Fulde-Ferrell superfluid phase in an optical lattice.

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Abstract

Recent developments in atomic physics like atom cooling and the use of optical lattices, equipped physicists with new tools for studying particle statistics and their interactions. It also provides the possibility to enter physical regimes that were impossible to reach until now. In this thesis we employ the effective Hubbard model for ultracold atoms in an optical lattice, a quite accurate and simple theoretical model for the regimes under consideration, in order to study the exotic superfluid Fulde-Ferrell phase. The difference in the physics of this superfluid phase is the possibility to have pairing at nonzero total momentum. In other words the Cooper pairs have a nonzero momentum. This phase transition is also important for high-energy physicists since, for example, it is believed to exist in the interior of cold compact stars. We consider an ultracold dilute Fermi gas with unequal populations for the spin up and spin down states. This situation is the analogue of a (neutral) superconductor under the influence of a Zeeman field. We start by explaining the creation of an optical lattice that is used to very accurately control the collisional properties of the atoms with the use of lasers. Next, for intuitive reasons, we study first the case of our fermi gas without an optical lattice. We then retrieve the normal to superfluid phase transition by placing our system in an optical lattice and, finally, move to the question of the existence of the Fulde-Ferrell phase.
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Chapter 1

Introduction

In the last sixteen years the development of atom cooling (William Philips, Claude Cohen-Tannoudji and Steven Chu, Nobel Prize in Physics 1997) the achievement of Bose-Einstein condensate (BEC) in 1995 (Anderson et al.; Bradley et al.; Davis et al.), and the achievement of Fermi degeneracy (DeMarco and Jin, 1999) in ultracold, dilute gases gave atomic and molecular physicists new tools for studying particle statistics and particle interactions, rather than the study of single atoms or phonons. As a result, for example, quantized vortices and vortex lattices (Abo-Shaeer et al., 2001; Madison et al., 2000) and molecular condensates with bound pairs of fermions in 2003 (Greiner et al.; Jochim et al.; Zwierlein et al.) were observed. The concept of superconductivity and superfluidity as a coherent macroscopic matter wave in an interacting many-body system appears in these phenomena.

Furthermore the last several years the use of Feshbach resonances in the collision of two atoms gave also the ability to tune the interaction strength in cold gases (Courtrille et al.; Inouye et al.; 1998), while the generation of strong periodic potentials for cold atoms through optical lattices (Greiner et al. 2002a) allowed to enter regimes where dilute gases exhibit strong correlations. Thus, at least theoretically, a quantum phase transition from superfluid to Mott insulating state was realized as well as the exploration of quantum Hall effect in fast rotating gases (Wilkin and Gunn, 2000) or of the crossover from a molecular BEC to BCS-superfluid of weakly bound Cooper pairs\(^1\) and the observation of fermionization in an one-dimensional Bose gas (Paredes B. et al. 2004, Kinoshita T. et al. 2004). These are again few examples of what the new achievements, the possibility of Feshbach resonances and of optical lattices, brought. Lately studies were extended to Fermi gases with unequal spin-up and spin-down densities (Partridge et al., Zwierlein et al. 2006). As we will see later on repulsive fermions in an optical lattice can be described by an ideal and tunable version of the Hubbard model. In this thesis we are interested in the case of a fermi gas with unequal spin population in an optical lattice and the existence of the Fulde-Ferrel phase \([14]\) in the zero temperature limit. For this we will assume

\(^1\)There are many references for this subject, some of them Batrenstein et al., Bourdel et al., Zwierlein et al. published in 2004
to have a deep optical lattice where a large band gap appears, also we will consider the interaction strength to be such that only a single (atom) channel Hamiltonian is going to be used. Before moving to the theoretical approach it will prove useful to see in a more detailed manner the generation of an optical potential and of an optical lattice. Next for intuitive and instructive reasons, before placing our system in an optical lattice, we will go through the simpler case of a vacuum. After having obtained the theoretical descriptions for this cases we will finally deal with the FF phase transition resulting exotic phases of superfluidity.

1.1 Optical Lattice

1.1.1 Optical Potentials

Initially we assume an atom subject to a laser field with electric field

\[ E(x, t) = E(x, t) \varepsilon e^{-i\omega_L t} \]

where \( \omega_L \) the laser frequency and \( \varepsilon \) the polarization vector of the laser. The amplitude of the electric field varies slowly in space \( x \) compared to the size of the atom and slowly in time compared to \( 1/\omega_L \). The interference of the atom with the laser beam induces an atomic dipole moment \( p = \alpha(\omega_L)E \), where \( \alpha(\omega_L) \) is called the polarizability. A consequence of this induced dipole is its interaction with the laser field, resulting a shift of the atomic levels. This is the well-known Stark shift. Through the Stark effect, the confinement of cold atoms with laser light happens due to the dipole force (ref [24])

\[ F = \frac{1}{2} \alpha(\omega_L) \nabla (|E(x)|^2) . \]

Because of the characteristics of the electric (laser) field in the dipole force only the time averaged intensity \( |E(x)|^2 \) enters while its direction depends on the sign of the polarizability. We assume the dominant contribution to the optical potential arises from an excited state \( |e\rangle \) with energy \( E_e = \hbar \omega_0 \). The polarizability in the vicinity of an atomic resonance from the ground state \( |g\rangle \) \(^2\) to an excited state has the standard form \( \alpha(\omega_L) \approx |\langle e|d|g\rangle|^2 /\hbar (\omega_0 - \omega_L) \) ref [9], with \( d \) the dipole operator in the direction of the field. If we define the difference between the laser frequency and the frequency of the atomic transition \( |e\rangle \leftrightarrow |g\rangle \) as the detuning, \( \delta = \omega_L - \omega_0 \), then we see that atoms are attracted to the nodes of the laser intensity for blue-detuned laser light \( (\delta > 0 \text{ or } \alpha(\omega_L) < 0) \) or from the anti-nodes for red-detuned light \( (\delta < 0 \text{ or } \alpha(\omega_L) > 0) \) as we see in Figure 1.1.

\(^2\)Actually atoms initially occupy a metastable internal state which defines the zero energy point of energy.
In a more formal way we can write the interaction between the atom and the electric field in the dipole approximation as $H_{\text{int}} = -\mathbf{d} \cdot \mathbf{E}$ and treat it as a perturbation. Here $\mathbf{d}$ is the electric dipole-moment operator defined as

$$
\mathbf{d} = -e \sum_i \mathbf{r}_i,
$$

while its quantum-mechanical expectation value is the atomic dipole moment $\mathbf{p}$. Applying perturbation theory and neglecting all quick oscillating terms through the rotating-wave approximation, that is neglecting terms where $\hbar (\omega_L + \omega_0)$ appears, the optical potential one obtains is ref [4]

$$
V(x) = \frac{|\Omega(x)|^2}{8\delta},
$$

where $\Omega(x) = E(x, t) \langle e | \mathbf{d} | g \rangle$ is the Rabi frequency responsible for the transition between the atomic levels. For this approximation we want to adiabatically eliminate the excited level and for this we require $\Omega(x) \ll \delta$. Thus the population transferred to the excited level, $|\Omega(x)|^2 / 8\delta^2$, is neglectable. Typically the depth of the optical potential (trapping potential) is linearly proportional to the laser power and ranges from nanokelvin to microkelvin scale.

1.1.2 Periodic Lattice

In order to generate an one-dimensional periodic (optical) lattice we start by overlapping two counterpropagating laser beams with $E_{\pm}(x, t) = E_0 e^{\pm ikx}$ with $k = 2\pi/\lambda$ propagating in the same direction. Here we assumed that they propagate in x-direction. We also choose the polarization of the light field to be linear or circular. Eventually we want to trap alkali atoms where only the valence electron matters so that the dipole operator (1.1) reduces to a single term. Then one can calculate for a two-level system in the dipole approximation the interaction Hamiltonian for the particular polarization, see ref [7] and references within, to be

$$
H_{\text{int}} = -eE(x) \cos(\omega t) r \sqrt{\frac{4\pi}{3}} Y_{1, m}^*(\theta, \phi) \epsilon_m,
$$
1.1 Optical Lattice

with $Y_{1,m}(\theta, \phi)$ the spherical harmonic functions, $r_m = r (4\pi / 3)^{1/2} Y_{1,m}(\theta, \phi)$ the corresponding position operators for electrons in spherical components appearing in the dipole operator (1.1) and $\epsilon_m$ the spherical unit vectors for $m = 0, \pm 1$. If we have $\epsilon_0 = \epsilon_z$ then we refer to linearly polarized light while for $m = \pm 1$ we refer to circularly polarized light. From first-order perturbation theory and because of parity reasons\(^3\) we see that the matrix element $E^1_g(x) = \langle g \vert H_{\text{int}} \vert g \rangle$ vanishes. The second order perturbation term

$$E^2_g(x) = \frac{|\Omega(x)|^2}{4} \left( \frac{1}{E_e - E_g - \hbar \omega_L} + \frac{1}{E_e - E_g + \hbar \omega_L} \right)$$

(1.2)

contributes to the energy implying that we are dealing with a second-order Stark effect. The shift is thus quadratic in the electric field. Again $\Omega(x)$ is the Rabi frequency, $\omega_L$ is the laser frequency and $E_g$ is the ground state $\vert g \rangle$ energy. We have also assumed that we have only one excited state $\vert e \rangle$ with energy $E_e$ but we can easy generalize for more by summing (1.2) over all the excited states.

The first term of equation (1.2) represents the stimulated absorption of a photon from the ground state while the second term the stimulated emission of a photon from the exited state. Furthermore we can make our approximation more precise if we take into account the lifetime $1/\Gamma_e$ of the excited state and replace $E_e$ with $E_e - i\hbar \Gamma_e / 2$. Then the real part represents the energy shift while the imaginary part represents the effective photon absorption rate of the atoms in the optical lattice. Following the reasoning of the previous section we use the rotating wave approximation (RWA) and obtain for the light shift

$$E_{Sh} = \frac{\left( \frac{|\Omega(x)|^2}{\Gamma_e} \right)^2}{1 + 4 \left( \frac{\delta}{\Gamma_e} \right)^2} \frac{\hbar \delta}{2}$$

(1.3)

Comparing with the optical potential of the previous section we see again that for positive detuning, i.e. the light is blue-detuned, atoms are repelled from the light and the potential minima coincide with the intensity minima. For negative detuning the light is red-detuned, atoms are attracted to the light and the potential minima coincide with the intensity maxima. We have also to mention that the excite state will have the opposite shift compared to the ground state.

The effective photon absorption rate of the atoms in the optical lattice in the RWA assuming $\delta > \Gamma_e$ is found to be

$$\Gamma_{e,eff} = \Gamma_e \left( \frac{|\Omega(x)|^2}{8\delta^2} \right)$$

This is a not so significant contribution. Thus we see an optical standing wave with period $\lambda / 2$ results due to the interference of the two laser beams. Observing

\(^3\)For the first order stark effect the unperturbed wave functions must have opposite parity. Here the dipole operator has odd parity while the ground state wavefunction has definite parity, so that this term drops out.
(1.3) we see that we have an optical potential proportional to the squared Rabi frequency so that we can roughly write ⁴

\[ V(x) \sim \cos^2(kx). \]

If we load the optical lattice with a BEC, atoms are trapped and oblate spheroid (disk-shaped) condensates are formed, Figure 1.2. The thickness of the disks is of the order of the wave length of the laser light while the diameter is of the order of the width of the laser beam. The average occupation number is a few hundred atoms per lattice site.

![Figure 1.2](image)

Figure 1.2 Generation of an one-dimensional optical lattice using a pair of counterpropagating laser beams, in the right figure the atomic density is presented. We can see that disk-shaped condensates are formed.

Two (square) and three (cubic) dimensional optical lattices can be created using one or two more pairs of laser beams in \( \hat{y} \) and \( \hat{y}, \hat{z} \) directions respectively. Thus for the two-dimensional case the optical potential is

\[ V(x) = V_{0x} \cos^2(kx) + V_{0y} \cos^2(ky), \]

and atoms are confined in prolate spheroid (cigar-shaped) condensates while the occupation number is of the order of 10-20 atoms per site ([5] and references therein). Finally in the 3D situation the trapping potential is of the form

\[ V(x) = V_{0x} \cos^2(kx) + V_{0y} \cos^2(ky) + V_{0z} \cos^2(kz) \]

and atoms are trapped to give spherical condensates, figure 1.3, but now only a few atoms per site can be loaded. The difference is that the number of sites has

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⁴Actually if we take into account the polarization of the laser beams, that is if the two running waves have an angle \( 2\phi \) between their polarization axes then the optical potential is of the form \( V_{\pm} \propto \cos^2(kx \pm \phi) \). Here by \( V_{\pm} \) we refer to circularly polarized light where plus sign corresponds to \( \epsilon_1 \) and minus sign to \( \epsilon_{-1} \) spherical unit vector. Since the two standing waves act as state dependent potentials and can be shifted by \( \Delta x = 2\phi/k \), we can choose the angle \( \phi \) such that the different internal hyperfine states are in opposite directions.
increased dramatically compared to the one-dimensional condensate. That means that we can load optical lattice with the same number of atoms independent of the dimension of the lattice we will have.

Furthermore different geometries and structures have been achieved, such as sheets of two-dimensional triangular lattices by changing the angle between the counter-propagating laser beams, two dimensional hexagonal lattices or face-centered-cubic [4],[5].

The advantage of an optical lattice is that it is a structure with no defects which means that we can create and study "pure crystals". Also the easy manipulation experimentally of the parameters like the periodicity of the lattice by changing the laser frequency as well as the interaction strength through Feshbach resonances gives the ability to have a perfect simulation and enter regimes prior impossible to reach. In table 1.1 we present typical values for optical lattices created by sodium $^{23}Na$ atoms and potassium $^{40}K$. Eventually we are going to be interested for dilute Fermi gases so that we may refer to the case of potassium atoms, later on we are also going to apply the proposal of Fulde and Ferrell to such a gas.
### 1.2 Atom - atom scattering processes in optical lattices.

#### 1.2.1 Qualitative approach

As already mentioned in this thesis we are interested in dilute Fermi gases in the zero temperature limit. Then only two-body processes take place in the gas and we can neglect three or higher-body processes. The interactions originate from the valence electron of the alkali atoms. Roughly speaking this is the situation of atom-atom scattering where the atoms have two internal states. First, before giving a more formal treatment, we will point out some important features of scattering theory and refer to a the simpler case of a square well\(^5\), for this we will follow the approach of ref. [6]. Let us assume two atoms of mass \(m\) and \(V(\mathbf{r})\) the interaction potential, that vanishes for large distances between the atoms. Moving to the center of mass frame the wave function \(\Psi(\mathbf{r})\) is determined by the Schrödinger equation

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right] \psi(\mathbf{r}) = E \psi(\mathbf{r}) ,
\]

where \(\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2\) for \(\mathbf{r}_1, \mathbf{r}_2\) the coordinates of the two atoms. At negative energies we have to look for the discrete values of the energy \(E\) such that \(\Psi(\mathbf{r})\) is square integrable, these eigenstates correspond to two-body bound states.

At positive energies we seek solutions with \(E = \hbar^2 \mathbf{k}^2 / m\), the kinetic energy of two atoms with momentum \(\hbar \mathbf{k}\). We expect the solution for \(r \to \infty\) to be a superposition of incoming and outgoing plane waves, since the interatomic interaction potential vanishes when the distance between the atoms becomes large. For smaller values of \(\mathbf{r}\) the scattering wave function suitable for describing a scattering experiment, is given by an incoming plane wave with relative momentum \(\hbar \mathbf{k}\) and an outgoing spherical wave with relative momentum \(\hbar \mathbf{k}'\), that possibly depends on the direction of observation \(\mathbf{r}/r\) through the scattering amplitude \(f(\mathbf{k}', \mathbf{k})\), and can be written as

\[
\psi(\mathbf{r}) \sim e^{i\mathbf{k}\mathbf{r}} + f(\mathbf{k}', \mathbf{k}) \frac{e^{i\mathbf{k}'\mathbf{r}}}{r} .
\]

\(^{5}\)In ref. [9] a useful toy model approach can be found, in which the van der Waals attraction at large distances is cutoff by a hard core at distance \(r_c\). This model captures the main features of scattering at low energies.
To see better the connection of $f(k', k)$ with the direction of the observation we have to keep in mind that the interatomic interaction potential is a function of $r$ thus the scattering amplitude depends only on the magnitude $k$ and the angle between $k'$ and $k$. Also because of energy conservation $k = k'$ and then $k' = kr$.

Next we will follow the partial-wave method and expand the incident plane wave in Legendre polynomials $P_l(\cos \theta)$ as

$$e^{ikr} = \sum_{l=0}^{\infty} \frac{(2l + 1)i^l}{kr} \sin \left[ kr - \frac{l\pi}{2} \right] P_l(\cos \theta),$$

while the scattering amplitude and the wave function are expanded as

$$f(k', k) = \sum_{l=0}^{\infty} f_l(k) P_l(\cos \theta)$$
$$\psi(r, \theta) = \sum_{l=0}^{\infty} R_l(k, r) P_l(\cos \theta) = \sum_{l=0}^{\infty} (u_l(k, r)/r) P_l(\cos \theta),$$

respectively. Here $R_l(k, r)$ is the radial wave function while $u_l(k, r)$ is the solution of the radial Schrödinger equation

$$\left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - \frac{mV(r)}{\hbar^2} + k^2 \right] u_l(k, r) = 0.$$

Plugging these expansions in equation (1.4) we deduce for the partial wave amplitudes $f_l(k)$

$$f_l(k) = \frac{2l + 1}{2ik} \left( e^{2i\delta_l(k)} - 1 \right). \tag{1.5}$$

In the last equation we introduced the so-called phase shift $\delta_l(k)$ of the $l$–th partial wave. Eventually we are going to deal with ultracold polarized fermion alkali atoms that their low energy generally does not overcome the centrifugal barrier. Then even the p-wave ($l = 1$) scattering is suppressed, generally speaking, and we have to take into account only the s-wave ($l = 0$) scattering. At this point we have to be careful, because the s-wave scattering is absent only for polarized fermion gas. In a non-polarized one due to Pauli principle s-wave scattering can take place as the dominant contribution in the scattering amplitude.

Setting $l = 0$ in (1.5) we obtain for the s-wave scattering amplitude

$$f_0(k) = \frac{1}{k \cot \delta_0(k) - ik}$$

and if we use the definition of the s-wave scattering length

$$a = -\lim_{k\to 0} \frac{\delta_0(k)}{k}, \tag{1.6}$$
then we can find that the scattering amplitude at zero-momentum is $f_0(0) = -a$.

Of course at this point it might seem that we have cheated using the definition (1.6),
but an explicit calculation, see ref. [6], even for the square well that switches off at
distance $R$ yields

$$
\delta_0(k) = -kR + \tan^{-1}\left[\frac{k}{k^<} \tan(k^<R)\right] \quad \text{with} \quad k^< = \sqrt{k^2 - mV_0/\hbar^2}.
$$

Assume for a moment that we have a hard core potential ( $V_0 \to \infty$ ) and use
the definition (1.6). Then we obtain for the s-wave scattering length $a = R$. This
means that at low energies the details of the potential are not important and we can
model it with an effective hard-core potential of radius $a$. Note that the scattering
length is positive for a repulsive interaction, negative for attractive interaction.

Next we are going to return to the square well potential and focus in the case
of a potential with attractive parts. In this situation the scattering length is deter-
dined by $a = R \left(1 - \tan\gamma/\gamma\right)$, where $\gamma = R\sqrt{m\vert V_0\vert/\hbar^2}$ a dimensionless constant.

Also it turns out [6], that there is a relationship between the energy of this bound
(molecular) state and the scattering length

$$
E_m = -\frac{\hbar^2}{ma^2},
$$

which actually is a quite general one. Note that a bound state has negative energy.

1.2.2 A more quantitative approach.

In a more formal treatment of the same problem we start by writing the Schrödinger
equation as

$$
[H_0 + V] |\psi\rangle = E |\psi\rangle,
$$

with $H_0$ the kinetic energy operator for the atoms and $V$ the interatomic interaction
potential that vanishes for large distances between the two atoms. In the absence of
$V$ the solution of the above Schrödinger is a plane wave. For a non zero interaction
potential we seek for solutions which asymptotically represent an incoming plane
wave, and an outgoing spherical wave\(^6\)

$$
|\psi_{k^+}^{(+)}\rangle = |k\rangle + \frac{1}{E^+ - H_0} V |\psi_{k^+}^{(+)}\rangle.
$$

Here $|k\rangle$ is the incoming plane wave and $E^+$ is the kinetic energy of the two
atoms as was previously defined with the only difference that now we have written
it as slightly complex using $E^+ = \lim_{\eta \to 0} E + i\eta$. Furthermore the scattering amplitude is

$$
f(k', k) = -\frac{m}{4\pi\hbar^2} \langle k' | V | \psi_{k^+}^{(+)}\rangle.
$$

\(^6\)A nice derivation of this formula with the use of Green’s function can be found in ref. [1].
It will prove useful to use the definition of the two-body transition matrix

\[ V|\psi_k^{(+)}\rangle = T^{2B} (E^+) |k\rangle. \tag{1.10} \]

The T-matrix summarizes all the quantum-mechanical amplitudes for two atoms that scatter with each other an arbitrary number of times. From this definition, performing some trivial calculations and taking advantage that the incoming plane waves form a complete set, we can derive the so-called Lippmann-Schwinger equation

\[ T^{2B} = V + V \frac{1}{z - H_0} T^{2B} (z) \tag{1.11} \]

the solution of which determines the scattering properties of the potential. This will become clear if we combine (1.09) and (1.10) and write

\[ f (k', k) = -\frac{m}{4\pi\hbar^2} \langle k'| T^{2B} (2\varepsilon_k) |k\rangle. \]

Our next step is to use perturbation theory in the potential. The solution of the Lippmann-Schwinger equation is the Born series

\[ T^{2B} = V + VG_0 (z) V + VG_0 (z) VG_0 (z) V + \ldots \tag{1.12} \]

where we denote

\[ G_0 (z) = \frac{1}{z - H_0}. \]

Since we are in the zero-temperature limit in order to reproduce the scattering amplitude at low energies and momenta, \( f_0 (0) = -a \), it is enough to keep only the first term of the Born series and replace the true interatomic interaction potential with the pseudopotential

\[ V (x - x') = \frac{4\pi a\hbar^2}{m} \delta (x - x'). \tag{1.13} \]

At this point we have to be really careful, since this pseudopotential can only be used in first-order perturbation theory and not for higher order terms. Up to now we saw how ultracold atoms can be trapped and with equation (1.13) we have a form of the interatomic potential, thus we have all the necessary tools to move to the theoretical treatment of our problem.
Chapter 2

BCS theory

As already mentioned in this thesis we will deal with a dilute Fermi gas with unequal population for the spin up and down components, in the weak-coupling regime so that we are away from Feshbach resonances and we will only need the single-channel Hamiltonian. That is we have to take into account only atoms and their interactions. Then this phenomenon can be described by the BCS Hamiltonian

\[ H = \sum_{\sigma=\uparrow, \downarrow} \int dx \Psi_{\sigma}^\dagger (x) \left( -\frac{\hbar^2 \nabla^2}{2m} - \mu_{\sigma} \right) \Psi_{\sigma} (x) + g \int dx \Psi_{\uparrow}^\dagger (x) \Psi_{\downarrow}^\dagger (x) \Psi_{\downarrow} (x) \Psi_{\uparrow} (x). \]  

(2.1)

Here \( \Psi_{\sigma} (x), \Psi_{\sigma}^\dagger (x) \) are the atom annihilation and creation operators respectively for the spin-up and spin-down component satisfying the anticommutation relation

\[ \left[ \Psi_{\sigma} (x), \Psi_{\sigma'}^\dagger (x') \right]_+ = \delta_{\sigma,\sigma'} \delta (x - x'). \]  

(2.2)

g is the atom-atom coupling, and \( \mu_{\sigma} \) the corresponding chemical potential.

For intuitive and simplicity reasons we will start by having our system in the absence of an optical lattice, since the approach for a dilute Fermi gas in an optical lattice we are interested in, is more the less the same. Having in mind the Hamiltonian (2.1) one can formulate the action from which we obtain, after performing a Hubbard-Stratonivich transformation using a complex field \( \Delta (x,t) \) that turns out to be the order parameter, the Green’s function for the fermions in momentum space. The poles of this Green’s function give us the condition

\[ \hbar \omega (k) = \sqrt{(\varepsilon (k) - \mu)^2 + |\Delta|^2}. \]  

(2.3)

where \( \mu \) is the average of the chemical potential for spin up and spin down components. This is the BCS dispersion relation where we can distinguish a gap of magnitude of the order parameter. It is known in this well studied theory that any attractive force between two electrons creates a bound state, the Cooper pair. The amount of energy required to break this pair into separate electrons is the energy
gap in the spectrum this bound state suggests. Also \( \varepsilon(k) \), the kinetic energy of a single atom, equals to \( \hbar^2 k^2 / 2m \).

## 2.1 Mean-field approach

Our next step is to apply the mean field approximation. The advantage of this theory is that it reproduces typical properties exhibited by a superfluid in terms of a set of parameters having a clear physical meaning and in a way that we also avoid heavy numerical estimations. Especially for dilute gases at zero temperature it is known that mean-field theories describe most of the effects of two-body interactions. Furthermore recent experimental results have shown that such a theory is quite accurate. Later on we will see that it really reproduces known phenomena (like the superfluid phase transition of Cooper pairs). In order to apply the mean field theory we start by assuming that we have found the saddle point\(^1\) solution and expanding around the order parameter \(-\frac{\Delta}{g} = \langle \Psi_\uparrow \Psi_\downarrow \rangle\) after performing a Hubbard-Stratonovich transformation, which in Fock space language is

\[
g\Psi_\uparrow \Psi_\downarrow \Psi_\downarrow \Psi_\uparrow \approx -\frac{|\Delta|^2}{g} - \Psi_\uparrow \Psi_\downarrow \Delta - \Delta^* \Psi_\downarrow \Psi_\uparrow, \tag{2.4}
\]

and Fourier transform. Then the mean field Hamiltonian we obtain is

\[
H_{MF} = \frac{1}{V} \sum_k \left( \Psi_{k,\uparrow} \Psi_{-k,\downarrow} \right) \left( \begin{array}{cc} \varepsilon(k) - \mu_\uparrow & -\Delta \\ -\Delta^* & \varepsilon(k) - \mu_\downarrow \end{array} \right) \left( \begin{array}{c} \Psi_{k,\uparrow} \\ \Psi_{-k,\downarrow} \end{array} \right)
\]

\[
+ \frac{1}{V} \sum_k \left( \varepsilon(k) - \mu_\downarrow \right) - \frac{|\Delta|^2}{g} \tag{2.5}
\]

### 2.2 Bogolioubov Transformation

Having formed the mean field Hamiltonian it will prove convenient to switch to quasiparticle description. In this picture the quasiparticles and holes complement the difference between the spin-up and spin-down densities, as it can be seen in Figure 2.1.

\(^1\)Actually the saddle-point Hamiltonian is synonymously known as the mean-field Hamiltonian.
Figure 2.1 "Transition" from spin-up and spin-down components to quasiparticles and holes picture.

Also in the case under consideration we simply mention that the effective mass of the Cooper pair is again $m$. In the next chapters (3 and 4) we will see in more detail manner the derivation of the effective mass and some interesting features of the quasiparticles. The reason that we are not doing this now is because it will be easier to see the physical differences of the quasiparticles and real ones in an optical lattice rather in the vacuum. Now we simply mention that with this approximation we consider electrons to be almost free and dressed by the interactions. This procedure works because the interactions can be integrated away in a sense of effective theories. Historically the idea of the quasiparticles comes from Landau’s theory of Fermi liquids. Mathematically this transition to the quasiparticle picture is possible with the use Bogolioubov transformation which diagonalizes the BCS Hamiltonian

$$b_{-k,\downarrow} = u(k) \Psi_{-k,\downarrow} + v(k) \Psi_{-k,\uparrow}$$

$$b_{k,\uparrow} = u(k) \Psi_{k,\uparrow} - v(k) \Psi_{-k,\downarrow} \quad (2.6)$$

for the quasiparticles and

$$\Psi_{k,\uparrow} = u(k) b_{k,\uparrow} + v(k) b_{-k,\downarrow}$$

$$\Psi_{-k,\downarrow} = u(k) b_{-k,\downarrow} + v(k) b_{k,\uparrow} \quad (2.7)$$

for the atoms.

The anticommutation rules for quasiparticle operator are the same as for the atom operators that is

$$[b_{k,\sigma}, b_{k',\sigma'}^\dagger]_+ = \delta_{\sigma,\sigma'},$$

which provides us with the condition $|u(k)|^2 + |v(k)|^2 = 1$. From now on we will use instead of the chemical potential $\mu_\sigma$ for the spin components, the sum and difference of the chemical potentials, introducing the parameters

$$\mu = \frac{\mu_\uparrow + \mu_\downarrow}{2}$$

$$\hbar = \frac{\mu_\uparrow - \mu_\downarrow}{2}. \quad (2.8)$$

Then one can also prove that

$$|v(k)|^2 = \frac{\hbar \omega(k) - (\varepsilon(k) - \mu)}{2\hbar \omega(k)}$$

$$|u(k)|^2 = \frac{\hbar \omega(k) + (\varepsilon(k) - \mu)}{2\hbar \omega(k)}$$

$$u(k) v(k) = \frac{|\Delta|}{2\hbar \omega(k)}$$
2.2 Bogolioubov Transformation

hold. The dispersion relation for the quasiparticles is

$$\hbar \omega \pm (k) = \hbar \omega (k) \pm \hbar.$$ (2.9)

Then we can write the mean field Hamiltonian in terms of

$$(b_{k,\uparrow}, b_{-k,\downarrow}^\dagger) = P \left( \Psi_{k,\uparrow}, \Psi_{-k,\downarrow}^\dagger \right), \quad P = \begin{pmatrix} u(k) & -v(k) \\ v(k) & u(k) \end{pmatrix}.$$ (2.10)

using the trivial diagonalization procedure where

$$P^\dagger D P = \begin{pmatrix} \varepsilon(k) - \mu \uparrow \\ -|\Delta| & - (\varepsilon(k) - \mu \downarrow) \end{pmatrix}, \quad D = \begin{pmatrix} \hbar \omega_-(k) & 0 \\ 0 & -\hbar \omega_+(k) \end{pmatrix}.$$ (2.11)

Then after some algebraic calculations we find that the mean field Hamiltonian for

the quasiparticles becomes

$$H_{MF} = \frac{1}{V} \sum_k \left( \hbar \omega_-(k) b_{k,\uparrow}^\dagger b_{k,\uparrow} + \hbar \omega_+(k) b_{-k,\downarrow}^\dagger b_{-k,\downarrow} \right)$$

$$+ \frac{1}{V} \sum_k (\varepsilon(k) - \mu - \hbar \omega(k)) - \frac{|\Delta|^2}{2},$$ (2.12)

where we can distinguish two terms. The first one corresponds to the Fermi-shell contributions while the second one corresponds to the vacuum contribution. In Figure 2.2 we see the dispersion relation and the thermodynamic potential. For these plots we assumed that we have a periodicity in the k-space with a period of $2\pi$, chosen such that we can compare with the case of an optical lattice where the first Brillouin zone is bounded in the interval $-\pi \leq k \leq \pi$. As a reminder $\varepsilon(k) = \hbar^2 k^2/2m$. The Fermi-shell contributions come from the condition $\hbar \omega_-(k) < 0$:

**Limits of integration**

After some algebra one can find for the atom energy of the Fermi shell

$$\varepsilon_A = \varepsilon(k_A) = \mu - \sqrt{\hbar^2 - |\Delta|^2}$$

$$\varepsilon_B = \varepsilon(k_B) = \mu + \sqrt{\hbar^2 - |\Delta|^2}.$$ (2.13)

From which we can deduce

- $\varepsilon_A, \varepsilon_B$ are real

$$\hbar^2 \geq |\Delta|^2.$$ (2.14)

---

2That is in other words the equivalent of a three dimensional BCS in a box with $k_\rho = \frac{2\pi}{L} h_\rho$ and $L = \pi$. 


• $\varepsilon_A \geq 0$ gives us
\[
\mu \geq \sqrt{h^2 - |\Delta|^2}
\] (2.15)

• Finally $\varepsilon_B \geq 0$ provided that if $\varepsilon_A < 0$ we set $\varepsilon_A = 0$ is always true if the previous conditions are satisfied.

\textbf{Figure 2.2} Dispersion relation dependance from momentum $k$ and thermodynamic potential against the order parameter. We see from the thermodynamic potential that we have of first-order phase transition. The contributions of the Fermi-shell to the thermodynamic potential arise for small values of $\Delta/t$, color-full lines. As parameter $h/t$ increases the fermi shell contributions increase the absolute value of the thermodynamic potential. We can also distinguish the equal population limit (standard BCS limit) for $h = 0$.

Even in this case we notice a first-order phase transition as chemical potential difference $h$ increases. We will study this transition in detail in an optical lattice, where we can actually tune the parameters (see Capter 3).
Chapter 3

BCS in an optical lattice

We place our system in an optical lattice and a band gap appears in the dispersion relation. Now because of the lattice periodicity we change the sum over momenta to a sum over the first Brillouin zone $|k| \in (-\pi/\hbar, \pi/\hbar)$ and a sum over bands. Here $\hbar = \lambda/2$ is the lattice spacing generated by a laser light of wavelength $\lambda$.

3.1 Tight-binding approximation

We assume the interaction energy is small so that we consider the effect of the lowest band of the lattice [12]. The higher bands cannot be excluded but can be treated as dispersionless or flat.

We consider a cubic lattice with the harmonic oscillator ground state where the nearest-neighbor hopping term is described by the Hubbard Hamiltonian. Because of the periodicity of the optical potential it will prove very convenient to use an alternative single-particle basis, the so-called Wannier functions (WF) that form for each band a complete basis. The annihilation operator for the particle can be expanded as

$$\Psi (x, t) = \sum_{b,i} a_{b,i} \chi_b (x - x_i) . \quad (3.1)$$

Here $i$ labels the lattice sites, $b$ the band and $a_{b,i}$ the annihilation operator for the Wannier states. The main difference with the Bloch functions is that Wannier functions do not diagonalize the Hamiltonian, they depend only on the relative distance $x \pm h_b$ while they are not necessarily well localized at site $x$. In our case though where the dominant contributions come from the lowest band WF they are indeed centered around the lattice site. The orthonormality relation for these functions is

$$\int d^3x \chi_b^* (x - x_i) \chi_{b'} (x - x_{i'}) = \delta_{b,b'} \delta_{x_i,x_{i'}} \quad (3.2)$$
Exploiting the periodicity of the potential \( x \rightarrow x + h \rho \) once again we can express the (spin independent) Wannier functions in terms of Bloch wave functions \( u_{b,k}(x) \) for \( N_s \) the volume of the space

\[
\chi_b(x - x_i) = \sum_k u_{b,k}(x) \frac{e^{-ikx_i}}{\sqrt{N_s}}.
\]  (3.3)

Finally we have to mention that choosing appropriate phases for the Bloch function there is a unique WF for each band, real and exponentially localized.

Next we return to the problem we are interested in. Plugging (3.1) in the BCS Hamiltonian (2.1) for the lowest band only, here on we are going to omit the band index \( b = 0 \), and define

\[
\varepsilon_{\sigma} = \int d\mathbf{x} \chi^\dagger(\mathbf{x}) \left( -\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{x}) \right) \chi(\mathbf{x}) = \frac{3\hbar \omega}{2},
\]

\[
t = -\int d\mathbf{x} \chi^\dagger(\mathbf{x} - \mathbf{h}_\rho) \left( -\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{x}) \right) \chi(\mathbf{x}),
\]  (3.4)

\[
U = g \int d\mathbf{x} \chi^\dagger_\uparrow(\mathbf{x}) \chi^\dagger_\downarrow(\mathbf{x}) \chi_\downarrow(\mathbf{x}) \chi_\uparrow(\mathbf{x}),
\]

where \( \varepsilon_{\sigma} \) is the ground state of the harmonic oscillator that we use for simplicity reasons, \( t \) is the hoping matrix element for nearest neighbors only, which corresponds to the so-called tight-binding approximation, while \( U \) is the coupling constant localized for a two-particle level. All these quantities can be easily calculated from atomic physics for the Fermi gas we want to have\(^2\). Then the Hubbard Hamiltonian in the second quantized form is

\[
H = \sum_{i,\sigma} a^\dagger_{i\sigma} (\varepsilon_{\sigma} - \mu_{\sigma}) a_{i\sigma} + U \sum_{i,\sigma} a^\dagger_{i\uparrow} a^\dagger_{i\downarrow} a_{i\downarrow} a_{i\uparrow} - t \sum_{\langle ij \rangle,\sigma} a^\dagger_{i\sigma} a_{j\sigma}. \]  (3.5)

We denote the distance between two nearest-neighbors as \( \mathbf{h}_\rho = \frac{1}{2} \mathbf{e}_\rho \) where \( \mathbf{e}_\rho \) is the unit vector in x,y,z direction. In the last equation we can see that we have to deal with only one site, except the last term where a summation over the nearest neighbor, for each lattice site, appears. We can avoid the existence of the different label site if we switch to the momentum space. To do this we perform a Fourier transform so that the Hamiltonian in k-space is

\[
H = \frac{1}{N_s} \sum_{k,\sigma,\rho} a^\dagger_{k\sigma} (\varepsilon_{\sigma} - \mu_{\sigma} - 2t \cos(\mathbf{k} \cdot \mathbf{h}_\rho)) a_{k\sigma} + \frac{U}{N_s} \sum_{k,\sigma} a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow} a_{-k\downarrow} a_{k\uparrow}. \]  (3.6)

\(^1\)That is Wannier functions do not approach the local harmonic oscillator ground state of each well but decay exponentially. The consequence of this, as we expected a Gaussian decay, is that they have modes in order to guarantee the orthonormality condition (3.2). But what makes this approximation valid is that the overlap between the WF and the Gaussian ground state is good enough even for a swallow optical potential.

\(^2\)For most of the numerical estimations we are going to rescale by dividing with the hoping matrix element so that we will not need to calculate explicitly these quantities. When we consider the case of \(^{40}\)K case we will see how these quantities can be estimated.
3.2 Mean-field approximation

Following the approach of the previous chapter one can find that the dispersion relation is again of the form

\[ \hbar \omega (k) = \sqrt{(\varepsilon (k) - \mu)^2 + |\Delta|^2}, \]  

but this time the atom energy in the optical lattice is

\[ \varepsilon (k) = -2t \sum_\rho \cos (k \cdot h_\rho). \]  

We will work in the mean-field approximation, after applying a Hubbard-Stratonovich transformation and neglecting quadratic terms in the fluctuations. Then the Hubbard Hamiltonian is

\[
H_{MF} = \frac{1}{N_s} \sum_k \left( a_{k,\uparrow}^\dagger a_{-k,\downarrow} + a_{k,\downarrow}^\dagger a_{-k,\uparrow} \right) \left( \begin{array}{cc}
\varepsilon_\sigma + \varepsilon (k) - \mu_{\uparrow} & 0 \\
0 & - (\varepsilon_\sigma + \varepsilon (k) - \mu_{\downarrow})
\end{array} \right) \left( \begin{array}{c}
a_{k,\uparrow}^\dagger \\
a_{-k,\downarrow}^\dagger
\end{array} \right) \\
+ \frac{1}{N_s} \sum_k (\varepsilon_k - \mu_{\downarrow}) - \frac{|\Delta|^2}{U},
\]

At this point for reasons that will become obvious later (equation (3.14)) we shift the chemical potential by the maximum hopping value and because we will describe only homogeneous lattices we also treat \( \varepsilon_\sigma \) as an another shift in the chemical potential, that is

\[ \mu \to \mu + 6t - \varepsilon_\sigma \]  

and rewrite \( \varepsilon (k) \) as

\[ \varepsilon (k) = -2t \sum_\rho (\cos (k \cdot h_\rho) - 1). \]  

in order to absorb the shift of the chemical potential.

3.3 Bogolioubov transformation

As we mention before the Wannier functions do not diagonalize the Hamiltonian. For this reason we turn to the quasiparticle description by means of the Bogolioubov transformation, that as it is well known diagonalizes the mean-field Hamiltonian. Using (2.6), (2.7) and after some trivial calculations we deduce for the mean-field Hamiltonian

\[
H_{MF} = \frac{1}{N_s} \sum_k \left( \hbar \omega_- (k) b_{k,\downarrow}^\dagger b_{k,\uparrow} + \hbar \omega_+ (k) b_{-k,\downarrow}^\dagger b_{-k,\uparrow} \right) \\
+ \frac{1}{N_s} \sum_k (\varepsilon (k) - \mu - \hbar \omega (k)) - \frac{|\Delta|^2}{U},
\]
As a reminder the dispersion relation for these particles is of the form

\[
\hbar \omega_\pm (k) = \hbar \omega (k) \pm \hbar,
\]

\[
\hbar \omega (k) = \sqrt{(\varepsilon (k) - \mu)^2 + |\Delta|^2}.
\] (3.13)

Now in the quasiparticle picture an atom travels through our atomic gas, interacts with the surrounding atoms and temporarily drags them along. In general the quasi-atoms can see an averaged potential while their effective mass can be calculated with various ways. One of these is to expand in Taylor series the energy of the quasiatom (3.11) neglect higher order terms, since \( T = 0 \) the momenta also holds \( k \to 0 \), and deduce \( \epsilon_k = t k^2 \lambda^2 / 2 \). Comparing with \( \epsilon_k = \hbar^2 k^2 / 2m^* \) it is easy to find that the effective mass is

\[
m^* = \frac{\hbar^2}{t \lambda^2}.
\] (3.14)

From now on we are going to use another way of calculating the effective mass in order to avoid the shift of the chemical potential like (3.10). We will take advantage that we are dealing with a lattice where we have assumed that only the contributions of the lowest band really matter and use

\[
\frac{1}{m^* (k)} = \frac{1}{\hbar^2} \frac{\partial^2 \varepsilon}{\partial k^2}.
\] (3.15)

Here the effective mass is defined through the curvature of the lowest energy band,[26]. Then we derive for the mass of the quasiatom

\[
m^* (k) = \frac{\hbar^2}{\lambda^2 t \cos (k \lambda / 2)}.
\]

From this we obtain the same result, equation (3.14), for the effective mass for \( k \to 0 \). Comparing this effective mass with the effective mass of the previous chapter, the one we had for the vacuum case, we see that now this mass depends on the laser frequency and the hopping matrix element, while before it was simply the electron mass.

### 3.4 Thermodynamic potential

Physically we are interested in is the free energy or thermodynamic potential, the minimization of which, as we know from statistical theory, with respect to the order parameter gives us the gap equation, while its partial derivative with respect to the chemical potential provides us with the equation of state. In order to obtain the thermodynamic potential we have to take the quantum-mechanical expectation value of the mean-field Hamiltonian in equation (3.12), which is actually the ground-state expectation value. The contribution from the number operator comes again
from $\hbar \omega_-(k) < 0$ where the fermi surface exists. The thermodynamic potential of the system then is

$$\Omega(\Delta, \Delta^*) = \langle \text{g.s.} \left| H_{MF} \right| \text{g.s.} \rangle = \frac{1}{N_s} \sum_{k \in \text{Fermi shell}} \hbar \omega_-(k) + \frac{1}{N_s} \sum_k (\varepsilon(k) - \mu - \hbar \omega(k)) - \frac{|\Delta|^2}{U}.$$  \hfill (3.16)

The form of the thermodynamic potential is such, that numerical approach is going to be used. For this we switch to the continuum limit by replacing

$$\frac{1}{N_s} \sum_k \rightarrow V \int \frac{dk}{(2\pi)^3} = \Omega_v \int \frac{dk}{(2\pi)^3}.$$

where $\Omega_v$ is the unit volume of the unit cell. We choose $\Omega_v = h^3 = \left(\frac{\lambda}{2}\right)^3$, with $h$ the lattice spacing. Then in the continuum limit $\Omega$ becomes

$$\Omega = \frac{\lambda^3}{(4\pi)^3} \int \frac{dk_x}{k_{Ax}} \int \frac{dk_y}{k_{Ay}} \int \frac{dk_z}{k_{Az}} \sqrt{(-\mu - 2t \left(\cos \frac{\lambda}{2} k_x + \cos \frac{\lambda}{2} k_y + \cos \frac{\lambda}{2} k_z - 3\right))^2 + |\Delta|^2} - \frac{\lambda^3}{(4\pi)^3} \int \frac{dk_x}{2\pi} \int \frac{dk_y}{2\pi} \int \frac{dk_z}{2\pi} (\varepsilon(k) - \mu - \hbar \omega(k)) - \frac{|\Delta|^2}{U}.$$ \hfill (3.17)

Due to the limitations of the (quasi)atom energy and more specific due to $\cos(k \cdot h_p) \in [-1,1]$ term, the limits for the Fermi shell $k_A, k_B$ have angular dependence. Setting $\hbar \omega_-(k)$ to zero yields for $k_A, k_B$

$$k_x = \frac{2}{\lambda} \arccos \left( \frac{1}{2t} \left( -\mu + 6t \pm \sqrt{h^2 - |\Delta|^2} \right) - \cos \left( \frac{\lambda}{2} k_y \right) - \cos \left( \frac{\lambda}{2} k_z \right) \right), \hfill (3.18)$$

while the amplitude of the momentum vector for the fermi shell can be found, at least as a first approximation, to be

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \frac{4}{\lambda^2 t} \left( \mu \pm \sqrt{h^2 - |\Delta|^2} \right), \hfill (3.19)$$

by expanding the cosine term up to first order.

### 3.5 Equation of state

Having the thermodynamic potential at hand we can compute the equation of state, as statistical theory tells us, by taking the partial derivative with respect to the averaged chemical potential

\[^3\text{This choice is made for consistency reasons, so that the dimensions of } U \text{ follow } [\Omega_v U^2] = [g^2]\]
\[ n = -\frac{\partial \Omega}{\partial \mu} \]  

(3.20)

and check whether we are on the right track by checking if our result for \( n \) is of the form

\[ n = 2 \sum_k |v_k|^2 + \text{shell contr.} \]

where the first term is the outcome of the well-known BCS theory. Using for the derivatives

\[ \frac{\partial \hbar \omega(k)}{\partial \mu} = -\frac{\varepsilon(k)-\mu}{\hbar \omega(k)} \]

\[ -\frac{\partial}{\partial \mu} \left( \frac{1}{N_s} \sum_{k \in \text{Fermi shell}} (\hbar \omega(k) - h) \right) = -\frac{1}{N_s} \sum_{k \in \text{Fermi shell}} \left( -\frac{\varepsilon(k)-\mu}{\hbar \omega(k)} \right), \]

and after some algebra\(^4\), we obtain for the equation of state:

\[ n = 2 \sum_k |v_k|^2 + \frac{1}{N_s} \sum_{k \in \text{Fermi shell}} \left( \frac{\varepsilon(k) - \mu}{\hbar \omega(k)} \right) \]  

(3.21)

We see that we retrieve the BCS result as expected and an extra term from the Fermi shell contribution. Now that we know that our model works well we can continue to the continuum limit, which can be worked for the numerical estimations. Making the shift similarly with the thermodynamic potential, the equation of state becomes

\[ n = \frac{\lambda^3}{(4\pi)^3} \int \frac{2\pi}{k_B} \int \frac{2\pi}{k_B} \int \frac{2\pi}{k_B} \left( 1 - \frac{\left( -2t(\frac{1}{2}k_x + \frac{1}{2}k_y + \frac{1}{2}k_z - 3) - \mu \right)}{\sqrt{\left( -\mu - 2t(\frac{1}{2}k_x + \frac{1}{2}k_y + \frac{1}{2}k_z - 3) \right)^2 + |\Delta|^2}} \right) \]

\[ + \frac{\lambda^3}{(4\pi)^3} \int \frac{k_B}{k_B} \int \frac{k_B}{k_B} \int \frac{k_B}{k_B} \left( -2t(\frac{1}{2}k_x + \frac{1}{2}k_y + \frac{1}{2}k_z - 3) - \mu \right) \]

\[ \left( -\mu - 2t(\frac{1}{2}k_x + \frac{1}{2}k_y + \frac{1}{2}k_z - 3) \right)^2 + |\Delta|^2 \]

(3.22)

here the first term corresponds to is the vacuum contributions while the second to the Fermi shell. Note that the momentum \( \mathbf{k} \) appears only in cosine terms which means that we can exploit the symmetry of this problem and use only the values of positive momenta.

\(^4\)Here we also used for the coherence factors

\[ |v_k|^2 = \frac{\hbar \omega(k) - (\varepsilon(k) - \mu)}{2\hbar \omega(k)}, \]

and

\[ |u_k|^2 = \frac{\hbar \omega(k) + (\varepsilon(k) - \mu)}{2\hbar \omega(k)}, \]

\[ u_k v_k = \frac{1}{2\hbar \omega(k)}. \]
3.6 Gap equation

Once again from statistical theory we know that the gap equation can be derived from the minimization of the thermodynamic potential with respect to the conjugate of the order parameter

$$\frac{\partial \Omega}{\partial \Delta^*} = 0,$$

(3.23)

in combination with

$$\frac{\partial h\omega(k)}{\partial \Delta^*} = \frac{\Delta}{2h\omega(k)},$$

provides us with the gap equation

$$\frac{1}{U} = \sum_{k \in \text{Fermi shell}} \left( \frac{1}{2h\omega(k)} \right) - \sum_k \left( \frac{1}{2h\omega(k)} \right).$$

(3.24)

The values of the momenta that constitute the Fermi shell are in general less than the periodicity of the Brioullin zone, or to put it differently, the first sum has less terms than the second one. Then our coupling constant is negative as expected for an attractive model such as BCS Hamiltonian we use. For the numerical estimation once again we go to the continuum limit by the usual procedure that yields

$$\frac{1}{U} = \frac{\lambda^3}{(4\pi)^3} \int \frac{dk_x}{k_{Ax}} \int \frac{dk_y}{k_{Ay}} \int \frac{dk_z}{k_{Az}} \frac{1}{2\sqrt{\left(-\mu - 2t\left(\cos \frac{\lambda}{2}k_x + \cos \frac{\lambda}{2}k_y + \cos \frac{\lambda}{2}k_z - 3\right)\right)^2 + |\Delta|^2}}$$

$$- \frac{\lambda^3}{(4\pi)^3} \int \frac{dk_x}{-\frac{2\pi}{k_{Ax}}} \int \frac{dk_y}{\frac{2\pi}{k_{Ay}}} \int \frac{dk_z}{\frac{2\pi}{k_{Az}}} \frac{1}{2\sqrt{\left(-\mu - 2t\left(\cos \frac{\lambda}{2}k_x + \cos \frac{\lambda}{2}k_y + \cos \frac{\lambda}{2}k_z - 3\right)\right)^2 + |\Delta|^2}}$$

(3.25)

It is easy to see that we have to calculate an integral that has even dependence with respect to the integration variable \(k\) since only cosine terms appear.

3.7 Magnetization

For reasons that will become obvious in the next chapter, we want to calculate the magnetization of the Fermi gas. This can be done by minimizing the thermodynamic potential with respect to the difference of the chemical potential \(h\). One can find that the magnetization depends only on the Fermi shell part by integrating by parts, in the continuum limit, and having in mind that the boundary terms drop out. Then
we can write

\[ m = \frac{\partial \Omega}{\partial h} = \frac{\partial}{\partial h} \left( \frac{1}{N_s} \sum_{k \in \text{Fermi shell}} (\hbar \omega(k) - h) \right) = -\frac{1}{N_s} \sum_{k \in \text{Fermi shell}} 1 \]

Our next step is to perform the numerical calculations that will give us some intuition, before moving to the Fulde-Ferrell phase transition. At this point we will use rescaled parameters.

### 3.8 Numerical estimates, one-dimensional Case.

For simplicity reasons, we will try to solve the one-dimensional case. It is important not to forget that one-dimensional optical lattices can be created and studied, that is our results correspond to a real situation even they refer to the somehow unnatural one dimension. For this we rescale the thermodynamic potential \( \Omega \), the chemical potential \( \mu \), gap \( \Delta \) and \( h \) as ratios over \( t \)

\[ \tilde{h} = \frac{h}{t}, \quad \tilde{\Delta} = \frac{\Delta}{t}, \quad \tilde{\mu} = \frac{\mu}{t}, \quad \tilde{\Omega} = \frac{\Omega}{t}. \]

(3.26)

#### 3.8.1 Dispersion relation.

Up to now we have calculated the physical quantities we are interested in for three dimensions. It is quite trivial to obtain the ones for one dimension, simply by dropping the summations over the three dimensions or the integrations over \( y, z \) directions. Using rescaled \( \tilde{\mu}, \tilde{\Omega}, \tilde{\Delta}, \tilde{h} \) the atom energy in the optical lattice and the dispersion relation simplifying (3.7) respectively is

\[ \varepsilon(k) = -2 \left( \cos \left[ \frac{\lambda}{2} k \right] - 1 \right) \]

\[ \hbar \omega(k) = \sqrt{\left( -\tilde{\mu} + 2 - 2 \cos \left[ \frac{\lambda}{2} k \right] \right)^2 + \tilde{\Delta}^2} \]  

(3.27)

while the condition \( \hbar \omega(k) = 0 \) provides us with the limits \( k_A, k_B \) for the Fermi shell

\[ k_{A,B} = \frac{2}{\lambda} \arccos \left( \frac{1}{2} \left( -\tilde{\mu} + 2 \pm \sqrt{\tilde{h}^2 - \tilde{\Delta}^2} \right) \right) \]

(3.28)

The arguments\(^5\)of cosine terms give us the conditions for the Fermi shell

\(^5\)While after expanding up to first order the cosine term

\[ k^2 = \frac{4}{\lambda^2} \left( \tilde{\mu} \pm \sqrt{\tilde{h}^2 - \tilde{\Delta}^2} \right) \]
• For $k_A$

$$4 + \sqrt{\tilde{h}^2 - \left|\tilde{\Delta}\right|^2} \geq \tilde{\mu} \geq + \sqrt{\tilde{h}^2 - \left|\tilde{\Delta}\right|^2}$$

For $k_B$

$$4 - \sqrt{\tilde{h}^2 - \left|\tilde{\Delta}\right|^2} \geq \tilde{\mu} \geq - \sqrt{\tilde{h}^2 - \left|\tilde{\Delta}\right|^2}$$

• Since $h, \Delta \geq 0$ it is $h \geq \Delta$

\[\text{Figure 3.1} \text{ Dispersion relation. Fermi shell exists for negative values of k}\]

One can see the similarity of this dispersion relation with the one of the previous model, as expected since $\epsilon(k) \propto \cos(k)$ for an optical lattice bears a similitude to $\epsilon(k) \propto \frac{\hbar k^2}{2m}$ in vacuum we studied in the previous chapter.

3.8.2 Thermodynamic potential

Simplifying (3.17) while taking advantage of the symmetry of our problem and integrating over $(\lambda/2)k$ instead of $k$, we can derive the thermodynamic potential, Figure 3.2, for one dimension and write

$$\tilde{\Omega} = \frac{1}{2\pi} \int_{k_A'}^{k_B'} \frac{d\left(\frac{\lambda}{2} k\right)}{k'} \sqrt{(-\tilde{\mu} + 2 - 2 \cos \left[\frac{\lambda}{2} k\right])^2 + \left|\tilde{\Delta}\right|^2 - \tilde{h}}$$

$$+ \frac{1}{\pi} \int_{0}^{\pi} \frac{d\left(\frac{\lambda}{2} k\right)}{k} \left(-\tilde{\mu} - \sqrt{(-\tilde{\mu} + 2 - 2 \cos \left[\frac{\lambda}{2} k\right])^2 + \left|\tilde{\Delta}\right|^2 - \frac{\left|\tilde{\Delta}\right|^2}{\tilde{V}}}\right)$$

(3.29)

with the new boundaries for the fermi shell to be

$$k'_{A,B} = \arccos \left[\frac{1}{2} \left(-\tilde{\mu} + 2 \pm \sqrt{\tilde{h} - \left|\tilde{\Delta}\right|^2}\right)\right]$$
Figure 3.2 Thermodynamic potential for $\mu/t = 1$ ratio and different values for $h/t$. A phase transition occurs when the second minimum coincides with the first minimum. In this case we have a phase transition at $h/t = 3.01551$.

Again one can notice the similarity of this picture with the one of figure 2.1 (b). The form of the thermodynamic potential is the one of a first-order phase transition. In this case we have normal to superfluid transition, which takes place when the second minimum coincides with the first minimum (i.e. for $\Delta/t = 0$). It is worth mentioning that we have no Fermi shell contributions in the free energy for the superfluid. The Fermi shell contributions can be observed, in analogy with the model of chapter 2, for small values of the order parameter and they are the ones that make the difference for the thermodynamic potential as $h/t$ varies.

### 3.8.3 Equation of state

Using exactly the same simplifications as for the gap equation we obtain for the equation of state

\[
n = -\frac{1}{\pi} \int_0^\pi d\left(\frac{\lambda}{2}k\right) \left(1 - \frac{(\tilde{\mu} + 2 \cos \frac{\lambda}{2}k)}{\sqrt{(-\tilde{\mu} + 2 \cos \frac{\lambda}{2}k)^2 + |\Delta|^2}}\right) \]

\[
\frac{1}{2\pi} \int_{k'}^{k_B} d\left(\frac{\lambda}{2}k\right) \frac{(\tilde{\mu} + 2 \cos \frac{\lambda}{2}k)}{\sqrt{(-\tilde{\mu} + 2 \cos \frac{\lambda}{2}k)^2 + |\Delta|^2}},
\]

We can compare this with ideal Fermi case

\[
\mu = \frac{\pi^2 \hbar^2}{2m} \left(\frac{3n}{\pi}\right)^{2/3}.
\]

The form of the ideal case is $n \propto \mu^{3/2}$
3.8 Numerical estimates, one-dimensional Case.

Figure 3.3 Form of the equation of state for an ideal fermi gas.

Here for this diagram we have set the constant $2m^{3/2}/(3\pi^2\hbar^3) = 1$. Later on we are going to return to this in the real case of the potassium $^{40}K$ fermi gas, where we would be able to see the real difference from the ideal case. In the next section we present the numerical results.

3.8.4 Gap equation

Following the same procedure as in the previous sections using the equation

$$\frac{\partial \Omega}{\partial \Delta^*} = 0,$$

and

$$\frac{\partial \hbar \omega (k)}{\partial \Delta^*} = \frac{\partial}{\partial \Delta^*} \left[ \sqrt{(-\tilde{\mu} + 2 - 2 \cos \left[\frac{\lambda}{2} k\right])^2 + |\tilde{\Delta}|^2} \right] = \frac{\Delta}{2 \sqrt{(-\tilde{\mu} + 2 - 2 \cos \left[\frac{\lambda}{2} k\right])^2 + |\tilde{\Delta}|^2}},$$

we deduce for the gap equation

$$\frac{1}{U} = \frac{\lambda}{4\pi} \int_{k_A}^{k_B} dk \frac{d}{2\sqrt{(-\tilde{\mu} + 2 - 2 \cos \left[\frac{\lambda}{2} k\right])^2 + |\tilde{\Delta}|^2}} - \frac{\lambda}{4\pi} \int_{-2\pi}^{-\frac{2\pi}{\lambda}} dk \frac{d}{2\sqrt{(-\tilde{\mu} + 2 - 2 \cos \left[\frac{\lambda}{2} k\right])^2 + |\tilde{\Delta}|^2}}$$

$$= \frac{1}{2\pi} \int_{k'_A}^{k'_B} d\left(\frac{\lambda}{2} k\right) \frac{d}{2\sqrt{(-\tilde{\mu} + 2 - 2 \cos \left[\frac{\lambda}{2} k\right])^2 + |\tilde{\Delta}|^2}} - \frac{1}{\pi} \int_0^{\frac{\pi}{\lambda}} d\left(\frac{\lambda}{2} k\right) \frac{d}{2\sqrt{(-\tilde{\mu} + 2 - 2 \cos \left[\frac{\lambda}{2} k\right])^2 + |\tilde{\Delta}|^2}}$$

(3.31)

We minimize the thermodynamic potential (3.29) with respect to the gap numerically for different values of the averaged chemical potential, figures 3.4 - 3.5, and

\[\text{6} \] All the minimization procedures as well as the diagrams where performed in Mathematica 6.
for different values of the coupling constant, figures 3.6 – 3.7, respectively. From these results we can also notice the quasiparticle-quasihole symmetry. Also from the phase diagram of the phase transition, see figure 3.6, we can see the separation between the normal and superfluid phase. We will return to these results in the next chapter where we study the FF phase transition we are interested in.

Figure 3.4 First diagram: $\Delta/t$ where the thermodynamic potential is minimized with $\mu/t$, for $h/t$ such that a phase transition occurs (critical $h/t$). Second diagram: $h_c/t$ with $\mu/t$. Notice the quasiparticle - hole symmetry for the shifted and rescaled chemical potential $0 \geq \mu/t > 2$ and $2 \geq \mu/t \geq 4$ respectively.

Figure 3.5 Rescaled order parameter with $h_c/t$. 
3.9 Weak-coupling limit.

Now we want to find an equation for the order parameter in the weak-coupling limit where, as we will see, we can use some approximations for the chemical potential and the atom energy. We start with the gap equation considering the superfluid solution (vacuum contributions) for which we have found

\[
\frac{1}{U} = -\frac{1}{\pi} \int_0^{\pi} d\left(\frac{\lambda}{2} k\right) \frac{1}{2 \sqrt{(-\tilde{\mu} + 2 - 2 \cos \left[\frac{\lambda}{2} k\right])^2 + |\tilde{\Delta}|^2}}. \tag{3.32}
\]

Near the edge of the band, where our shifted chemical potential is \(\tilde{\mu} = \delta \tilde{\mu}\), and small gap the main contribution comes from the term \( -\delta \tilde{\mu} + 2 - 2 \cos \left[\frac{\lambda}{2} k\right] \approx 0 \) which now can be expanded as

\[-\delta \tilde{\mu} + 2 - 2 \cos \left[\frac{\lambda}{2} k\right] = -\delta \tilde{\mu} + 2 \left(1 - \frac{1}{2} k^2\right) + \ldots.\]

Setting \(\varepsilon = k^2\) in the gap equation (3.32), in this limit becomes

\[
\frac{1}{U} = -\frac{1}{4\pi} \int_0^{\pi^2} d\varepsilon \frac{1}{\sqrt{\varepsilon} \sqrt{(\varepsilon - \delta \tilde{\mu})^2 + |\tilde{\Delta}|^2}}.
\]
We see that for $\varepsilon = \delta\tilde{\mu}$ the above integral diverges to infinity. If we expand $\varepsilon$ around this value, as $\Delta \to 0$,

$$\sqrt{\varepsilon} \approx \sqrt{\delta\tilde{\mu} + \frac{(\varepsilon - \delta\tilde{\mu})}{2\sqrt{\delta\tilde{\mu}}} - \ldots}$$

and set $x = \frac{\varepsilon - \delta\tilde{\mu}}{\Delta}$ the above integrant can be written as

$$\frac{1}{\tilde{U}} = -\frac{1}{4\pi\sqrt{\delta\tilde{\mu}}\Delta} \int_{\frac{\delta\tilde{\mu}}{\Delta}}^{\frac{\varepsilon - \delta\tilde{\mu}}{\Delta}} \frac{dx}{\sqrt{x^2 + 1}}.$$

Finally setting $x = \sinh y$ gives us

$$\frac{1}{\tilde{U}} = -\frac{1}{4\pi\sqrt{\delta\tilde{\mu}}} \int_{\sinh^{-1}\left(\frac{\delta\tilde{\mu}}{\Delta}\right)}^{\sinh^{-1}\left(\frac{\varepsilon - \delta\tilde{\mu}}{\Delta}\right)} \frac{\cosh(y)dy}{\sinh(y)} \quad (3.33)$$

In order to simplify (3.33) we will use

$$\ln \left[ x + \sqrt{x^2 + 1} \right] = \sinh^{-1}(x),$$

and after some algebra we obtain

$$\frac{1}{\tilde{U}} = -\frac{1}{4\pi\sqrt{\delta\tilde{\mu}}} \left[ \ln \frac{1}{\Delta^2} + \ln \left( 4\delta\tilde{\mu} \left( \pi^2 - \delta\tilde{\mu} \right) \right) \right] \quad (3.34)$$

where we used the expansion around $\Delta = 0$ for

$$\left( \pi^2 - \delta\tilde{\mu} + \sqrt{(\pi^2 - \delta\tilde{\mu})^2 + \Delta^2} \right) \left( \delta\tilde{\mu} + \sqrt{\delta\tilde{\mu}^2 + \Delta^2} \right) \approx 4\delta\tilde{\mu} \left( \pi^2 - \delta\tilde{\mu} \right) + \frac{(\pi^2 - \delta\tilde{\mu})^2}{\delta\tilde{\mu} (\delta\tilde{\mu} - \pi)} \Delta^2 + \ldots.$$  

From equation (3.34) we can find the expression for the gap

$$\Delta = 2\sqrt{\mu \left( \pi^2 - \mu \right)} \exp \left[ 2\pi\sqrt{\mu} \frac{t}{\tilde{U}} \right]$$

or equivalently

$$\ln \hat{\Delta} = \ln \left( 2\sqrt{\mu \left( \pi^2 - \mu \right)} \right) + \frac{2\pi\sqrt{\mu}t}{\tilde{U}} \quad (3.35)$$

In figure 3.7 we have a logarithmic plot of the gap resulting from the minimization of the thermodynamic potential versus the $t/\tilde{U}$ (purple color line). Blue colored line corresponds to equation 3.35. In the case drawn we have only 0.0058% difference from the theoretical value $2\pi\sqrt{\mu} = 1.40496$, in Table 3.1 we present for different coupling regimes the difference from the theoretical value. From the results obtained we can say that in the weak-coupling limit we have agreement.
3.9 Weak-coupling limit.

<table>
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<th>$t/U$</th>
<th>Numerically</th>
<th>Difference%</th>
</tr>
</thead>
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<td>23.31</td>
</tr>
<tr>
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<td>1.41558</td>
<td>0.7556</td>
</tr>
<tr>
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<td>0.0058</td>
</tr>
<tr>
<td>0.6</td>
<td>1.345</td>
<td>4.267</td>
</tr>
</tbody>
</table>

Table 3.1 Difference of the numerical estimations from theoretical (1.40496) value for different values of the rescaled coupling constant.

![Logarithmic plot of $\Delta/t$ with $t/U$ for $\mu/t = 0.05$ near the edge of the band so that the approximation is more accurate. Theoretical line 3.35, blue color, and numerical results (Figure 3.5 in a logarithmic plot), purple line.](image)

Figure 3.7 Logarithmic plot of $\Delta/t$ with $t/U$ for $\mu/t = 0.05$ near the edge of the band so that the approximation is more accurate. Theoretical line 3.35, blue color, and numerical results (Figure 3.5 in a logarithmic plot), purple line.

3.9.1 Magnetization

As we saw the magnetization, Figure 3.8, depends only on the Fermi shell contribution and is given by

$$\tilde{m} = -\frac{\lambda}{4\pi} \int_{k_A}^{k_B} dk = -\frac{1}{2\pi} (k_B - k_A).$$  \hspace{1cm} (3.36)
Upon looking in the magnetization along the phase transition we see that the magnetization is zero, as expected since as we have already mentioned in §3.7 the magnetization arises because of the Fermi shell. In this first order phase transition the contributions come from the vacuum term, that is we do not have a Fermi shell.
Chapter 4

Fulder-Ferrell and Larkin-Ovchinnikov phase.

In this project we are dealing with Fermi gases with unequal population for the spin up and down states. We can say that this problem is the analogous to the physics of a superconductor under the influence of a Zeeman field, resulting the difference in the chemical potential for the spin up and down species. Up to know we studied the situation for a Fermi gas where the Fermi surfaces for \( k, -k \) of the Cooper pairs match. It is possible though that the Fermi surfaces of the pairing spin particles \( k, -k \) do not match. Historically, in the weak-coupling limit Sarma,[13], showed that at zero temperature the uniform state with population imbalance is unstable. Furthermore he concluded by comparing the free energy of the normal and the superconducting state, in the normal equal population limit, that as the magnetic field is increased there is a first-order phase transition from the superconducting state to the normal one. Not long after this proposal Fulde and Ferrell (FF),[14], and independently one year later Larkin and Ovchinnikov (LO),[15], proposed better alternatives with even lower free energy than the states considered by Sarma (see also Figure 4.4 where we indicate the Sarma phase. It is clear that the free energy in the FF phase is lower than the one of the Sarma phase).

The basic idea of FF and LO is the same and it is based in the possibility, as we wrote to the previous paragraph, that it might be favorable to have pairing at non-zero total momentum. The Fermi surfaces are relatively shifted and the Cooper pairs are given momentum. This is, as mentioned in most of the literature, the FFLO phase. At this point we will not follow the mainstream of the literature and distinguish the FF phase from the LO phase. We justify this choice because of the many differences, except the main idea of pairing at non-zero momentum, between these two proposals. One of the main differences is that FF proposed that the order parameter is of the form \( \Delta (x) = \Delta_0 e^{ipx} \) while LO’s choice was an order parameter of the form \( \Delta (x) = \Delta_0 \cos (px) \). The last leads to an inhomogeneous problem where we have to involve the Bogoliubov-De Gennes equation. The first that showed the suitable attention was Yip,[16]. However, in both these cases one can show that
the free energy is lower than the one Sarma proposed. Now the Cooper pairs given momentum \( p \) have energy \( \varepsilon(p + k) \) and \( \varepsilon(p - k) \) while the total energy of the pair is

\[
\frac{\varepsilon(p + k) + \varepsilon(p - k)}{2}
\]

such that for \( p \to 0 \) the energy is \( \varepsilon(k) \).

Finally we have to mention that the unequal population for spin up and down particles is not the only case of the FF (or LO) phase transition. Studies for different mass between the spin up and spin down states have been performed, [23], and happen to be very interesting, leading to similar results. Also this phase transitions is not very interesting for condensed matter physicists only but also for high energy ones, since the FF (or LO better as we will see) phase transition is expected to exist in neutron stars. The most restive readers we refer to [27].

### 4.1 Fulde-Ferrell phase in BCS.

First we will consider the case of the homogeneous BCS Hamiltonian (2.1)

\[
H = \sum_{\sigma = \uparrow, \downarrow} \int dx \Psi_{\sigma}^\dagger(x) \left( -\frac{\hbar^2 \nabla^2}{2m} - \mu_{\sigma} \right) \Psi_{\sigma}(x) + g \int dx \Psi_{\uparrow}^\dagger(x) \Psi_{\downarrow}^\dagger(x) \Psi_{\downarrow}(x) \Psi_{\uparrow}(x)
\]

and use the Fourier transform

\[
\Psi_{\uparrow} = \frac{1}{\sqrt{V}} \sum_k \Psi_{\uparrow, p + k} e^{-i(p + k) \cdot h} \quad \Psi_{\downarrow} = \frac{1}{\sqrt{V}} \sum_k \Psi_{\downarrow, p + k} e^{i(p + k) \cdot h}
\]

\[
\Psi_{\uparrow} = \frac{1}{\sqrt{V}} \sum_k \Psi_{\uparrow, p - k} e^{-i(p - k) \cdot h} \quad \Psi_{\downarrow} = \frac{1}{\sqrt{V}} \sum_k \Psi_{\downarrow, p - k} e^{i(p - k) \cdot h}
\]

We see, after performing a Hubbard-Stratonovich transformation, that the only terms affected in the BCS Hamiltonian (2.1), except the energy of the Cooper pairs, are the ones proportional to the gap

\[
\Delta \Psi_{\uparrow, p + k} \Psi_{\uparrow, p - k} \to \Delta \Psi_{\uparrow, p + k} \Psi_{\uparrow, p - k} e^{-i(p + k) \cdot h} e^{-i(p - k) \cdot h} = \Delta \Psi_{\uparrow, p + k} \Psi_{\uparrow, p - k} e^{-2i(p + k) \cdot h}
\]

\[
\Delta^* \Psi_{\uparrow, p - k} \Psi_{\uparrow, p + k} \to \Delta^* \Psi_{\uparrow, p - k} \Psi_{\uparrow, p + k} e^{i(p - k) \cdot h} e^{-i(p - k) \cdot h} = \Delta^* \Psi_{\uparrow, p - k} \Psi_{\uparrow, p + k} e^{2i(p - k) \cdot h}
\]

So we can multiply the pair function by a plane-wane \( \Delta = -g \langle \Psi_{\uparrow, p - k} \Psi_{\downarrow, p + k} \rangle e^{2i(p \cdot h)} \) and write the corresponding mean field Hamiltonian as

\[
H_{MF}^{FF} = \frac{1}{\sqrt{V}} \sum_k \left( \Psi_{\uparrow, p + k} \Psi_{\uparrow, p - k} \right) \left( \begin{array}{cc}
\varepsilon(p + k) - \mu_{\uparrow} & -\Delta \\
-\Delta^* & -\varepsilon(p - k) - \mu_{\downarrow}
\end{array} \right) \left( \Psi_{\uparrow, p + k} \Psi_{\uparrow, p - k} \right)
\]

\[
+ \frac{1}{\sqrt{V}} \sum_k \left( \varepsilon(p + k) + \varepsilon(p - k) - \mu_{\downarrow} \right) - \frac{|\Delta|^2}{\mu_{\downarrow}}
\]

(4.2)

\[\text{1The sums over } k \text{ can be replaced by sums over } p + k \text{ so that the energy of an atom is not effected and only the off-diagonal terms (Cooper pairs) depend on momentum } p.\]
4.2 Fulde-Ferrell phase in an optical lattice.

The dispersion relation now is

$$\hbar \omega (k, p) = \sqrt{\left( \frac{\varepsilon (p + k) + \varepsilon (p - k)}{2} - \mu \right)^2 + \Delta^2}. \quad (4.3)$$

We switch again to the quasiparticle description by means of the Bogoliubov transformation that diagonalizes the mean field Hamiltonian and is of the same form as in the previous chapters, while the coherence factors have a bit different form than before

$$u^2 (k, p) = \frac{\hbar \omega (k, p) + \varepsilon (p + k) + \varepsilon (p - k) - \mu}{2\hbar \omega (k, p)},$$

$$v^2 (k, p) = \frac{\hbar \omega (k, p) - \varepsilon (p + k) - \varepsilon (p - k) - \mu}{2\hbar \omega (k, p)}, \quad (4.4)$$

$$u (k, p) v (k, p) = \frac{\Delta}{2\hbar \omega (k, p)}.$$ 

The diagonalized Hamiltonian is then

$$H_{MF}^{FF} = \frac{1}{V} \sum_k \begin{pmatrix} b_{\uparrow, p+k}^\dagger & b_{\downarrow, p-k} \end{pmatrix} \begin{pmatrix} \hbar \omega_- (k, p) & 0 \\ 0 & -\hbar \omega_+ (k, p) \end{pmatrix} \begin{pmatrix} b_{\downarrow, p+k} \\ b_{\uparrow, p-k} \end{pmatrix}$$

$$+ \frac{1}{V} \sum_k \left( \varepsilon (p + k) + \varepsilon (p - k) - \mu \right) - \frac{\hbar^2}{U} - \frac{\Delta^2}{U}. \quad (4.5)$$

with the quasiparticle dispersion relation

$$\hbar \omega_\pm (k, p) = \hbar \omega (k, p) \pm \left( \frac{\varepsilon (p - k) - \varepsilon (p + k)}{2} + h \right). \quad (4.6)$$

For the thermodynamic potential we expect again the contributions for the Fermi shell from the condition $\hbar \omega_- (k, p) < 0$ and it is

$$\Omega_{MF}^{FF} = \frac{1}{V} \sum_{k \in \text{Fermi Shell}} \hbar \omega_- (k, p)$$

$$+ \frac{1}{V} \sum_k \left( \frac{\varepsilon (p + k) + \varepsilon (p - k)}{2} - \mu - \hbar \omega (k, p) \right) - \frac{\hbar^2}{U} - \frac{\Delta^2}{U}. \quad (4.7)$$

4.2 Fulde-Ferrell phase in an optical lattice.

In an optical lattice, as we saw in chapter 2, the energy for the shifted chemical potential (3.8), is

$$\varepsilon (k) = -2t \sum_{\rho} (\cos |k \cdot h_{\rho}| - 1)$$
For the FF phase we will shift the chemical potential this time as

$$\mu \to \mu - \epsilon_\sigma$$

absorbing only the energy $\epsilon_\sigma$ since we consider homogeneous gases. The energy for an atom now depends also in the center-of-mass momentum $p$ and it is

$$\varepsilon_k \to \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} = \frac{-2t\sum_{\rho} \cos[(p+k) \cdot h_{\rho}]-2t\sum_{\rho} \cos[(p-k) \cdot h_{\rho}]}{2}$$

$$= -t\sum_{\rho} (\cos [(p+k) \cdot h_{\rho}] + \cos [(p-k) \cdot h_{\rho}]) = -2t\sum_{\rho} (\cos [p \cdot h_{\rho}] \cos [k \cdot h_{\rho}]).$$

Following the same procedure as in §3.1 we find the thermodynamical potential

$$\Omega_{MF}^{FF} = \frac{1}{N_s} \sum_{k \in \text{Fermi Shell}} \hbar \omega_k (k, \mathbf{p}) + \frac{1}{N_s} \sum_{k} \left( \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} - \mu - \hbar \omega_k (k, \mathbf{p}) \right) - \frac{\Delta^2}{U}.$$ (4.8)

In order to obtain the numerical estimation we have to switch again to the continuum limit by

$$\frac{1}{N_s} \sum_{k} \to \left( \frac{\lambda}{2} \right)^3 \int \frac{dk}{(2\pi)^3},$$

and the thermodynamic potential is

$$\Omega_{MF}^{FF} = \frac{\lambda^3}{(4\pi)^3} \int \frac{dk}{k_{Ax}} \int \frac{dk}{k_{Ay}} \int \frac{dk}{k_{Az}} \left[ \hbar \omega_k (k, \mathbf{p}) - \left( \frac{\varepsilon(p-k)+\varepsilon(p+k)}{2} + \hbar \omega_k (k, \mathbf{p}) \right) \right]$$

$$+ \frac{\lambda^3}{(2\pi)^2} \int \frac{dk}{k_{Ax}} \int \frac{dk}{k_{Ay}} \int \frac{dk}{k_{Az}} \left( \varepsilon(p+k)+\varepsilon(p-k) \right) - \mu - \hbar \omega_k (k, \mathbf{p}) - \frac{\Delta^2}{U}. $$ (4.10)

The Fermi shell boundaries are given by the condition $\hbar \omega_- (k, \mathbf{p}) = 0$ and become

$$\sum_{\rho} (-2t \cos [(p-k) \cdot h_{\rho}] - \mu + h) (-2t \cos [(p+k) \cdot h_{\rho}] - \mu - h) + \Delta^2 = 0.$$ (4.11)

Notice that in this case we can no longer have symmetry between $k$ and $-k$, because of the form of the atom energy. The quasiparticle effective mass can be calculated by using equation (3.15) which is valid again since we consider the effects of the lowest band while this equation defines the effective mass exactly from the curvature of the lowest band. The difference now is that the effective mass depends also in the imposed momentum $\mathbf{p}$ and it is

$$m^*(k, \mathbf{p}) = \frac{\hbar^2}{2}\lambda^3 \cos (\lambda k / 2) \cos (\lambda p / 2).$$ (4.12)
As we will see in the numerical results the values of the imposed momentum are very small, that is we can replace in the above formula \( k \to 0 \) and \( p \to 0 \), which means that the effective mass is again \( m^* = (\lambda^2 t/\hbar^2)^{-1} \).

### 4.2.1 Equation of state

The equation of state for the Fulde-Ferrell phase can be derived again from the equation \( n = -\partial \Omega/\partial \mu \) and it is

\[
\begin{align*}
\frac{1}{U} &= \lambda^3 \int_{k_{Ax}}^{k_{Bx}} dk_x \int_{k_{Ay}}^{k_{By}} dk_y \int_{k_{Az}}^{k_{Bz}} dk_z \frac{1}{2 \sqrt{\left( \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} - \mu \right)^2 + \Delta^2}} \\
&\quad + \lambda^3 \int_{k_{Ax}}^{k_{A}} dk_x \int_{k_{Ay}}^{k_{A}} dk_y \int_{k_{Az}}^{k_{A}} dk_z \frac{1}{2 \sqrt{\left( \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} - \mu \right)^2 + \Delta^2}}.
\end{align*}
\]

(4.13)

### 4.2.2 Gap equation

The gap equation, using \( \partial \Omega/\partial \Delta^* = 0 \), is

\[
\begin{align*}
\frac{1}{U} &= \lambda^3 \int_{k_{Ax}}^{k_{Bx}} dk_x \int_{k_{Ay}}^{k_{By}} dk_y \int_{k_{Az}}^{k_{Bz}} dk_z \frac{1}{2 \sqrt{\left( \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} - \mu \right)^2 + \Delta^2}} \\
&\quad + \lambda^3 \int_{k_{Ax}}^{k_{A}} dk_x \int_{k_{Ay}}^{k_{A}} dk_y \int_{k_{Az}}^{k_{A}} dk_z \frac{1}{2 \sqrt{\left( \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} - \mu \right)^2 + \Delta^2}}.
\end{align*}
\]

(4.14)

### 4.2.3 Magnetization

The magnetization once again comes only from the Fermi shell contributions and will be

\[
m = -\lambda^3 \int_{k_{A}}^{k_{B}} d^3k
\]

(4.15)

### 4.3 Numerical Estimations, one-dimensional case

Again for the numerical estimation, since the three dimensional case appears to have many difficulties, we are going to use an one dimensional optical lattice. The dispersion relation is given by equation (4.3) and after the Bogoliubov transformation,
equation (4.6) is still valid. The boundaries for the Fermi shell are
\[
\sqrt{\left(\frac{\varepsilon (p + k) + \varepsilon (p - k)}{2} - \mu \right)^2 + \Delta^2} = \left(\frac{\varepsilon (p - k) - \varepsilon (p + k) + h}{2}\right)
\]

or after some algebra
\[
\left(-2t\cos \left[\frac{\lambda}{2} (p - k)\right] - \mu + h\right) \left(-2t\cos \left[\frac{\lambda}{2} (p + k)\right] - \mu - h\right) + \Delta^2 = 0. \quad (4.16)
\]

The dispersion relation now depends on the momentum \(p\) which makes significant changes in the Fermi shell boundaries. In figure 4.1 we plot \(\hbar \omega(k)\) for different combinations of the rescaled parameters \(\mu/t, \Delta/t, h/t, \lambda p/2\). Notice that the dispersion relation is periodic in the momentum \(p\) with a period of \(2\pi\). Also at \(p = 0\) we expect to have agreement with the model of §3.8. In Appendix B one can see the dependence of the dispersion relation on the different parameters.

![Figure 4.1](image)

**Figure 4.1** Dispersion relation for different combinations of the rescaled parameters \(\mu/t, \Delta/t, h/t, \lambda p/2\).

### 4.3.1 Thermodynamic Potential

The thermodynamic potential simplifying the one of the three dimensional case yields
\[
\Omega_{MF}^{FFLO} = \frac{\lambda}{4\pi} \int_{k_A}^{k_B} dk \left[\hbar \omega (k; p) - \left(\frac{\varepsilon (p - k) - \varepsilon (p + k)}{2} + h\right)\right]
\]
\[+ \frac{\lambda}{4\pi} \int_{-\frac{2\pi}{\lambda}}^{\frac{2\pi}{\lambda}} dk \left(\frac{\varepsilon (p + k) + \varepsilon (p - k)}{2} - \mu - \hbar \omega (k)\right) - \frac{\Delta^2}{U} \quad (4.17)
\]

It appears that \(p\) contribution differs between the weak and strong coupling. In figure 4.2 and 4.3 the thermodynamic potential is plotted for these two cases. For the weak coupling we plot detail for values of the imposed momentum \(\lambda p/2\) near the corresponding value of the phase transition. It appears the we have the same picture for four different intervals of \(\lambda p/2\), see Appendix B for details.
4.3 Numerical Estimations, one-dimensional case

Figure 4.2 Thermodynamic potential for $U/t = -3.99$ plotted for various values of $h/t$ before (first two diagrams) and after (third diagram) the superfluid phase transition.

Figure 4.3 Thermodynamic potential for $U/t = -1.08$ plotted for various values of $h/t$ before (first two diagrams) and after (third diagram) the Fulde-Ferrell superfluid phase transition.

4.3.2 Equation of state

Simplifying equation of state for the one dimension we get

\[
\begin{align*}
n &= \frac{\lambda}{4\pi} \int_{-2\pi}^{2\pi} dk \left( 1 - \frac{\left( \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} - \mu \right)}{\sqrt{\left( \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} - \mu \right)^2 + \Delta^2}} \right) \\
&\quad + \frac{\lambda}{4\pi} \int_{k_A}^{k_B} dk \frac{\left( \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} - \mu \right)}{\sqrt{\left( \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} - \mu \right)^2 + \Delta^2}}.
\end{align*}
\]  

(4.18)

4.3.3 Gap equation

Similarly the gap equation is

\[
\begin{align*}
\frac{1}{U} &= \frac{\lambda}{4\pi} \int_{k_A}^{k_B} \frac{dk}{k_B} \left( \frac{1}{2\sqrt{\left( \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} - \mu \right)^2 + \Delta^2}} + \Delta \right) \\
&- \frac{\lambda}{4\pi} \int_{-2\pi}^{2\pi} \frac{dk}{2\pi} \left( \frac{1}{2\sqrt{\left( \frac{\varepsilon(p+k)+\varepsilon(p-k)}{2} - \mu \right)^2 + \Delta^2}} + \Delta \right).
\end{align*}
\]  

(4.19)
By minimizing the thermodynamic potential (numerically) with respect to $\Delta/t$ and $p = 0$ we obtain the same results with the model of §2.8 as expected. If we minimize with respect to $p$ and the rescaled gap $\Delta/t$ simultaneously we notice no difference in the phase diagram for strong coupling (for values of $U/t \geq -1.3$) while for weak coupling (i.e. $0 \leq U/t \leq -1.3$) we find a new minimum for $p \neq 0$ and different $\Delta/t$. In figure 3.4 we plot the thermodynamic potential and the corresponding contour plots for the minimum in weak and strong coupling. This minimum appears in for four intervals of the imposed momentum $p$, since the dispersion as we saw is periodic in $p$ with $2\pi$ period. The reader can see in detail the evolution of the thermodynamic potential for different values of the net magnetization $h/t$ in Appendix B.

Figure 4.4. Thermodynamic potential for $U/t = -3.99$ and $U/t = -1.08$ for critical $h/t$. In the first case we see that we have the minimum corresponding to the normal superfluid first order phase transition ($p = 0$), while in the weak coupling case we have different global minimum for $h/t$ smaller than the previous superfluid phase. This minimum corresponds to the Fulde-Ferrell phase. The black point points the so-called Sarma phase (see Chapter 4, page 37).
4.3 Numerical Estimations, one-dimensional case

We find that the new minimum coincides with the superfluid minimum, thus becomes the global minimum, for $h/t$ values smaller than those of the superfluid phase. The phase diagram for this case is the one of figure 4.5.

![Phase diagram for the superfluid phase (p = 0, Yellow Color) the Fulde-Ferrell phase (p ≠ 0, red color) and the normal phase (light blue color). Detail including only the weak coupling regime i.e $0 \leq U/t \leq -1.3$.](image)

**Figure 4.5.** Phase diagram for the superfluid phase ($p = 0$, Yellow Color) the Fulde-Ferrell phase ($p \neq 0$, red color) and the normal phase (light blue color). Detail including only the weak coupling regime i.e $0 \leq U/t \leq -1.3$.

The new phase transition appears to be a second-order phase transition, since the new minimum approaches the previous SF minimum continuously as seen in figure 4.6. The FF phase is restricted in a narrow region, similarly found in ref. [21], [22],[17],[18], next to the normal state. As it is emphasized in ref. [16], that in this case the phase separation occurs throughout the rest of the entire region where the uniform phase is unstable. More specific it is mentioned that for small population differences it has been found that phase separation occurs unless it holds $U \to -\infty$ for the coupling constant. But this is an artifact of the FF which cannot smoothly go into the completely paired equal population state [19]. In other words the state at least for the zero temperature limit should be the LO phase.
4.3.4 Fermi Shell Topography.

As we have already seen in §3.8 the first order transition happens for an order parameter such that the thermodynamic potential has only vacuum contributions while we have no Fermi shell. In the FF phase transition though the thermodynamic potential has contributions from both terms, the majority of which still comes from the vacuum. In figure 4.7 we plot the dispersion relation for $-\pi \leq p \leq \pi$ and $p$ and for the FF phase. Notice that the Fermi shell is independent of the coupling term. We also have to mention that the form of the Fermi shell remains the same within the Fulder-Ferrell phase.

**Figure 4.6** The minimum for $p \neq 0$ in the $\Delta - \lambda/2p$ plane. We see that this phase transition looks like a second order one. The minimum approaches continuously $(0,0)$ point as $U/t$ decreases. Second diagram the thermodynamic potential against the rescaled parameter for the value of the imposed momentum $\lambda p/2$ corresponding to the FF phase transition.

**Figure 4.7.** Dispersion relation for the FF phase for $-\pi \leq \lambda p/2 \leq \pi$ and in the second line near the value of $\lambda p/2$ corresponding to a FF phase transition for $U/t = -1.08$. 
4.3.5 Magnetization

The magnetization contributions come again only from the Fermi Shell and it is

\[ m = -\frac{\lambda}{4\pi} \int_{k_A}^{k_B} d^3k = -\frac{\lambda}{4\pi} (k_B - k_A). \] (4.20)

Again for \( p \neq 0 \) the magnetization differs. We know from §3.8 that for \( p = 0 \) the magnetization equals to zero for \( \Delta/t \) values smaller than \( h/t \) (see §3.8.1). In the new \( h/t \) critical (FF phase) we have non-zero magnetization unlike the superfluid phase. Of course this is something expected since in the FF phase we have contributions in the thermodynamic potential from the Fermi shell. Once more the magnetization does not depend on the interaction parameter \( U \).

Figure 4.8. Magnetization for \( U/t = -3.99 \) and \( U/t = -0.45 \) for \(-\pi \leq p \leq \pi\), detail of the magnetization for the \( p \) of \( U = -0.45 \) (FF phase). In the FF phase the magnetization is non-zero and for this case is \( M/t = -0.00627 \) (black point).
Appendix A

Large momenta treatment

In this thesis we deal only with the $T = 0$ limit. When we move to higher energies, so that we can compare with our results, then we have to take into account the renormalization procedure.

A.1 Renormalization

For large momenta the term $\epsilon(k) - \mu - \hbar \omega(k)$ diverges because of the form of the dispersion relation. If we expand $\hbar \omega(k)$ up to first order the divergence becomes

$$\epsilon(k) - \mu - \hbar \omega(k) \approx -|\Delta|^2 \frac{2\epsilon(k)}{2\epsilon(k)}.$$  \hfill (A.1)

Instead of introducing a cutoff $\Lambda$ we subtract this divergence from the mean field Hamiltonian and renormalize the coupling constant from $g^{-1}$ to $(T^{2B})^{-1}$ where

$$\frac{1}{T^{2B}} = \frac{1}{g} + \frac{1}{V} \sum_{k} \frac{1}{2\epsilon(k)}.$$  \hfill (A.2)

A.2 Thermodynamic potential

In order to find the thermodynamic potential we have to calculate the ground state expectation value. The only contribution to the number operator comes from $\hbar \omega_-(k) < 0$ (where we are within the Fermi surface). It is then

$$\Omega(\Delta, \Delta^*) = \langle \text{g.s.} | H_{MF} | \text{g.s.} \rangle$$

$$= \frac{1}{V} \sum_{k \in \text{Fermi shell}} \hbar \omega_-(k) + \frac{1}{V} \sum_{k} \left( \epsilon(k) - \mu - \hbar \omega(k) + \frac{|\Delta|^2}{2\epsilon(k)} \right) - \frac{|\Delta|^2}{T^{2B}}.$$  \hfill (A.3)
A.2 Thermodynamic potential

A.2.1 Unitarity and equal population limit

In the unitarity limit where $g \to \infty$ the renormalization term drops out and at equal population the condition $\hbar \omega_-(\mathbf{k}) = 0$, that gives us the boundaries of the Fermi sea $\mathbf{k}_A, \mathbf{k}_B$, give

$$k^2_A = \frac{2m}{\hbar^2} \left( \mu - \sqrt{\hbar^2 - |\Delta|^2} \right), \quad k^2_B = \frac{2m}{\hbar^2} \left( \mu + \sqrt{\hbar^2 - |\Delta|^2} \right)$$  \hspace{1cm} (A.4)

and has imaginary solutions and there is no fermi sea in this limit. The thermodynamic potential becomes then

$$\Omega(=) = \frac{1}{V} \sum_k \left( \varepsilon(k) - \mu - \hbar \omega(k) + \frac{|\Delta|^2}{2\varepsilon(k)} \right).$$  \hspace{1cm} (A.5)

Since there is no angular dependence in this expression we can shift to the continuum

$$\Omega(=) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dk k^2 \left( \varepsilon(k) - \mu - \hbar \omega(k) + \frac{|\Delta|^2}{2\varepsilon(k)} \right)$$  \hspace{1cm} (A.6)

where we introduced scalar variables $\tilde{\varepsilon} = \frac{\varepsilon}{\mu}, \tilde{\Delta} = \frac{\Delta}{\mu}$

A.2.2 $\varepsilon$ cutoff

Again $\hbar \omega(k)$ term diverges. We introduce a large cutoff $\Lambda$ beyond which we perform the integral analytically in a large $\varepsilon$ approximation. It is then

$$\sqrt{(\tilde{\varepsilon} - 1)^2 + |\tilde{\Delta}|^2} \approx \tilde{\varepsilon} - 1 + \frac{|\tilde{\Delta}|^2}{2\tilde{\varepsilon}} \left( 1 + \frac{1}{\tilde{\varepsilon}} \right),$$

so that the thermodynamic potential for these limits becomes

$$\Omega(=) = 2K \int_0^\Lambda d\tilde{\varepsilon} \sqrt{\tilde{\varepsilon}} \left( \tilde{\varepsilon} - 1 - \sqrt{(\tilde{\varepsilon} - 1)^2 + |\tilde{\Delta}|^2} + \frac{|\tilde{\Delta}|^2}{2\tilde{\varepsilon}} \right) + 2K \int_\Lambda^\infty d\tilde{\varepsilon} \sqrt{\tilde{\varepsilon}} \left( -\frac{|\Delta|^2}{2\varepsilon} \right)$$

$$= 2K \int_0^\Lambda d\tilde{\varepsilon} \sqrt{\tilde{\varepsilon}} \left( \tilde{\varepsilon} - 1 - \sqrt{(\tilde{\varepsilon} - 1)^2 + |\tilde{\Delta}|^2} + \frac{|\tilde{\Delta}|^2}{2\tilde{\varepsilon}} \right) + \frac{2K|\Delta|^2}{\sqrt{\Lambda}},$$  \hspace{1cm} (A.7)

with $K = \frac{\mu^{5/2}}{8\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2}$
A.2.3 Unitarity limit and unequal populations

Again renormalization term drops out but now $k_A, k_B$ have real solutions and we have contributions from the Fermi shell term. The thermodynamic potential then is

$$\Omega = \frac{1}{V} \sum_{k \in \text{Fermi shell}} \hbar \omega_-(k) + \Omega(=)$$

(A.8)

and again has no angular dependence.

Following same procedure as in previous section in the continuum limit $\Omega$ is

$$\Omega = \frac{\mu^{5/2}}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_{\epsilon_A}^{\epsilon_B} d\tilde{\epsilon} \sqrt{\tilde{\epsilon}} \left( \tilde{\epsilon} - 1 - \sqrt{(\tilde{\epsilon} - 1)^2 + |\tilde{\Delta}|^2 + \frac{|\Delta|^2}{2\epsilon}} \right) + \Omega(=)$$

(A.9)

$$= 2K \int_{\epsilon_A}^{\epsilon_B} d\tilde{\epsilon} \sqrt{\tilde{\epsilon}} \left( \tilde{\epsilon} - 1 - \sqrt{(\tilde{\epsilon} - 1)^2 + |\tilde{\Delta}|^2 + \frac{|\Delta|^2}{2\epsilon}} \right)$$

$$+ 2K \int_0^\Lambda d\tilde{\epsilon} \sqrt{\tilde{\epsilon}} \left( \tilde{\epsilon} - 1 - \sqrt{(\tilde{\epsilon} - 1)^2 + |\tilde{\Delta}|^2 + \frac{|\Delta|^2}{2\epsilon}} \right) + \frac{2K|\Delta|^2}{\sqrt{\Lambda}},$$

with

$$\epsilon_A = \epsilon(k_A) = \mu - \sqrt{\hbar^2 - |\Delta|^2}$$

$$\epsilon_B = \epsilon(k_B) = \mu + \sqrt{\hbar^2 - |\Delta|^2}.$$  (A.10)
Appendix B

Tables and Figures.

In this appendix we present the dependence of the dispersion relation, thermodynamic potential and the magnetization from the various parameters so that the reader can have a better view of the problem.

Dispersion relation dependence from the chemical potential.

Figure B.1. Dispersion relation plotted against $\lambda k/2, \lambda p/2$ with fixed rescaled chemical potential difference $h/t = 1$ and rescaled order parameter $\Delta/t = 1$. We vary the rescaled averaged chemical potential as $-2 \leq \mu \leq 2$. 
Dispersion relation dependence from the order parameter.

**Figure B.2.** Dispersion relation plotted against $\lambda k/2, \lambda p/2$ with fixed rescaled averaged chemical potential $\mu/t = 1$ and rescaled chemical potential difference $h/t = 1$. We vary the rescaled order parameter $\Delta/t$, as $0 \leq \Delta/t \leq 4$.

Dispersion relation dependence from the total momentum of the Cooper pair and $h/t = constant$.

**Figure B.2.** Dispersion relation plotted against $\lambda k/2, \Delta/t$ with fixed rescaled averaged chemical potential $\mu/t = 1$ and rescaled chemical potential difference $h/t = 1$. We vary the total momentum of the Cooper pair $-\pi \leq \lambda p/2 \leq \pi$.
Dispersion relation dependence from the total momentum of the Cooper pair and $\Delta/t = \text{constant}$.

Figure B.3. Dispersion relation plotted against $\lambda k/2, h/t$ with fixed rescaled averaged chemical potential $\mu/t = 1$ and rescaled order parameter $\Delta/t = 1$. We vary the total momentum of the Cooper pair $-\pi \leq \lambda p/2 \leq \pi$.

Thermodynamic potential for normal to superfluid phase transition.
Figure B.4. Thermodynamic potential plotted against $\Delta/t$, $\lambda_p/2$ with fixed $\mu/t = -1.9$ and $U/t = -3.99$. We vary the difference of the chemical potentials from zero to 1.56. We see that in this case the nonzero total momentum of the Cooper pairs doesn’t play any role. In the first figure of the second row we have the thermodynamic potential for $h_c/t = 0.73886$.

Thermodynamic potential for normal to FF to superfluid phase transition.

Figure B.5. Thermodynamic potential plotted against $\Delta/t$, $\lambda_p/2$ with fixed $\mu/t = -1.9$ and $U/t = -1.08$. We vary the difference of the chemical potentials from zero to 0.119. We see that in this case the nonzero total momentum of the Cooper pairs plays significant role and a Fulde-Ferrell phase is formed. In the second figure of the second row we have the thermodynamic potential for $h_c/t = 0.0798997$ where the new phase transition appears.

FF minimum behaviour.
Figure B.6. First figure: Critical values of the order parameter and the chemical potential difference of spin up and down particles obtained by minimizing the thermodynamic potential for different values of the coupling constant. Second figure: Minimizing the thermodynamic potential for different values of the chemical potential difference of spin up and down particles for a specific value of the rescaled coupling constant ($U/t = -1.08$). For both diagram we have fixed the rescaled chemical potential at the value $\mu/t = -1.9$.

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<th>$h_{cr}/t$ FF</th>
<th>Difference %</th>
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<td>0.00195</td>
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</tr>
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<td>-1.25</td>
<td>0.10351</td>
<td>0.10237</td>
<td>1.115</td>
</tr>
</tbody>
</table>

Table 2. Characteristic values of $h$ critical for the FF, SF phase and their difference.

Magnetization varying the order parameter.

Figure B.7. Magnetization plotted against $\lambda p/2, h/t$ with fixed $\mu/t = -1.9$. We vary the rescaled order parameter $\Delta/t$, as $\Delta/t = 0.0099, 1.2362132077474337, 4$.

Magnetization varying the chemical potential.

Figure B.8. Magnetization plotted against $\lambda p/2, h/t$ with fixed $\Delta/t = 0.8$. We vary the rescaled averaged chemical potential $\mu/t$, as $\mu/t = -1.9, -1.0$. 
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