this expression, the metric is split up in a time piece and a space piece to accommodate the structure of the diffusion equation. The inverse metric tensor \( g^{\mu\nu} = n^\mu n^\nu + \gamma^{\mu\nu} \), where \( n^\mu = (e^{-\phi}, 0, 0, 0) \) and \( \gamma^{\mu\nu} \) is the inverse metric with the first row and column replaced by zero’s, as to represent the spacial part. In the second line we used integration by parts to rewrite the second term in the Lagrangian.

Varying this action does not yield the diffusion equations (3.3) and it is therefore incorrect. The time derivative drops out at some point. A logical consequence of not being invariant. In Langevin’s theory the action is the equation of motion squared, with an appropriate constant in front. In our case the action is

\[ S = \int d^4 x \frac{\sqrt{-g}}{C_V} \frac{1}{2} \left\{ \left( C_V n^\mu \partial_\mu - \frac{\kappa}{\sqrt{-g}} \partial_\mu \sqrt{-g} \gamma^{\mu\nu} \partial_\nu \right) \phi + Q \right\}^2. \]  
(3.6)

In a dissipative system there is noise and this noise determines the constant in the action. The underlying theory is described by Kobe et al [12].

Varying the action (3.6) obviously yields the equation of motion,

\[ \left( C_V n^\mu \partial_\mu - \frac{\kappa}{\sqrt{-g}} \partial_\mu \sqrt{-g} \gamma^{\mu\nu} \partial_\nu \right) \phi + Q = 0. \]  
(3.7)
It should now be mentioned that the scalar field, which is of course related to our temperature, is in fact $T e^\phi$, the redshifted temperature, since this is a scalar. In general relativistic theory not $T = \text{constant}$, but $T e^\phi = \text{constant}$ defines an isothermal configuration. Also, $Q = e^\phi q$. Writing out the equation of motion to find the energy equations we use that
\[
\sqrt{-g} = \sqrt{-\det g_{\mu\nu}} = e^{\phi + \lambda} r^2 \sin \theta.  \quad (3.8)
\]
Now, because only the $\mu = 0$ element of $n^\mu$ is non-zero and only the time and radial derivatives of $T$ are non-zero, we have
\[
C_V e^{-\phi} \partial_t T e^\phi + e^\phi q = \frac{\kappa}{e^{\phi + \lambda} r^2 \sin \theta} \partial_r \left( (e^{\phi + \lambda} r^2 \sin \theta e^{-2\lambda} \partial_r T e^\phi) \right). \quad (3.9)
\]
A little rewriting, multiplying with $4\pi$ and placing the conductivity $\kappa$ inside the derivative gives,
\[
4\pi r^2 e^\lambda \left\{ e^{2\phi} q + C_V \partial_t T e^\phi \right\} = \partial_r \left( e^{\phi - \lambda} 4\pi r^2 \kappa \partial_r T e^\phi \right). \quad (3.10)
\]
The term within brackets on the right hand side of this equation we define to be the luminosity, $L e^{2\phi}$. For clarity, we rewrite the equations governing the energy throughout the star into the form we need to solve for the neutron star cooling,
\[
\frac{dT e^\phi}{dt} = -\frac{1}{C_V} \left\{ \frac{dLe^{2\phi}}{dr} \frac{1}{4\pi r^2 e^\lambda} + e^{2\phi} q \right\}, \quad (3.11a)
\]
with,
\[
Le^{2\phi} = -\frac{dT e^\phi}{dr} e^\phi 4\pi r^2 \kappa. \quad (3.11b)
\]
As boundary conditions for these equations we require that the derivative of temperature with respect to radius is zero in the stellar centre at $r = 0$. The second constraint is that no flux can go through the stellar centre at $r = 0$.

In addition to the equations above, describing diffusion within the star, coupled to neutrino emissions, we add a photon luminosity term. At the stellar boundary the star emits photons and we incorporate this term in an extra luminosity term at $r = R$. We assume a black body spectrum which, with the appropriate redshift and proper distance functions, yields the following photon luminosity
\[
L^\gamma e^{2\phi}(R) = 4\pi R^2 \sigma T^4 e^{2\phi} \frac{e^{2\phi}}{e^\lambda}, \quad (3.12)
\]
where $\sigma$ is the Stefan-Boltzmann constant.

Apart from the photon luminosity, the equations (3.11) are completely general for neutron star cooling. No matter what model is chosen to describe the star, these are the equations to be used for calculating the diffusion and emission of energy. The parameters used in these equations however, do depend on the chosen model of the star.
Chapter 4
Parameters

In the energy equations \((3.11)\), describing the cooling of our neutron star, three parameters appear: the heat capacity per unit volume \(C_V\), the thermal conductivity \(\kappa\) and the neutrino emissivity \(q\). These parameters all depend on the model used to describe the star, as mentioned in the introduction and chapter 2. Apart from the equation of state, it is in these parameters that the physics of the neutron star is incorporated. As described, our model consists of neutrons only and since other particles are forbidden, the only interaction allowed is neutron-neutron bremsstrahlung.

Before we look into the three parameters themselves we quickly introduce the effective mass and the Fermi velocity, which we will use along the way. The effective mass, which is defined as

\[
m^* = \frac{p_F}{v_F}.
\] (4.1)

depends on the Fermi velocity, which in turn is defined as

\[
v_F = \left(\frac{d\epsilon}{dp}\right)_{p=p_F}.
\] (4.2)

Using that \(p = \hbar k\), we find

\[
m^* = \frac{\hbar^2 k_F}{\left(\frac{d\epsilon}{dk}\right)_{k=k_F}} = \frac{1}{c^2} \sqrt{c^2 \hbar^4 k^2 + m^2 c^4} = mc^2 \sqrt{an^2/3 + 1} = \frac{\epsilon_F}{c^2}.
\] (4.3)

In the last equation, the factor \(a\) is, as before,

\[
a = \frac{(3\pi^2)^{2/3}\hbar^2}{m^2 c^2},
\] (4.4)

and \(n\) is the particle density.

4.1 Heat capacity

In the model of a star consisting only of neutrons, the heat capacity is that of the Fermi liquid of neutrons, given as a low temperature expansion. We are interested in the first order term to find a heat capacity scaling linearly with temperature \(T\).
Chapter 4. Parameters

In general, the heat capacity, per unit volume, is given as

\[ C_V = \frac{1}{V} \frac{\partial E}{\partial T}, \]  

(4.5)

so we must find the energy of the Fermi liquid, as a function of temperature,

\[ \frac{E}{V} = \int_0^\infty d\epsilon \, f(\epsilon) \rho(\epsilon). \]  

(4.6)

Here \( \rho(\epsilon) \) is the density of states with energy \( \epsilon \) and \( f(\epsilon) \) is the Fermi distribution of particles with energy \( \epsilon \) and chemical potential \( \mu \),

\[ f(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} + 1}. \]  

(4.7)

To compute the integral (4.6) we follow the approach of Sólyom \[14\] and write

\[ \epsilon \rho(\epsilon) = g(\epsilon) \text{ with,} \]

\[ G(\epsilon) = \int_0^\epsilon d\epsilon' g(\epsilon'), \quad g(\epsilon) = \frac{dG(\epsilon)}{d\epsilon}. \]  

(4.8)

Integrating by parts gives,

\[ \frac{E}{V} = [f(\epsilon)G(\epsilon)]_0^\infty + \int_0^\infty d\epsilon \, G(\epsilon) \left( -\frac{df(\epsilon)}{d\epsilon} \right), \]  

(4.9)

where the boundary term vanishes since \( f(\epsilon) \) goes to zero as \( \epsilon \) goes to infinity and \( G(0) = 0 \) by definition.

In the limit where the temperature goes to zero, the derivative of the Fermi distribution becomes a delta function. However, we are interested in low but finite temperatures and therefore expand the function \( G(\epsilon) \) around \( \epsilon = \mu \); the so called Sommerfeld expansion,

\[ G(\epsilon) = G(\mu) + G'(\mu)(\epsilon - \mu) + \frac{1}{2} G''(\mu)(\epsilon - \mu)^2 + \ldots. \]  

(4.10)

This yields,

\[ \frac{E}{V} = G(\mu) \int_0^\infty d\epsilon \, \left( -\frac{df(\epsilon)}{d\epsilon} \right) + G'(\mu) \int_0^\infty d\epsilon \, (\epsilon - \mu) \left( -\frac{df(\epsilon)}{d\epsilon} \right) \]

\[ + \frac{1}{2} G''(\mu) \int_0^\infty d\epsilon \, (\epsilon - \mu)^2 \left( -\frac{df(\epsilon)}{d\epsilon} \right) + \ldots. \]  

(4.11)

In the first term, the integral over the derivative of the Fermi distribution is 1, if we integrate from minus infinity to plus infinity. This can be done since the derivative is sharply peaked around \( \epsilon = \mu \). At the odd-order terms of \( (\epsilon - \mu) \) then become zero because the integral is odd, and we are left with

\[ \frac{E}{V} = G(\mu) + \frac{1}{2} G''(\mu) \int_{-\infty}^\infty d\epsilon \, (\epsilon - \mu)^2 \left( -\frac{df(\epsilon)}{d\epsilon} \right) + \ldots. \]  

(4.12)

Finally, the remaining integral can be done,

\[ \frac{1}{2} \int_{-\infty}^\infty d\epsilon \, (\epsilon - \mu)^2 \left( -\frac{df(\epsilon)}{d\epsilon} \right) = (k_B T)^2 \int_0^\infty dx \, \frac{x^2 e^x}{(e^x + 1)^2} \]

\[ = (k_B T)^2 \zeta(2) = (k_B T)^2 \frac{\pi^2}{6}, \]  

(4.13)
4.1. HEAT CAPACITY

and for the energy, up to second order in temperature, we find,

$$\frac{E}{V} = \int_0^\mu d\epsilon \rho(\epsilon) + \frac{\pi^2}{6} (k_B T)^2 \left[ \frac{d}{d\epsilon} \rho(\epsilon) \right]_{\epsilon = \mu}. \quad (4.14)$$

Expanding this equation again around $\mu = \epsilon_F$, we get

$$\frac{E}{V} = \int_0^{\epsilon_F} d\epsilon \rho(\epsilon) + (\mu - \epsilon_F) \rho(\epsilon_F) + \frac{\pi^2}{6} (k_B T)^2 \left[ \rho(\epsilon_F) + \epsilon_F \rho'(\epsilon_F) \right]. \quad (4.15)$$

At this point, we need an expression for the chemical potential $\mu$ in terms of the Fermi energy, again as a low temperature expansion. To do this we use the expression for the particle density and make another Sommerfeld expansion.

$$n = \int_0^\infty d\epsilon f(\epsilon) \rho(\epsilon) = \int_0^\infty d\epsilon \mathcal{P}(\epsilon) \left( -\frac{df(\epsilon)}{d\epsilon} \right)$$

$$= \mathcal{P}(\mu) + \frac{1}{2} \mathcal{P}''(\mu) \int_{-\infty}^\infty d\epsilon (\epsilon - \mu)^2 \left( -\frac{df(\epsilon)}{d\epsilon} \right) + \ldots$$

$$= \int_0^\mu d\epsilon \rho(\epsilon) + \frac{\pi^2}{6} (k_B T)^2 \left[ \frac{d}{d\epsilon} \rho(\epsilon) \right]_{\epsilon = \mu}$$

$$= \int_0^{\epsilon_F} d\epsilon \rho(\epsilon) + (\mu - \epsilon_F) \rho(\epsilon_F) + \frac{\pi^2}{6} (k_B T)^2 \rho'(\epsilon_F). \quad (4.16)$$

In this calculation $\mathcal{P}$ is the integral of the density of states $\rho$, and the rest of the expansions and integrals are the same as for the energy, computed above. Since the first term in the final expression, the integral of the density of states, is just the particle density $n$, the following terms must cancel. From this we find

$$\mu - \epsilon_F = -\frac{\pi^2}{6} (k_B T)^2 \frac{\rho'(\epsilon_F)}{\rho(\epsilon_F)}. \quad (4.17)$$

This expression for the chemical potential minus the Fermi energy, we can insert back into equation (4.15) which gives

$$\frac{E}{V} = \int_0^{\epsilon_F} d\epsilon \rho(\epsilon) - \frac{\pi^2}{12} (k_B T)^2 \rho(\epsilon_F) + \frac{\pi^2}{6} (k_B T)^2 \left[ \rho(\epsilon_F) + \epsilon_F \rho'(\epsilon_F) \right]$$

$$= \int_0^{\epsilon_F} d\epsilon \rho(\epsilon) + \frac{\pi^2}{6} (k_B T)^2 \rho(\epsilon_F). \quad (4.18)$$

Finally, we can take the derivative of the energy to temperature to find the heat capacity,

$$C_V = \frac{1}{V} \frac{\partial E}{\partial T} = \frac{\pi^2}{3} \rho(\epsilon_F) k_B^2 T. \quad (4.19)$$

In this expression for the heat capacity we have not yet used the form of the density of states, which depends on the dispersion relation. For a Fermi gas, the dispersion relation is different for the classical and the relativistic case. To find the relativistic density of states for our star, we consider neutrons with an energy between $\epsilon$ and $\epsilon + d\epsilon$, which fill a shell in $k$-space between $k$ and $k + dk$, of volume

$$\frac{4\pi}{3} (k + dk)^3 \approx 4\pi k^2 dk. \quad (4.20)$$
The number of allowed $k$-points in this region is
\[ V = \frac{4\pi k^2}{(2\pi)^3} \, dk, \quad (4.21) \]
and thus, the number of states per unit volume in this region is
\[ \rho(\epsilon)d\epsilon = \frac{2}{(2\pi)^3} 4\pi k^2 \, dk. \quad (4.22) \]
The extra factor 2 comes from the two possible spin states.
Since we know how the energy $\epsilon$ depends on momentum through the dispersion, equation (2.21), the density of states at the Fermi energy becomes
\[ \rho(\epsilon_F) = \frac{k_F}{\pi^2 c^2 \hbar^2} \sqrt{\frac{2}{\pi^2} \frac{k_F^2}{\hbar^2} + \frac{m^* c^4}{\pi^2 \hbar^2}} \approx \frac{m^* k_F}{\pi^2 \hbar^2}. \quad (4.23) \]
Using this, the heat capacity becomes
\[ C_V = \frac{k_F}{3\pi^2 c^2 \hbar^2} k_B T \approx \frac{m^* k_F}{3\pi^2 \hbar^2} k_B T \approx \frac{m}{3\pi^2} \sqrt{\pi^2 n^{1/3} + 1} \left(3\pi^2 n^{1/3} k_B^2 T\right). \quad (4.24) \]
a function of density and proportional to temperature, as desired.
Using the density profile from the TOV equations and a constant temperature of $T = 10^8$ K, a reasonable initial temperature for a neutron star, we find the curve as depicted in Figure 4.1(a), at the end of this chapter.
For more sophisticated stellar models, consisting of more types of particles, the heat capacity is simply the sum of the heat capacities of each particle species present. This makes expanding the model relatively simple when the heat capacity is concerned.

4.2 Thermal conductivity

A temperature gradient in a medium drives a current. The coefficient relating the temperature gradient and the current is the thermal conductivity $\kappa$, through
\[ \mathbf{J} = -\kappa \nabla T. \quad (4.25) \]
All currents in the system can be expressed in terms of the distribution function. The particle current and the energy current are
\[ \mathbf{J}_N = \int \frac{dk}{4\pi^3} \mathbf{v}_k f(k) \quad \text{and} \quad \mathbf{J}_E = \int \frac{dk}{4\pi^3} \epsilon_k \mathbf{v}_k f(k), \quad (4.26) \]
respectively. The Fermi velocity $\mathbf{v}_k$ is the particle velocity corresponding to the Fermi momentum. From the two currents above we can find the heat current using the thermodynamic identity $dQ = dE - \mu dN$. Expressed in currents this yields
\[ \mathbf{J}_Q = \mathbf{J}_E - \mu \mathbf{J}_N = \int \frac{dk}{4\pi^3} (\epsilon_k - \mu) \mathbf{v}_k f(k). \quad (4.27) \]
We now have to find the correct distribution function $f$ to calculate the heat current and thus the thermal conductivity. In thermal equilibrium, where $f = f_0$, the current is of
course zero. Since we are interested in the non-equilibrium situation where a temperature gradient drives a current we have to find the non-equilibrium distribution function for this system. To do so we look at the Boltzmann equation, which describes the change of the distribution function due to all the processes in the medium. It is our aim to find an expression for the distribution function expanded in terms of $\nabla T$, to match the diffusion equation, (4.25), and to find the thermal conductivity $\kappa$.

### 4.2.1 Boltzmann equation

To determine the non-equilibrium distribution function, the phase-space motion of the particles inside the volume element $d\mathbf{r}d\mathbf{k}$ around point $\mathbf{r}, \mathbf{k}$ at a certain time $t$ is considered. If no collisions occur all the particles will, after time $dt$, be in the region $d\mathbf{r'}d\mathbf{k'}$ at position $\mathbf{r'}, \mathbf{k'}$, with $\mathbf{r'} = \mathbf{r} + \mathbf{v}dt$ and $\mathbf{k'} = \mathbf{k} + \mathbf{v}dt$. Because of the conservation of particle number,

$$f(\mathbf{r}, \mathbf{k}, t)d\mathbf{r}d\mathbf{k} = f(\mathbf{r} + \mathbf{v}dt, \mathbf{k} + \mathbf{v}dt, t + dt)d\mathbf{r'}d\mathbf{k'},$$

(4.28)

and because $d\mathbf{r}d\mathbf{k} = d\mathbf{r'}d\mathbf{k'}$, from Liouville’s theorem, we can write for small time differences, in the linear order expansion

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{k}} = 0,$$

(4.29)

which is of course the continuity equation in phase-space.

Note that

$$\mathbf{v}_k = \frac{1}{\hbar} \frac{\partial \epsilon_k}{\partial \mathbf{k}} \quad \text{and} \quad \mathbf{k} = -\frac{e}{\hbar} (\mathbf{E} + \mathbf{v}_k \times \mathbf{B}).$$

(4.30)

Since we assume no electric or magnetic field in our star, the second term vanishes. If we also look at the semi-statical case, where the distribution function does not change with time, we are only left with the middle term of equation (4.29), containing the derivative with respect to position.

The next step is to expand the actual distribution function around the local equilibrium function $f_0$ with a small perturbation, $f = f_0 + \delta f$. This perturbation is small and can be assumed not to vary with position. If we further assume the chemical potential not to vary with position, the partial derivative can be written as

$$\frac{\partial f_0}{\partial \mathbf{r}} = \frac{\epsilon_k - \mu}{T} \nabla T \left( -\frac{\partial f_0}{\partial \epsilon_k} \right).$$

(4.31)

In our medium however, there are interactions since particles collide with other particles. This can scatter particles in and out of their phase-space trajectories and on the right side of equation (4.29) a term is added to incorporated this,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{k}} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}},$$

(4.32)

the so-called collision integral. Equation (4.32) is the Boltzmann equation.

Using the expansion for the distribution function and the above assumptions, the Boltzmann equation simplifies enormously and we are left with

$$\frac{\epsilon_k - \mu}{T} \left( -\frac{\partial f_0(k)}{\partial \epsilon_k} \right) \mathbf{v}_k \cdot \nabla T = \left( \frac{\partial f(k)}{\partial t} \right)_{\text{coll}}.$$

(4.33)
4.2.2 Collision integral

It is now time to look into the collision integral itself. We consider the interaction of two particles colliding and thus the collision integral describes the process of two particles coming in, interacting through some potential and two particles going out,

\[
\left( \frac{\partial f(k)}{\partial t} \right)_{\text{coll}} = 2 \int \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \frac{dk_4}{(2\pi)^3} \frac{2\pi}{\hbar} (T(\theta))^2 \\
\times (2\pi)^3 \delta(k + k_2 - k_3 - k_4) \delta(\epsilon_k + \epsilon_{k_2} - \epsilon_{k_3} - \epsilon_{k_4}) \\
\times [ f(k)f(k_2) (1 - f(k_3)) (1 - f(k_4)) \\
- (1 - f(k)) (1 - f(k_2)) f(k_3)f(k_4) ] .
\]

(4.34)

In this expression \( T(\theta) \) is the transition matrix element, with \( \theta \) the angle between the incoming particle momenta, \( k \) and \( k_2 \). The factor 2 in front of the integral accounts for the two spin states of the neutrons, the delta functions impose momentum and energy conservation and the factors between square brackets describe Pauli blocking for both the incoming and the reverse process.

If we insert for \( f(k) \) the equilibrium function plus a perturbation, and for all other functions the equilibrium function, the Pauli blocking terms become

\[
[f_0(k) + \delta f(k)] f_0(k_2) (1 - f_0(k_3)) (1 - f_0(k_4)) \\
- (1 - f_0(k) - \delta f(k)) (1 - f_0(k_2)) f_0(k_3)f_0(k_4)
\]

\[
= [ f_0(k)f_0(k_2) (1 - f_0(k_3)) (1 - f_0(k_4)) \\
- (1 - f_0(k)) (1 - f_0(k_2)) f_0(k_3)f_0(k_4)] \\
+ \delta f(k) [ f_0(k_2) (1 - f_0(k_3)) (1 - f_0(k_4)) \\
+ (1 - f_0(k_2)) f_0(k_3)f_0(k_4)] .
\]

(4.35)

In equilibrium, the collision integral becomes zero since the in- and out scattering is equal and both terms cancel each other. That means the first part of the resulting expression in equation (4.35), with only equilibrium distribution functions, yields zero in the collision integral and only the second term, proportional to \( \delta f(k) \) contributes. We can now pull this perturbation out of the integral and write

\[
\left( \frac{\partial f(k)}{\partial t} \right)_{\text{coll}} = - \frac{\delta f(k)}{\tau(k)}
\]

(4.36)

with

\[
\frac{1}{\tau(k)} = 2 \int \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \frac{dk_4}{(2\pi)^3} \frac{2\pi}{\hbar} (T(\theta))^2 \\
\times (2\pi)^3 \delta(k + k_2 - k_3 - k_4) \delta(\epsilon_k + \epsilon_{k_2} - \epsilon_{k_3} - \epsilon_{k_4}) \\
\times [ f_0(k_2) (1 - f_0(k_3)) (1 - f_0(k_4)) + (1 - f_0(k_2)) f_0(k_3)f_0(k_4) ] ,
\]

(4.37)

the relaxation time of the interaction. This approach allows us to write the perturbation \( \delta f \) in terms of the equilibrium distribution function and the relaxation time, using the Boltzmann equation. Inserting equation (4.36) into equation (4.33) we get

\[
\delta f(k) = - v_k \frac{\partial}{\partial \epsilon_k} \left( - \frac{\partial f_0(k)}{\partial \epsilon_k} \right) \nabla T \\
+ \frac{1}{T} .
\]

(4.38)
4.2. THERMAL CONDUCTIVITY

Using this expression we can compute the heat current. Since the equilibrium distribution does not contribute to the current, we only need the perturbation, given by equation (4.38),

\[
J_Q = \int \frac{dk}{4\pi^3} (\epsilon_k - \mu) v_k \delta f(k)
\]

\[
= -\frac{1}{\pi} \int d\epsilon \rho(\epsilon) \frac{1}{3} \epsilon_k^2 \tau(\epsilon)(\epsilon_k - \mu)^2 \left(-\frac{\partial f_0(k)}{\partial \epsilon_k}\right) \frac{\nabla T}{T}
\]

\[
= -\frac{1}{3} \rho(\epsilon_F) v_F^2 \tau(\epsilon_F) \frac{\pi^2}{3} (k_B T)^2 \frac{\nabla T}{T}, \quad (4.39)
\]

where in the second step the density of states \(\rho(\epsilon)\) was used to change the integration variable, \(\rho(\epsilon)d\epsilon = k^2dk/\pi^2\). The factor \(\frac{1}{3} v_k^2\) comes from the angular integral over the dyadic product \(v_k \otimes v_k\), which is diagonal and isotropic. In the last step the integral was done by a Sommerfeld expansion, as seen before. Because the perturbation of the distribution function was already expressed in terms of the temperature gradient the thermal conductivity immediately follows,

\[
\kappa = \frac{\pi^2}{9} \rho(\epsilon_F) v_F^2 \tau(\epsilon_F) k_B^2 T
\]

\[
= \frac{1}{3} C_V v_F^2 \tau(\epsilon_F). \quad (4.40)
\]

4.2.3 Relaxation time

The final step is to compute the relaxation time \(\tau\). In principle this can be done from equation (4.37). However, since the arguments of the energy delta-function all depend on the momenta, as well as the angles between them, this is an agonising thing to do. Instead we turn to the literature on Fermi-liquid theory, in particular Baym and Pethick [15].

In their approach not the relaxation time method but a variational method is applied to linearize the collision integral in the Boltzmann equation. The variation to the distribution function is then written as

\[
\delta f_i = \frac{\partial f_0}{\partial \epsilon_i} \Phi_i, \quad (4.41)
\]

with the \(\Phi_i\) functions to be determined. To do so, a large mathematical apparatus is wielded in which the expression is expanded in spherical harmonics after which the eigenvalues of the \(\Phi_i\) functions are calculated.

Before looking into the result we have to match our expressions to those of Baym and Pethick. In their linearized collision integral, the angular integration decouples in such a way that only a weighted angular integral over the transition matrix remains. Matching notations we have that our transition matrix \(T\) is coupled to their scattering probability \(W\), through

\[
\frac{2\pi}{\hbar} T(\theta)^2 = W(\theta, \phi). \quad (4.42)
\]

The weighted angular integral then yields an average scattering probability

\[
\langle W \rangle \equiv \int \frac{d\Omega}{4\pi} \frac{W(\theta, \phi)}{\cos \theta/2} = \int \frac{d\Omega}{4\pi} \frac{2\pi}{\hbar} T(\theta)^2 \cos \theta/2, \quad (4.43)
\]
which plays a central role in the derivation. In this expression \( d\Omega = \sin \theta \, d\theta \, d\phi \).

To be able to compute this probability we use the transition matrix \( T \), which is defined as

\[
T = \int \! d\mathbf{r} \, V(r) e^{i\mathbf{q} \cdot \mathbf{r}}, \tag{4.44}
\]

where \( V \) is the interaction potential and \( \mathbf{q} = \mathbf{k}_1 - \mathbf{k}_2 \). Because both \(|\mathbf{k}_1|\) and \(|\mathbf{k}_2|\) are \( k_F \), \( q^2 = 2k_F^2(1 - \cos \theta) \), \( \theta \) being the angle between \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \). The potential \( V(r) \) is the following Yukawa potential with a repulsive and an attractive part \cite{19},

\[
V(r) = V_R e^{-\mu_R r} - V_A e^{-\mu_A r}, \tag{4.45}
\]

of which the parameters \( V_R, V_A, \mu_R \) and \( \mu_A \) are experimentally known. The transition matrix is thus

\[
T(\theta) = 4\pi \int \! dr \, r V(r) \frac{\sin(qg(\theta))}{g(\theta)}

= 4\pi \int \! dr \, \left\{ V_R e^{-\mu_R r} - V_A e^{-\mu_A r} \right\} \frac{\sin(qg(\theta))}{g(\theta)}

= 4\pi \left\{ \frac{V_R}{q^2 + \mu_R} - \frac{V_A}{q^2 + \mu_A} \right\}

= 4\pi \left\{ \frac{V_R}{2k_F^2(1 - \cos \theta) + \mu_R^2} - \frac{V_A}{2k_F^2(1 - \cos \theta) + \mu_A^2} \right\}. \tag{4.46}
\]

This expression can be inserted in equation (4.43) and the integral can be done. The result is a cumbersome, yet analytic expression depending only on the Fermi momentum \( k_F \), and the interaction parameters, which are just numbers.

Baym and Pethick then turn the crank to determine the functions describing the perturbations in the distribution functions and recast the result in a form containing a relaxation time \( \tau \) as to match equation (4.40),

\[
\tau = \frac{48\pi^2 h^6}{m^3 \langle W \rangle (k_B T)^2} \sum_{\nu=4,6,...} \frac{2\nu + 1}{\nu(\nu + 1)(\nu(\nu + 1) - 2\lambda_k)}. \tag{4.47}
\]

The sum comes from the solution of the eigenvalue equation of the functions \( \Phi_i \), which is a polynomial of which only the even terms contribute. In this sum, \( \lambda_k \) is another weighted angular integration over the scattering probability,

\[
\lambda_k = \frac{1}{\langle W \rangle} \int \! d\Omega \, W(\theta, \phi) (1 + 2 \cos \theta). \tag{4.48}
\]

Since we know \( \langle W \rangle \), we can compute \( \lambda_k \), do the sum (numerically) and find \( \tau \), completing all we need for the thermal conductivity \( \kappa \).

Since the relaxation time goes with the inverse of temperature squared and the heat capacity is proportional to temperature, the thermal conductivity is proportional to the inverse of temperature.

Using the density profile from chapter 2 and the initial temperature \( T = 10^8 \) K, we find for the thermal conductivity \( \kappa \), the curve shown in Figure 4.1(a) at the end of this chapter.
4.3 Neutrino emission

Depending on the model chosen for the neutron star, there are many processes that generate neutrinos. Since the mean free path length of the neutrinos is much larger than the star radius we assume that they do not interact again and are essentially lost. Therefore, the loss in temperature is directly proportional to the neutrino emissivity in the energy equations.

The most fundamental process in which neutrinos and anti-neutrinos are created is the so called direct Urca process [17],

\[ n \rightarrow p + e + \nu, \quad p + e \rightarrow n + \nu, \] \hspace{1cm} (4.49)

which is just beta and inverse-beta decay. Allegedly, the term Urca was taken from a now defunct, but once glamorous, casino of that name in Rio de Janeiro [18].

The direct Urca process is only possible if the proton fraction exceeds a certain critical value below which energy and momentum conservation can not be fulfilled. If the star only consists of neutrons, protons and electrons, this fraction amounts to about 11%. If muons and hyperons are included this critical value increases slightly to 13%. Modern, advanced equations of state and accompanying stellar models predict high enough proton fractions for the direct Urca process to be possible [19], but the resulting cooling simulations show that these stars do not fit observations [20].

Another, similar neutrino producing process, which does not have a threshold like the direct Urca process, is the modified Urca process which can go through two channels,

\[ n + n \rightarrow p + n + e + \bar{\nu}, \quad p + n + e \rightarrow n + n + \nu, \] \hspace{1cm} (4.50a)
\[ n + p \rightarrow p + n + e + \bar{\nu}, \quad p + p + e \rightarrow n + p + \nu, \] \hspace{1cm} (4.50b)

the ‘neutron’ and the ‘proton’ channel. These processes are still considered as the dominant cooling mechanism in neutron stars. Their emission rates were calculated by Friman and Maxwell in 1979 [21], and later slightly improved by Yakovlev et al in 2001 [22]. The modified Urca process has an emission rate of about five orders of magnitude smaller than the direct Urca process. Due to this large difference the latter is placed in the category of enhanced cooling. As said, observations seem to rule out any enhanced cooling in neutron stars with the current models.

A third significant process in which neutrinos and anti-neutrinos are produced is bremsstrahlung,

\[ n + n \rightarrow n + n + \nu + \bar{\nu}, \quad n + p \rightarrow n + p + \nu + \bar{\nu}, \quad p + p \rightarrow p + p + \nu + \bar{\nu}. \] \hspace{1cm} (4.51)

The emission rates are again calculated by Friman and Maxwell and improved by Yakovlev et al and were found to be about two orders of magnitude smaller then the modified Urca process.

Although the three processes stated above are the dominant cooling mechanisms of neutron stars containing neutrons, protons and electrons, a large variety of processes is introduced if we look at more complex stellar models and equations of state. We can, for instance, incorporate other baryons alongside the neutrons and protons, such as \( \Sigma, \Delta, \Xi \) and \( \Lambda \) particles. These particles interact in an Urca, or modified Urca like fashion, producing neutrinos and anti-neutrinos. Furthermore, most modern models predict a quark-gluon plasma in the stellar centre. In this plasma quark beta-decay, very similar to the reactions above but with quarks instead of nucleons, can be present. Details on these reactions and their emission rates are given by Maxwell et al [23], Thorsson et al [21] and Duncan et al [25].
4.3.1 Neutron bremsstrahlung

Since we have only neutrons in our stellar model and do not allow any other particles, only neutron-neutron bremsstrahlung is permitted as an interaction creating neutrinos in our star,

\[ n + n \rightarrow n + n + \nu + \bar{\nu}. \]  

(4.52)

This reaction goes through the weak neutral currents and produces neutrinos of every flavour. The reaction and the Feynmann diagrams describing it are given by Friman and Maxwell, 1979 [21]. Contrary to many other particle interactions there is no threshold associated to momentum conservation for this reaction and it operates at any density. The neutrino emission from the neutron-neutron bremsstrahlung is given by

\[
q = \frac{1}{4} \int \left[ \prod_{j=1}^{4} \frac{d{p_j}}{(2\pi)^3} \right] \frac{d{p}_\nu}{(2\pi)^3} \frac{d{p}_{\bar{\nu}}}{(2\pi)^3} (\epsilon_{\nu} + \epsilon_{\bar{\nu}}) s |M|^2 \times (2\pi)^4 \delta(p_f - p_i) \delta(E_f - E_i) [f(p_1)f(p_2)(1 - f(p_3))(1 - f(p_4))].
\]

(4.53)

The integrals are over the four neutron momenta (two incoming and two outgoing) and the two outgoing neutrinos. The \( \epsilon_{\nu} \) and \( \epsilon_{\bar{\nu}} \) terms are the energy of the neutrino and anti-neutrino respectively, the factor 1/4 in front is to avoid double counting and \( M \) is the transition matrix. The delta functions are over the total momenta \( p_f \) and \( p_i \), and the total energy \( E_f \) and \( E_i \), where \( f \) and \( i \) denote the final and initial state. The functions \( f(p) \) are again the Pauli blocking functions.

Since the integral for the neutrino emission looks very similar to the integral in the thermal conductivity, but with two more particles and corresponding integrals, we again turn to the literature for the result, in this case to Yakovlev et al who described many neutrino emitting processes in neutron stars in 2001, [22].

In order to simplify the integral, Yakovlev et al argue that it is sufficient to use the squared matrix element averaged over the directions of the neutrino momenta. This way the integration over neutrino momenta and the angular integrals decouple. In their work they use for the transition matrix element not the simple attractive and repulsive Yukawa potential, but a matrix derived by Maxwell, consisting of a one-pion-exchange tensor part and a short range part parameterized with nuclei Fermi liquid parameters.

The resulting numerical expression for the neutrino emissivity is

\[
q \approx 7.5 \times 10^{19} \left( \frac{n^4}{m^4} \right) \left( \frac{n}{n_0} \right)^{1/3} \alpha \beta \mathcal{N}_\nu T_9^8 \text{ erg cm}^{-3} \text{ s}^{-1}. \]

(4.54)

The neutron particle density is denoted by \( n \) and the neutrons mass and effective mass are \( m \) and \( m^* \), respectively. The factors \( \alpha \) and \( \beta \) are introduced by Maxwell to account for numerous effects like correlations, repulsive parts in the interaction, etc, and \( \mathcal{N}_\nu \) is the number of neutrinos, which we will take as 3. The values for the correction factors are \( \alpha = 0.59 \) and \( \beta = 0.56 \). The reference particle density \( n_0 = 0.16 \text{ fm}^{-3} \) and the notation \( T_9 \) denotes the temperature in units of \( 10^9 \text{ K} \), to the eighth power. This, together with our known expression for the effective mass gives the following result for the neutrino emissivity,

\[
q \approx 13.69 \times 10^6 \left( a n^{2/3} + 1 \right)^2 n^{1/3} T_9^8 \text{ erg cm}^{-3} \text{ s}^{-1}
\]

(4.55)
where $a = (3\pi^2)^{2/3}\hbar^2/m^2c^2$, as before. Intuitively, the neutrino emissivity would depend on $n^2$, but this is altered by the Pauli blocking factors in the integral (4.53).

We can directly use equation (4.55) in our code to calculate cooling. Again, using the density profile from our neutron star and the initial temperature $T = 10^8$ K, we find the curve as shown in Figure 4.1(c).
Figure 4.1: The heat capacity $C_V$ (a), the thermal conductivity $\kappa$ (b) and the neutrino emissivity $q$ (c), as function of radius, using the density profile of a the maximum mass star consisting of only neutrons. The temperature is taken to be $T = 10^8$ K, a reasonable initial temperature for a neutron star.
Chapter 5

Calculations

Now that the theory, describing the physics of our neutron star, is complete, we have to construct the algorithm to compute the cooling. To calculate how the temperature throughout the star changes in time we discretize the equations describing the energy flow in our star, equations (3.3). We introduce a time step alongside our radial distance step and write the equations in a way that the temperature at time $t_j$ depends on the temperature profile and parameters at time $t_{j-1}$. This way, starting from an initial temperature profile, the temperature throughout the star can be found for each moment in time.

From supernova theory it is deduced that newly created neutron stars have temperatures of $10^8 \sim 10^{10}$ K, relatively isothermal throughout the star. We use these temperatures as initial temperature profiles for our calculations.

When calculating the stellar profiles for mass and particle density as described in chapter 2, we needed a very accurate grid to avoid numerical errors at the stellar boundary. However, for the cooling simulations computer time becomes a considerable problem. To reduce computer time for the calculations we thin out the grid from 876 grid points to 25 points. This leaves enough points to guarantee a smooth profile, while drastically reducing computer time. We thus have a radial grid $r_i$ with $i = 0, 1, 2, \ldots, N$ where $N = 24$ and $r_{i+1} - r_i = d$. We have to note that $r_0$ is not zero, but the first non-zero point. This rules out any numerical problems when dividing by $r$.

The equations we have to solve to compute the cooling of our star, equations (3.11), are given here again,

$$\frac{dT e^\phi}{dt} = -\frac{1}{C_V} \left\{ \frac{dLe^{2\phi}}{dr} \frac{1}{4\pi r^2 e^\lambda} + e^{2\phi} q \right\},$$

(5.1a)

with

$$Le^{2\phi} = -\frac{dT e^\phi}{dr} \frac{e^\phi}{e^\lambda} \kappa \frac{4\pi r^2}{r}.$$  

(5.1b)

Using Euler derivatives, the discretized version of these equations would be

$$\tilde{T}_i = \tilde{T}_{i-1} - \frac{\Delta t}{C_V} \left\{ \left( \tilde{L}_i - \tilde{L}_{i-1} \right) \frac{1}{4\pi r^2 e^\lambda} + e^{2\phi} q \right\},$$

(5.2a)
with
\[ \tilde{L}_i = -\left( \frac{\tilde{T}_{i+1} - \tilde{T}_i}{d} \right) \frac{e^\phi}{e^\lambda} \approx 4\pi r^2, \] (5.2b)

where \( j \) denotes the time index and \( \tilde{L} = L e^{2\phi} \) and \( \tilde{T} = T e^\phi \), for notational convenience.

In the derivatives a forward difference is used in the radial derivative of the luminosity and a backwards difference in that of the temperature, assuring a correct combined second derivative.

When solving this set of numerical equations, care needs to be taken at the boundaries of the grid. One of the reasons the energy equations are written in two equations instead of one is that only first derivatives appear, which makes the boundary problems considerably more tractable. As boundary conditions we demand that the first derivative of the temperature profile is zero at \( r = 0 \), guaranteeing a smooth profile through the stellar centre. The second constraint is that no flux can go through the stellar centre at \( r = 0 \). If we look at the expression for the luminosity we see that this effectively sets \( L_{-1} = 0 \). At the stellar surface, \( r = R \), we set the luminosity of the heat flow \( L_N^{(d)} \), to zero. The photon luminosity at \( r = R \) is non-zero and yields a loss term.

The parameters present in the energy equations, described in chapter 4, depend on both density and temperature. Since the density profile of the star does not change in time we can split of the temperature dependance and calculate each parameter’s dependance on density at the beginning of the calculations. This needs to be done only once and during the cooling simulations we can calculate the remaining function on temperature in each time step.

### 5.1 Numerical diffusion

The energy equations (5.1) are just diffusion equations with a sink term and calculating diffusion numerically is notorious for it’s instability. In the discretized system, equations (5.2), the algorithm calculates the luminosity to and from each grid point depending on the temperature grid in the previous time step. This determines if energy is transported to the grid point or away from it, and whether it cools or warms up. This is a continuous process, discretized by doing a calculation at a certain time and multiplying the result by the time step \( \Delta t \). However, when the time step is too large, compared to the gradient in the temperature field driving the current, the algorithm becomes inaccurate. Overshooting might occur, which means a grid point gains or loses so much energy in one step that it shoots through the equilibrium solution, creating an opposite gradient which might result in overshooting back. This can result in wild fluctuations in the temperature profile. In the continues limit, the net temperature difference becomes smaller when the gradient reaches an equilibrium value and this damping effect negates overshooting.

To make the algorithm as accurate and stable as possible we use a four-point Runge-Kutta method, instead of Euler, when determining the new temperature profile at each step. This is an iterative approach to approximate solutions of ordinary differential equations. The algorithm looks like this,

\[ T^j = T^{j-1} + \frac{1}{6} \Delta t (k_1 + 2k_2 + 2k_3 + k_4), \] (5.3)
5.1. NUMERICAL DIFFUSION

with

\[ k_1 = f(t^j, T^j), \]
\[ k_2 = f(t^j + \frac{1}{2}\Delta t, T^j + \frac{1}{2}\Delta t \cdot k_1), \]
\[ k_3 = f(t^j + \frac{1}{2}\Delta t, T^j + \frac{1}{2}\Delta t \cdot k_2), \]
\[ k_4 = f(t^j + \Delta t, T^j + \Delta t \cdot k_3). \]

In this scheme equation (5.3) takes over the role of equation (5.2a). The function \( f \) is given by

\[ f(t^j, T^j) = -\frac{1}{C_V} \left\{ \frac{\tilde{L}_i(T^j) - \tilde{L}_{i-1}(T^j)}{d} \right\} \frac{1}{4\pi r^2 e^\lambda} + e^{2\phi} q, \] (5.4)

and the luminosity \( \tilde{L}_i \) is as in equation (5.2b), but with the variable inserted in the function \( f \) as the temperature field.

Where Euler uses the slope of the function at the beginning of an interval to find the value of the function at the end of the interval, Runge-Kutta uses the slope at the beginning (\( k_1 \)) to find the value at the midpoint of the interval. Then calculating the value again at the midpoint but with the slope at the midpoint (\( k_2 \)), providing a new slope at the midpoint, (\( k_3 \)). The value at the end of the interval is determined by this last slope (\( k_3 \)) and a concluding slope at the end point is found (\( k_4 \)). Finally, all slopes are averaged, giving a greater weight to the midpoint values. This averaged slope is used in an Euler like fashion to find the temperature value for the next step. This is equation (5.3). The scheme as depicted above is executed for the entire profile at each step, which means the \( k_i \) are also complete radial profiles.

The error per step in the Runge-Kutta algorithm is \( O(\Delta t^5) \), where it is \( O(\Delta t^2) \) with Euler. Although the Runge-Kutta method obviously requires more computer time then the simple Euler derivatives, the increased accuracy makes the diffusion calculations much more accurate, allowing for a larger time step.

Now that we have an accurate algorithm for the temperature profile, we still need to assure stability by choosing the time step small enough. To increase computer efficiency however, we need the largest time step possible. When looking at the energy equations, a condition for the time step can be established. Combining equations (3.3a) and (3.3b) we have the total diffusion equation

\[ C_V \frac{dT e^\phi}{dt} = \kappa e^\phi \left\{ \frac{d^2 T e^\phi}{dr^2} + \frac{d T e^\phi}{dr} \left( \phi' - \lambda' + \frac{2}{r} + \frac{\kappa'}{\kappa} \right) \right\} - e^{2\phi} q. \] (5.5)

The dominant term on the right hand side is the second derivative of temperature to radial distance. Ignoring the other terms we can use the resulting expression, which relates the time derivative to the second radial derivative, to find how the time step \( \Delta t \) and the radial distance step \( d \) should compare. For the errors in the temperature to decrease and not to blow up, we need

\[ \frac{e^{2\lambda} C_V}{e^\phi \kappa} d^2 > \Delta t. \] (5.6)

Since we have two very involved extra terms in our equation (5.5), we take \( \Delta t \) another factor of one hundredth smaller, to accommodate for their disturbing effect on the temperature profile. This factor is determined by trial and error. We can use equation (5.6) as a condition for our time step, \( \Delta t \).
Both the heat capacity $C_V$ and the thermal conductivity $\kappa$, depend on density and temperature and we take the minimum value of the left hand side of equation (5.6), throughout the radial grid, to set the time step. However, although $d$ is a constant chosen for our grid, the heat capacity and the thermal conductivity are temperature dependent and thus change during the calculations: $C_V \propto T$ and $\kappa \propto 1/T$. This results in the time step scaling with temperature squared,

$$\Delta t \propto T^2. \quad (5.7)$$

Although the time step can be very large at first, it has to become smaller and smaller since the temperature ever decreases through cooling of the star. This is a fundamental problem for our calculation since the algorithm becomes slower and slower after more and more computer time.

Before going into this problem of time step versus computer time, we look at the temperature profiles as calculated by the algorithm described above. Figure 5.1 shows the temperature profiles as a function of radius for various times at the beginning of the stars life. If the initial temperature of the star is taken to be $10^8$ K, we see in the very beginning of the calculation that the star loses energy at its boundary due to photon radiation. Through diffusion, heat flows outward until an equilibrium is reached. At a certain time the temperature profile obtains a certain shape which remains stationary. This is the isothermal configuration in the curved space of the star. Due to the energy losses of the star the entire profile is lowered in temperature each step but the shape of the profile does not change anymore.

In Figure 5.1 the profiles get closer and closer together as the simulation continues. This is caused by reaching the equilibrium, but also because the time step is getting smaller and smaller with time. Figure 5.2 shows how the time step $\Delta t$ behaves as a function of actual cooling time up to 0.8 years.

The constant value of 100 seconds at the beginning of the calculation is a maximum value to keep the temperature gradient in check. Especially at the beginning, the temperature is so high, the loss term due to photon luminosity at the stellar boundary can
5.2. Isothermal profile algorithm

When the temperature profile does not change shape anymore, we replace the algorithm described above into another, much more simple, algorithm. According to Page in 2006 [26], we can, when treating the temperature profile ‘isothermal’, substitute the energy equations (3.3) by

\[ C \frac{dT e^\phi}{dt} = -(L^\nu + L^\gamma)e^{2\phi}, \]

(5.8)

where \( L^\nu \) and \( L^\gamma \) are now the total neutrino luminosity and photon luminosity away from the star and \( C \) is the total heat capacity of the star. The photon luminosity is just the luminosity at the stellar boundary, equation (3.12),

\[ L^\gamma e^{2\phi} = 4 \pi R^2 \sigma T^4 \frac{C_{\phi}}{e^\lambda}. \]

(5.9)

For the total neutrino luminosity we have to take the integral over the stellar volume,

\[
L^\nu e^{2\phi} = \int_0^R e^{2\phi} q \frac{4 \pi r^2 e^\lambda}{e^\lambda} \, dr = \sum_{i=0}^N q_i \frac{4 \pi r^2}{e^\lambda} \, d.
\]

(5.10)

**Figure 5.2:** Time step \( \Delta t \) as a function of time. Initially the time step is kept at a maximum of 100 seconds to restrain the temperature gradient. The time step, scaling with temperature squared is then lowered due to the cooling of the star.
The temperature profile at the moment it becomes static, at $t = 0.92$ years. This is the isothermal configuration in the curved space of the star. The diffusion algorithm is substituted by the isothermal profile algorithm at this point. This curve is for a maximum mass star with an initial temperature of $10^8$ K.

![Figure 5.3: The temperature profile at the moment it becomes static, at $t = 0.92$ years. This is the isothermal configuration in the curved space of the star. The diffusion algorithm is substituted by the isothermal profile algorithm at this point. This curve is for a maximum mass star with an initial temperature of $10^8$ K.](image)

The same holds for the total heat capacity,

$$ C = \int_0^R C_V \cdot 4\pi r^2 e^{\lambda r} dr $$

$$ = \sum_{i=0}^{N} C_V \cdot 4\pi r_i^2 e^{\lambda r_i} d. \quad (5.11) $$

In both integrals the proper volume element is used by inserting the factor $e^{\lambda r}$.

This calculation is much simpler than the first algorithm and since no diffusion is involved, we can use Euler for the time derivative, which makes the discretized version of equation (5.8),

$$ \tilde{T}_j^j = \tilde{T}_j^{j-1} - \frac{\Delta t}{C}(\tilde{L}^\nu + \tilde{L}^\gamma). \quad (5.12) $$

Additionally, there are no more limits for the time step $\Delta t$, making this algorithm much faster.

The criterion for which the diffusion algorithm may be substituted by this second algorithm is that the temperature profile does not change shape during the calculation. In other words, the difference in temperature each time step must be equal for all gridpoints throughout the star. We approximate this by switching algorithms when the maximum and the minimum temperature differences throughout the star are within 0.5 K of each other. For the calculations done in this thesis this happens after approximately one year of cooling time.

The temperature profile at the moment of algorithm switching is depicted in Figure 5.3. This curve is calculated for a maximum mass star, starting at $10^8$ K.
5.3 Cooling curves

To see how a neutron star cools in time we look at the surface temperature of the star as a function of time. To correspond with current publications on neutron star cooling we will present the surface temperature as measured ‘at infinity’, \( T^\infty = T(R) e^{\delta(R)} \). This is also the quantity used in the energy equations and thus a direct output of our model.

Figure 5.4 shows the surface temperature versus time for three different initial temperatures, on a double logarithical scale. The stars started at \( 10^8 \), \( 10^9 \) and \( 10^{10} \) K to examine the effect of the initial temperature on the cooling of the star. Well within a year all three curves join and become indistinguishable.

Especially for the higher temperatures it can be seen that within \( 10^{-6} \) years (about 31 seconds), the temperature has already dropped a substantial amount from the initial temperature. The last part of the curves is a straight line in this plot.

All three curves are made using the density profile for the maximum mass star, \( M = 0.767 M_\odot \), consisting of a Fermi gas of neutrons, as described in the previous chapters.

After about one year the internal temperature profiles become ‘isothermal’ and the diffusion algorithm is replaced by the isothermal profile algorithm. This is indicated by the dashed, vertical orange line in the figure. The exact times at which the algorithms change are 0.9227, 0.9358 and 0.9359 years for initial temperatures \( 10^8 \), \( 10^9 \) and \( 10^{10} \) K, respectively.

**Figure 5.4:** The surface temperature, as measured at infinity, \( T^\infty \), versus time. Three initial temperatures were used: \( 10^8 \) K (blue), \( 10^9 \) K (red) and \( 10^{10} \) K (green). All curves join well within a year. Just under a year the temperature profile becomes isothermal and the algorithms switch. This is indicated by the vertical dashed, orange line.
Chapter 6

Discussion

With Figure 5.4 showing how the surface temperature of a neutron star changes with time the goal of the cooling simulation is achieved. We now turn to analysing the results.

The first readily attainable characteristic of the cooling curves observed, is that the initial temperature of our neutron star does not influence temperature and cooling at timescales after about a year. Obviously, the internal temperature profile reaches a certain equilibrium within that time, independent of the precise profile at the beginning of the calculations. The fact that the temperature profiles become ‘isothermal’ just within a year for all three initial temperatures supports this. Once this equilibrated isothermal profile is reached all history of the cooling until then becomes inconsequential.

Next, we look at the effects of the neutrino emissions and the photon emission respectively. Although the neutrino emission from the star is strongest in the centre of the star due to the higher densities and temperatures, we see from Figure 5.1 that most of the energy loss is at the stellar boundary. This suggests that photon emission is the dominant cooling process for our star. To investigate this we artificially increase the neutrino emission rate, \( q \). Again from Yakovlev [22] we find an expression for the neutrino emission rate for stars containing neutrons, protons and electrons. The dominant neutrino creating process is then the modified Urca process,

\[
\begin{align*}
    n + n &\rightarrow p + n + e + \bar{\nu}, & p + n + e &\rightarrow n + n + \nu, \\
    n + p &\rightarrow p + p + e + \bar{\nu}, & p + p + e &\rightarrow n + p + \nu.
\end{align*}
\]

The rate at which this process creates neutrinos depends exactly the same on number density and temperature as the neutrino bremsstrahlung, but the numerical pre factor is about two orders of magnitude larger: \( 11.46 \times 10^8 \) instead of \( 13.69 \times 10^6 \).

The cooling curves for stars with this artificially simulated ‘modified Urca’ are shown in combination with the already displayed curves, calculated with neutrino bremsstrahlung only, in Figure 6.1.

In this figure, the lines for the different neutrino processes differ very little, putting us in a position to state that the neutrino emissions are irrelevant in the cooling of our neutron star. To further investigate this we examine the energy losses from both neutrinos and photons. Figure 6.2 shows the internal energy of a star starting at \( 10^8 \) K, as well as the total energy lost by neutrino emission and by photon emission separately. This energy loss is accumulative and thus depicts the integrated energy lost from the star since the beginning of the calculations. The lost energy is precisely the difference between the total energy and the internal stellar energy. The difference between the
amount of energy lost through photons and neutrinos is about ten orders of magnitude. For an initially hotter star of $10^{10}$ K the difference decreases due to the stronger power law dependance on temperature of the neutrino emission. Still, the gap remains about two orders of magnitude, by which the neutrino emissions persists as a minor effect.

The energy loss per second from neutrinos and photons is showed in Figure 6.3 again for a star of initially $10^8$ K. In this plot it is clearly shown that the energy loss from photons is many orders of magnitude larger then the loss contributed by neutrinos, at each moment of the cooling. The beginnings of these curves are relatively straight because the internal temperature is not yet in equilibrium at this stage. Then, starting at around 0.01 year the different power laws of the neutrino and photon radiation become apparent. Neutrino emission, with an eighth power dependance on temperature plummets much faster then the photon emission which goes as temperature to the fourth.

With the cooling trajectories discussed, we briefly turn to the assumptions made in the model used. In our star, where no crust or envelope was used, the equation of state of the Fermi gas of neutrons was applied to the entire star, up to the very surface. Densities at the surface however, do not concur with that assumption. More accurate would be to add an envelope to be able to use a correct equation of state at each layer in the star.

The maximum mass and radius already show that the chosen model and subsequent equation of state are inadequate. The maximum mass of 0.767 $M_\odot$ is to too low and the radius of 8.77 km is too small to agree with observations. The ideal Fermi gas of neutrons has no interactions incorporated. Repulsive interactions add to the pressure counteracting gravity, causing the total mass and radius to be larger. Ignoring these interactions turns out to be too much a simplification. The effect the increased density
**Figure 6.2:** The internal energy (orange), the integrated neutrino energy loss (red) and the integrated photon energy loss (green), as well as the total energy of the system (blue), as a function of time. The energy loss due to neutrino radiation is about ten orders of magnitude smaller than that of photon emission.

**Figure 6.3:** The energy loss per second from both neutrino emissions (blue) and photon radiation (red) as a function of time. The neutrino loss is negligible to that of the photon radiation at all times.
from a larger total star mass would be on the parameters involved in the cooling is
difficult to say, but it can be substantial.

The zero temperature assumption on the other hand can be seen as accurate. The
Fermi energy is between 0.0015 erg (surface) and 0.0020 erg (centre), which is 936 to
1248 Mev. This is comparable with a temperature of \(1.0 \sim 1.4 \times 10^{13}\) K. With the initial
temperatures of the star taken between \(10^8\) and \(10^{10}\) K, this makes treating the star at
zero temperature, or with a low temperature expansion, correct.

This also shows that the cooling of the star has no effect on the equation of state.
Computing the stellar profiles once, using the TOV equations and the zero temperature
equation of state is efficient and accurate.
Chapter 7

Conclusions and Outlook

The conclusion of our analysis is that a star purely consisting of neutrons cools mainly through photon radiation. Neutrino emissions contribute only a fraction of the energy loss throughout the life of the star. This particular model of only neutrons is clearly too simple for relevant cooling calculations and it does not produce a physical description of a neutron star.

We can however, conclude from our model that the initial temperature is of little or no influence to the cooling trajectory of the star, after about one year.

The next step would be to enhance the stellar model. The theory in this thesis invariably consists of two parts. A general description of the effect of the part under discussion on neutron star cooling and a specific calculation for our particular stellar model of only neutrons. The first part is written and worked out in such a way that the model is easily enhanced or extended by adding other particles and corresponding interactions. A logical approach would be to first add protons and electrons, then include a range of baryons and finally deconfined quark matter.

The effect of enhancing the model by incorporating more particles is profound. The equation of state is altered, leading to a different mass and density profile. All the parameters in the cooling equations are altered. The heat capacity has to be calculated for all particles and the thermal conductivity has to be recalculated using all the interaction potentials between the various particles. Finally the neutrino emission rate has to incorporate all the neutrino producing reactions possible. This reveals why increasing the neutrino emission by a factor as to simulate a neutron-proton-electron star with the modified Urca process is not sufficient.

Apart from the choice of particles and the interaction model, subdividing the star into a core, a crust and an envelope can have a big effect on the cooling simulation as well. The main advantage of this approach is that a suitable model with corresponding equation of state can be chosen for each region in the star. Since the equation of state for the envelope and the crust are relatively well known for most (advanced) particle models, they can be glued to a more uncertain interior model and comparisons can be made. Since the envelope is so thin, it is sufficient to take a parallel plane approximation, making the numerics very straightforward.

The relevance of adding an envelope might be larger then only incorporating a correct equation of state for the densities present [27]. Acting as a thermal insulator between the hot interior of the star and the surface, the envelope has a strong effect on surface temperature and thus on photon luminosity. The result from our model, that the energy
loss due to photon radiation is far greater than the loss from neutrino emission, might change dramatically in the case were a suitable envelope is glued to the outside of the interior model.

With a suitable envelope it would also be possible to incorporate photon luminosity inside the star. In the stellar core, which becomes isothermal very quickly, this has no effect, but in the envelope photon radiation can contribute to the total luminosity and thus influence heat conduction.

Another effect ignored in this study, which has enormous effects on neutron star cooling, is superfluidity of the matter within the star. The effective nucleon-nucleon interaction contains a strong repulsive, short range part and a weaker, attractive long range part. Depending on the average interparticle distance, the matter may become superfluid. It is believed [19] that neutrons form $^1S_0$ superfluid pairs in the density range between neutron drip and saturation density of nuclear matter. At supernuclear densities, in the core of the star, the repulsive interaction becomes dominant and the $^1S_0$ gap closes. However, the tensor and spin-orbit interaction become attractive at such densities and the formation of $^3P_2$-neutron pairs might become possible. Also, the proton concentration in the core region may reach a particle density similar to that of the pair-condensed neutron gas at subnuclear densities, enabling the protons to form $^1S_0$ superfluid pairs. Since the protons are charged, this superfluid is also a superconductor.

The hyperons and quarks that might be present in advanced neutron star models might form superfluid states too. Diederix describes the formation of quark Cooper pairs in his work [4] and thus the color superconductivity within the neutron star core.

When the matter inside a neutron star becomes superfluid the parameters used in the cooling model, the heat capacity, the thermal conductivity and the neutrino emissivity, are reduced by an exponential factor of the gap energy over $k_BT$. Unfortunately, the gap energies are hard to find or unknown for more involved models. The breaking and formation process of the Cooper pairs, on the other hand, produces additional neutrinos which add to the neutrino emissivity. It is impossible to say beforehand what the effect of introducing superfluidity into the neutron star model would be on the cooling scenario.

Despite the many uncertainties about the effect of the alterations to the model proposed on neutron star cooling, we believe the neutrino emissivity will increase, due to the added particle interactions producing neutrinos, and the photon luminosity will drop, due to the thermal insulation of an added envelop. This will cause the neutron star to cool primarily through neutrino emission in the beginning of it’s life. The power law dependence on temperature of the neutrino emission rate remains much stronger then that of the photon radiation which is always temperature to the fourth. After a certain time, the energy loss through neutrinos will equal that of the loss through photons, after which the cooling is again dominated by photon radiation.

At the moment a contradiction seems to exist between theoretical models and observations of neutron stars. The theory and models of matter under extremely dense conditions predict exotic neutron star interiors with condensates of pions or kaons or a superconducting quark phase. The cooling simulations for these stars all indicate that these matter manifestations induce very fast cooling of the star [19]. The densities are so high the direct Urca process can become the dominant neutrino producing process, not only for neutrons and protons but for quarks as well. This increases the neutrino emissivity by many orders of magnitude making the star cool very strongly.

Observations, on the other hand, moderately favour the models where none of the enhanced cooling occurs. These models, mostly without hyperons or quarks, but often with superfluidity included, cool slower and seem to fit better with known neutron stars.
whose thermal emissions have been clearly detected [6, 20]. The combined uncertainties in temperature and age of the measurements, the spread of the observations and all the unknowns in the equation of state, the neutrino emissivity and the superfluid gaps, however, keep the discussion wide open and a final verdict is yet to be spoken.
Bibliography


