

Probing the short-scale geometry of space-time by looking at the spectrum of the Laplacian

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22nd Januari 2007

Masters thesis
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Abstract

This thesis is inspired by the quantized space-times emerging from the Causal Dynamical Triangulations approach to quantum gravity. Using this quantization method, it was found out in numerical calculations that at small length scales space-time behaves differently than at the classical large length limit. We believe that space-time can be described effectively by a generalized kind of geometry, which incorporates some fractal properties reminiscent of those of ensembles of branched polymers in the continuum limit.

We want to probe the short-scale geometry of these structures by looking at the spectrum of the Laplacian on their dual graph. We first look at a small subclass of these geometries, the homogeneously subdivided simplices. Their dual graphs are equal to the dual Sierpinski gaskets. The spectra of these graphs can be calculated in an analytical way. We will compare these spectra with the spectra of regular lattice graphs. We will also apply these results to more general geometries. We are not able to calculate the spectra of these general geometries in an analytical way. Nevertheless, we can derive some geometrical properties of these geometries, like the number of cycles and triangles.

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1 Introduction

In quantum gravity one tries to look for a way to unify quantum theory with Einstein's theory of general relativity. The problem is that these two theories are incompatible due to their conceptual differences and different ranges of applicability. Originally, quantum gravity was studied as the problem of quantising general relativity perturbatively, but this led to meaningless quantum field theories which were non-renormalizable. Nowadays there are many different non-perturbative programmes in quantum gravity, but still none of these attempts to construct a quantum gravity theory has managed to solve the original problem. For a nice overview and a more thorough introduction on quantum gravity, I would like to refer to two articles of Isham ([10] and [11]).

An interesting question is how space-time looks at small length scales. When probing space-time at smaller and smaller length scales, one needs ever larger energies. Extrapolating from Einstein's theory of general relativity, space-time itself will fluctuate at the Planck scale because of the energy-momentum uncertainties. Hence, what looks smooth to the naked eye, may have an intricate and highly fluctuating structure at small length scales.

Forming a model to describe quantum gravity is difficult because of the lack of empirical data. Nowadays it is not possible to directly test quantum gravity experimentally, because of the extremely short length scales that are involved. Because of this, there exist many different approaches to construct theories that possibly describe quantum gravity.

Classical 4-dimensional space-time is defined as a 4-dimensional manifold with Lorentzian metric. One problem of quantum gravity is that it is not clear how much of this classical structure should be held fixed. It is possible to keep the manifold fixed, but to make the metric quantum fluctuate. Other approaches allow quantum fluctuations of the manifold structure, of the topology of the manifold or they start with a new mathematical background structure. Beside the question of how much of the space-time structure should be held fixed, the approaches differ in many other aspects, like the role of time and causality.

There are many different programmes in quantum gravity. Two major programmes in quantum gravity are superstring theory (see [13] for an introduction) and Ashtekar's theory of canonical quantum gravity. The quantization method that we will use in this thesis is that of causal dynamical triangulations ([2] and [3]), which builds on a simplicial discretization of the space of all space-time geometries. It is invented by R. Loll and J. Ambjørn.

By doing numerical calculations on these causal dynamical triangulations, they found out that at small length scales space-time behaves different than at the classical large length limit. It is believed that space-time can be described

effectively by a generalized kind of geometry, which incorporates some fractal properties reminiscent of those of ensembles of branched polymers (or stacked spheres) in the continuum limit.

In this thesis we want to probe the geometry of these structures. By using the spectrum of the Laplacian on the dual graphs of these stacked sphere configurations we try to get some quantitative information about the geometrical structure of these configurations.

To get a better understanding about causal dynamical triangulations, we first give an introduction to simplicial complexes and triangulations in section 2. A triangulation describes the subdivision of a manifold in simplices. Simplices are triangular building blocks. Simplicial complexes are complexes built up from simplices, which are glued together. See subsection 2.1.

In the same section, two applications of simplicial complexes will be described. These are the classical Regge calculus (subsection 2.2) and dynamical triangulations (subsection 2.3).

In section 3 causal dynamical triangulations is described. It is a quantization method in which four-dimensional simplices are used to construct 4-dimensional space-time. By doing numerical experiments on these triangulations R. Loll, J. Ambjørn and J. Jurkiewicz managed to measure some observables on model universes.

They managed to implement discrete diffusion through these causal dynamical triangulations and found out that at small length scales space-time has an intricate structure (section 3.1). By looking at the geometry of spatial slices of the quantum universe, one finds that in the continuum limit the simplicial geometries exhibit some properties of branched polymers (section 4).

The geometry of these branched polymer or stacked sphere configurations is studied by looking at the dual graphs of these configurations (section 5). We get geometrical information of these dual graphs from the spectrum of the Laplacian (section 6).

Before we look at a small class of stacked sphere configurations, the k -stage homogeneously subdivided simplices, we first look at the spectrum of lattice graphs (section 7), because the spectrum of these graphs have some similarities with the spectrum of the dual graph of a k -stage homogeneously subdivided simplices. A regular dual Sierpinski gasket is the dual graph of a k -stage homogeneously subdivided simplex. In section 8 and 8.1 the construction and labelling of regular dual Sierpinski gaskets is described.

The spectra of these gaskets are calculated first, because they gives us information about the spectra of general stacked sphere configurations. The spectra of two-dimensional regular dual Sierpinski gaskets are calculated in subsection 9.1. These calculations are generalized to arbitrary dimensions in subsection 9.2. Some general results of the regular dual Sierpinski gaskets are described in subsection 9.3 and the spectra of the dual Sierpinski gaskets and lattice graphs are compared in subsection 9.4.

These results are applied on another special class of stacked sphere configurations, the triangular stacked sphere configurations. The spectra of the dual graph of these stacked sphere configurations can be related with the spectra of the consolidated graph of the dual graph of these stacked sphere configurations (see subsection 10). In subsection 11 some results on general stacked sphere configurations are described.

In the last section (section 12) some open problems are discussed. We were not able to calculate the spectrum of general stacked sphere configurations. This section also contains some references to articles from people who are working on this subject.

2 Triangulation of manifolds

2.1 Simplicial manifolds

In order to talk about the simplicial discretization of space-time, we first need to understand what simplicial manifolds [1] are. To construct these manifolds we use a certain type of building blocks, which are called simplices.

Definition 2.1 (Simplex) An n -simplex $\sigma^n = \{x_0, \dots, x_n\}$ is a subspace in \mathbb{R}^d with $d \geq n$, given in the following way

$$\sigma^n = \sum_{i=0}^n \lambda_i x_i$$

with x_0, \dots, x_n points in \mathbb{R}^d and $\sum_{i=0}^n \lambda_i = 1$.

For $n = 1$, simplex σ^n is a vertex, for $n = 2$ it is an edge, for $n = 3$ it is a triangle and for $n = 4$ it is a tetrahedron. We see that the points x_0, \dots, x_n are vertices. See figure (1).



Figure 1: From left to right: a vertex, an edge, a triangle and a tetrahedron

A face of a n -simplex σ^n is a $n - 1$ -simplex whose vertices are a subset of those of σ^n . We can make a structure built up by these simplices

Definition 2.2 (Simplicial Complex) A simplicial complex K is a finite collection of simplices in \mathbb{R}^d with the following composition rules:

- If $\sigma_1^n \in K$, then all faces of σ_1^n are in K
- If $\sigma_1^n, \sigma_2^n \in K$, then $\sigma_1^n \cap \sigma_2^n$ is either a face or an empty set

A simplicial complex is also called a complex.

A simplicial complex forms a triangulation of an underlying space. To see this we first define

$$|K| = \bigcup_{\sigma \in K} \sigma.$$

Introduce a topology on $|K|$, such that the embedding of each simplex into $|K|$ is continuous. In other words, $A \subset |K|$ is closed if and only if $A \cap \sigma^n$ is closed in σ^n for all $\sigma^n \in K$. $|K|$ is called a geometric carrier of complex K and $|K|$

is said to be triangulated by K . A triangulation of a space M is a complex K together with a homeomorphism $|K| \rightarrow M$.

Definition 2.3 (Simplicial Manifold) *A simplicial complex is a n -dimensional simplicial manifold if every neighbourhood of a vertex is homeomorphic to a n -dimensional ball. Here, we mean by neighbourhood of a vertex, the set of simplices that contains the vertex.*

If we have a finite set of simplices and an associated set of faces, a glueing is a choice of pairs of faces together with the simplicial identification of these faces, such that every face is in exactly one pair. We can build up a complex in this way by glueing together a finite set of n -simplices.

By the glueing operation we do not always get a simplicial manifold. In general we get a pseudo-manifold.

Definition 2.4 (Pseudo-manifold) $T \rightarrow M$ is a pseudo-manifold if:

- every simplex of T is either a n -simplex or a face of a n -simplex
- every $(n - 1)$ -simplex is a face of at most two n -simplices
- for any two simplices σ^n, τ^n there exists a simplicial path of n -simplices $\sigma^n = \sigma_0^n, \sigma_1^n, \dots, \sigma_k^n = \tau^n$, such that σ_i^n and σ_{i+1}^n have a $(n - 1)$ -simplex as a face in common. Here $k+1$ is the length of the simplicial path.

In pseudo-manifold there are regular points and singular points. A regular point is a point which has a neighbourhood homeomorphic to a n -dimensional simplex. A point that is not regular is called singular. The set of regular points in a pseudo-manifold is dense and connected and the set of singular points in a pseudo-manifold is of dimension less than $n - 1$.

Two triangulations T^1 and T^2 of the same manifold M are said to be equivalent whenever there is a one-to-one mapping of the vertices, edges, faces, and higher dimensional simplices of T^1 onto vertices, edges, faces and higher dimensional simplices of T^2 which preserves the incidence relations. When there is not such a mapping, the two triangulations are distinct.

2.2 Classical Regge Calculus

Regge calculus is a mathematical description to produce simplicial approximations of space-time that are solutions of the field equations of Einstein. It was invented by T. Regge ([15]).

We now introduce metric properties in the simplicial complexes. By definition space-time inside the simplices is flat, such that curvature is concentrated around particular simplices, the bones. A bone of a n -dimensional complex is a $n - 2$ -dimensional subsimplex that n -simplices have in common. In 2 dimensions a bone is the vertex where triangles meet. Space-time curvature of simplicial

manifolds can be described by deficit angles. The deficit angle $r(B)$ at bone B is given by

$$r(B) = 2\pi - \phi(B).$$

Here $\phi(B)$ is the rotation angle, defined by the number $n(B)$ of simplices that are incident to bone B and the angle $\delta_j(B)$ between two faces of a simplex σ_j^n incident to the bone B ,

$$\phi(B) = \sum_{j=1}^{n(B)} \delta_j(B).$$

For two-dimensional simplicial manifolds, the Gaussian curvature is locally defined as the parallel transportation of a test vector around a closed loop around the bone. It is given by

$$\kappa = \frac{r(B)}{A^*}$$

with A^* the area of the loop. This loop is dual to the bone itself. If the deficit angle is positive, we have a positive curvature. If the deficit angle is negative, we have a negative curvature.

Regge proved that the simplicial equivalent of Hilbert's action for a lattice space-time is given [14] by

$$\frac{1}{16\pi} \int_M \sqrt{-g} R d^4x \longrightarrow I_R = \frac{1}{8\pi} \sum_{i \in \{\text{bones}\}} r(i) A_i$$

with A_i the area of bone i and I_R the simplicial equivalent of Hilbert's action.

By variational calculus, we get the Regge equations, which are given by

$$0 = \frac{\delta I_R}{\delta l_j} = 2 \sum_{i \in \{\text{bones}\}} r(i) \frac{\delta A_i}{\delta l_j}$$

with l_j the edge lengths of the simplices. In the derivation of this equation we set

$$\sum_i \frac{r(i)}{l_j} A_i = 0$$

which is known as the Regge identity.

From the Regge equations we finally get equations of motions for the edge lengths of every edge in our simplicial manifold.

2.3 Dynamical triangulations

The difference between Regge calculus and dynamical triangulations, is the fact that the edge lengths of the simplices are fixed to one value in the case of Dynamical triangulations. There is no classical theory of dynamical triangulations. It does not describe a simplicial version of classical general relativity in the way classical Regge calculus does. This is because we can not perform variational calculus on the edge lengths as they are fixed. It is used as a quantization method.

Definition 2.5 (Dynamical Triangulation) *For $a \in \mathbb{R}^+$, a dynamical triangulation K_a of a carrier $|K|$ is a division of a simplicial manifold $|K|$ generated by glueing along their adjacent faces $N_n(K_a)$ equilateral simplices with edge lengths equal to a . $N_n(K_a)$ is the number of simplices σ^n*

3 Causal dynamical triangulations

Causal dynamical triangulations is an approach to construct a theory of quantum gravity. It was invented by R. Loll and J. Ambjørn and is based on the concept of dynamical triangulations (see subsection 2.3). In the theory of causal dynamical triangulations, we want to perform a path integral over causal geometries.

We foliate space-time globally into discrete intervals of length $\Delta\tau$. In this way we can make a pancake structure of space-time slices of constant time $n\Delta\tau$ with $n \in \mathbb{N}$. Now we look at one space time slice of thickness $\Delta\tau$, bounded by two of those space-time slices. Such a slice will be built up by certain 4-dimensional simplices (see Figure (2))

A $(4,1)$ -simplex is a 4-dimensional simplex. Four vertices of this simplex are on a spatial slice of constant time τ and are connected with edges with each other to form a spatial tetrahedron. The fifth vertex is on a spatial slice of constant time $\tau + \Delta\tau$ and is connected by edges to the four vertices on the spatial slice of constant time τ .

A $(3,2)$ -simplex is a 4-dimensional simplex. Three vertices of this simplex are on a spatial slice of constant time τ and are connected with edges with each other to form a spatial triangle. The fourth and fifth vertices are on a spatial slice of constant time $\tau + \Delta\tau$ and are connected by an edge to each other to form a spatial edge. These two vertices are also connected by edges to the three vertices on the spatial slice of constant time τ .

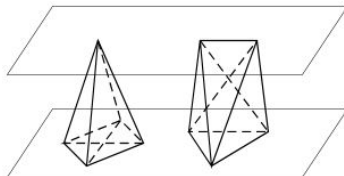


Figure 2: *Left a $(4,1)$ -simplex, right a $(3,2)$ -simplex. These simplices and their inversed time counterparts are placed between two spatial slices of constant time*

We can invert the time orientation of these $(4,1)$ -simplices and $(3,2)$ -simplices. We get $(1,4)$ - and $(2,3)$ -simplices in this way. By definition the space-like edges have fixed squared edge lengths $l_{spatial}^2 = a^2$ and the time-like edges have fixed squared edge lengths $l_{spatial}^2 = -\alpha a^2, \alpha > 0$. α is kept variable to allow a scaling between the time- and space-like edges.

The space-time slices of thickness $\Delta\tau$ are built up by glueing these $(4,1)$ -

and (3, 2)-simplices and their inversed time orientation counterparts together. Two space-time slices of thickness $\Delta\tau$ can be glued together whenever their spatial geometry matches.

With geometries we mean configurations of the 3-simplices on spatial slices of constant time. The topology of these geometries is fixed to S^3 . Let the geometry at spatial slice of constant time $t = 0$ be the initial geometry and the geometry at spatial slice of constant time $t = n\tau, n \in \mathbb{N}$ the final geometry. The path integral over geometries is the integral over all possible geometry-paths with given initial and final geometry. A geometry-path is also called a universe. A geometry-path is built up from geometries at spatial slices of constant time $t = n\tau, n \in \{1, \dots, m - 1\}$. These geometries are connected by 4-simplices in the same way as explained above. We perform the path integral by summing over all possible geometries at spatial slices of constant time $t = n\tau, n \in \{1, \dots, m - 1\}$. The propagator between the initial and final geometry gives us the complete dynamical information of the quantum theory.

Let $N_4^{(4,1)}$ be the number of (4, 1)-simplices, $N_4^{(3,2)}$ the number of (3, 2)-simplices and N_0 be the number of vertices. The Euclidean action for pure gravity is given by [2]

$$S_E = -(\kappa_0 + 6\Delta)N_0 + \kappa_4(N_4^{(4,1)} + N_4^{(3,2)}) + \Delta(2N_4^{(4,1)} + N_4^{(3,2)}).$$

where κ_0, κ_4 and Δ are variables, which depend on the bare inverse Newton constant $k^{(b)}$, the bare cosmological constant $\lambda^{(b)}$ and α . The Δ is a measure for the asymmetry between the space-like and time-like edges. $\Delta = 0$ corresponds to $\alpha = 1$.

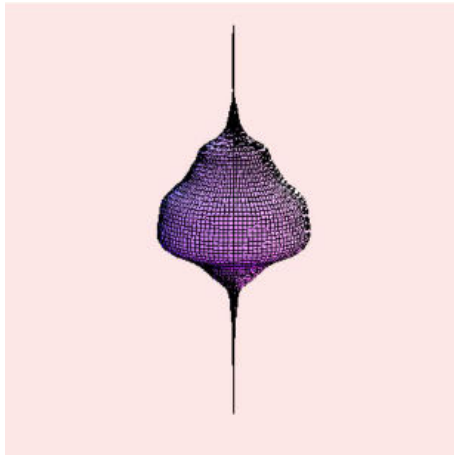


Figure 3: A snapshot of a universe. For this snapshot 91.100 4-simplices over a time extent of $t=40$ are used. Figure is from [3]

For the use in numerical calculations, the geometry-paths are constructed by performing a random walk in the complete ensemble of causal geometries of time extension t and fixed volume. The random walk is generated by a set of moves that changes the geometry locally, without altering the topological properties.

By doing numerical experiments, R. Loll, J. Ambjørn and J. Jurkiewicz managed to measure certain observables in this quantum ensemble. As an example, they managed to measure the dimensionality and other geometrical properties of this ensemble [3]. A snapshot of an universe built up by 91.100 4-simplices over a time extent of $t=40$ can be seen in figure (3). We see a stable 4-dimensional geometry here. It turns out that the macroscopic dimensionality of spacetime emerges with the classical value. This will be clear from the next subsection.

3.1 Diffusion through Causal Dynamical Triangulations

By diffusing particles in a space, we can probe the geometrical properties, like the dimensionality, of this space. We will see how this is done.

The diffusion equation on a d -dimensional manifold with fixed Riemannian metric $g_{ab}(\xi)$ is given by

$$\frac{\delta}{\delta\sigma} K_g(\xi, \xi_0; \sigma) = \Delta_g K_g(\xi, \xi_0; \sigma)$$

with σ the diffusion time, Δ_g the Laplace operator of metric g , and $K_g(\xi, \xi_0; \sigma)$ the probability density of diffusion from ξ to ξ_0 in time σ .

We are looking at piecewise linear structures defined by the dynamical triangulation T . Therefore we use another definition for the probability density of diffusion. $K_T(i, i_0; \sigma)$ is the probability density of discrete diffusion from simplex i_0 to i in discrete diffusion time σ .

We start with a diffusing particle positioned in simplex i_0 . Our initial probability density is given by

$$K_T(i, i_0; \sigma = 0) = \delta_{i, i_0}.$$

with i and i_0 4-simplices. This density is 0 everywhere except at an arbitrary simplex i_0 . To let the particle diffuse, we use an evolution rule. The probability that the particle goes to one of the 5 neighbouring simplices is equal for all simplices. Hence the evolution rule becomes

$$K_T(j, i_0; \sigma + 1) = \frac{1}{5} \sum_{k \rightarrow j} K_T(k, i_0; \sigma)$$

where simplex j is neighbouring to simplex k and by taking the sum $\sum_{k \rightarrow j}$ we take the sum over all simplices k that are a neighbour of j . The total probability is conserved.

$$\sum_j K_T(j, i_0; \sigma) = 1$$

because no diffusing particle is leaving the system.

We define the return probability $P_T(i_0; \sigma)$ to be the probability that a particle that starts at simplex i_0 will return to i_0 in time σ .

$$P_T(i_0; \sigma) = K_T(i_0, i_0; \sigma).$$

The quantum average is defined as

$$P_{N_4}(\sigma) = \frac{1}{Z_E(N_4)} \sum_{T_{N_4}} e^{-S_E(T_{N_4})} \frac{1}{N_4} \sum_{i_0 \in T_{N_4}} K_{T_{N_4}}(i_0, i_0; \sigma)$$

where we take the average over all simplices i_0 in the triangulations T_{N_4} with N_4 4-simplices. $S_E(T_{N_4})$ is the action, $Z_E(N_4)$ is the partition function.

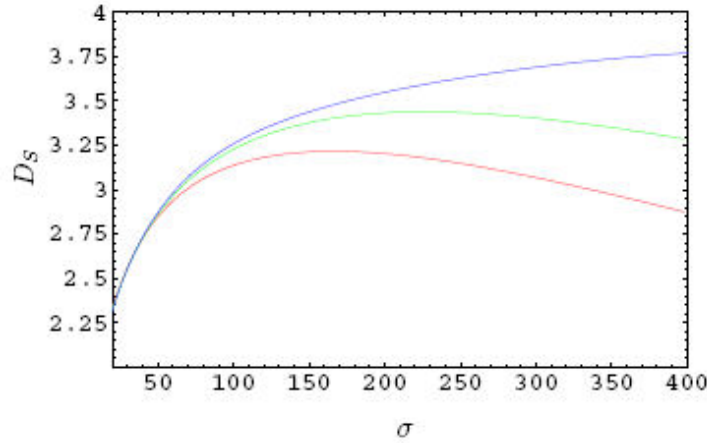


Figure 4: Plot of the spectral dimension D_s of a universe against the diffusion time σ . The lower plot is a measurement with 20k 4-simplices, the middle plot a measurement with 40k 4-simplices and the upper plot a measurement with 80k 4-simplices. The top curve is believed to be a good approximation to the infinite volume limit. The figure is from [2]

We know that the return probability for diffusion on fractal structures has the form

$$P_{N_4}(\sigma) = \sigma^{-D_s/2} F\left(\frac{\sigma}{N_4^{2/D_s}}\right) \quad (1)$$

with D_s the spectral dimension and F a function.

We can measure D_s by taking the logarithmic derivative of (1),

$$D_s(\sigma) = -2 \frac{d \log P_{N_4}(\sigma)}{d \log \sigma} + \text{finite-size corrections.} \quad (2)$$

In this way we get information about the dimensionality from diffusion.

The spectral dimension has been measured on the universes created by causal dynamical triangulation techniques. By averaging over 400 different diffusion processes we get the curve $D_s(\sigma)$ which is plotted in figure (4). These diffusion processes differ in the choice of triangulations and starting simplices i_0 .

We see that the curve $D_s(\sigma)$ slowly approaches an asymptotic value for large σ . Diffusion time is related to the distance that a particle moves. Hence, for large distances, the curve approaches 4.02 ± 0.1 , which is near the classical value of four. This phenomenon is called scale dependence of the spectral dimension. For small length scales, the curve approaches 1.80 ± 0.25 .

We conclude that the large-scale dimension of space-time is four, within measuring accuracy. At small length-scales, the value of the spectral dimension approaches approximately two, which signals an intricate non-classical structure.

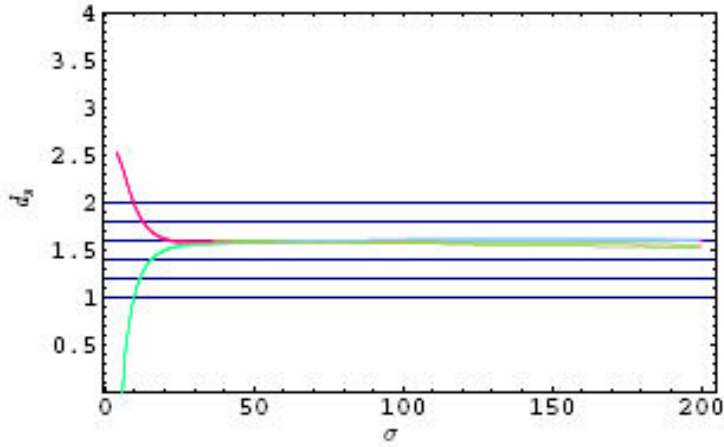


Figure 5: *Plot of the spatial spectral dimension d_s of spatial slice of constant time against the diffusion time σ . The measurement is done with a universe with 91.1k simplices over a time $t = 40$. The figure is from [2]*

Similar calculations have been done for the three-dimensional spatial slices. By averaging over different starting simplices i_0 and different triangulations, we get a curve for the spatial spectral dimension $d_s(\sigma)$ which is plotted in figure (5). The odd and even times behave differently for small times. However, these are mere short-distance lattice artifacts. That is the reason why we see two lines at small diffusion time. For large diffusion times d_s becomes stable around $d_s = 1.56 \pm 0.1$, which is roughly half the expected classical value. From this and a number of other measurements of the spatial slices we conclude that the spatial slices are fractal. In section 4 we will see that this fractal structure shows some similarity with branched polymers.

4 Stacked spheres

It turns out that in the infinite-volume limit some of the geometric properties of the spatial slices coincide with those of branched polymers, as we will explain.

A branched polymer is an one-dimensional polymer which admits branching and does not contain any loops. Hence, if we cut any edge of such a graph, we get two disconnected graphs. At each vertex, an edge is allowed to branch into $n-1$ other edges with branching probability p_n . We are interested in (ensembles of) such branched polymers in the infinite-volume limit.

The partition function for branched polymers is given by

$$Z_{BP}(N) = \sum_{BP_N} \rho(BP_N),$$

$$\rho(BP_N) = \prod_{n=1}^N p_n,$$

with $Z_{BP}(N)$ the partition function for branched polymers with N vertices which is calculated by taking the sum over all branched polymers with N vertices of the weights $\rho(BP_N)$. A weight $\rho(BP_N)$ is the product of all branching probabilities p_n .

For (rooted) branched polymers, the leading behaviour of the infinite-volume limit of the partition function is given by

$$Z_{BP}(N) \sim N^{\gamma-2} e^{-\mu_0 N} \tag{3}$$

with μ_0 a non-universal constant measuring the exponential growth and γ a critical exponent that measures the fractal structure of the geometry [2]. A similar behaviour has been seen in studies of simplicial three-dimensional quantum gravity [2] where we have to take $\gamma - 3$ instead of $\gamma - 2$. Measurements have been done to determine the γ for the spatial slices in the causal dynamical triangulations and it turns out that the value is equal to $\gamma = 0.35 \pm 0.09$.

The spectral dimension of branched polymers is given by

$$d_s = \frac{2}{1 + \gamma}. \tag{4}$$

If we identify the γ in equation 4 with the γ -value that was measured for the spatial slices, we get a value for d_s which comes close to our measured value of $d_s = 1.56 \pm 0.1$.

It turns out that there is a particular subclass of branched polymers which has a γ -value equal to the γ -value that we measured for the spatial slices. This subclass consists of branched polymers with a branching probability

$$p_n \sim \frac{1}{n^\alpha}, n \gg 1 \tag{5}$$

with α a constant.

Under certain technical conditions, the generic value of γ for this class of branched polymers is given by $\frac{1}{3}$. For this γ -value, the spectral dimension becomes equal to $d_s = \frac{3}{2}$ which is, again within measuring accuracy, equal to the value measured for spatial slices. It turns out that this dimension, but also other dimensions and critical exponents are precisely the numbers that characterize the spatial geometries of the spatial slices.

Although the deeper relation between these two types of structures is currently unknown, the similarity between the dimensions and critical exponents motivates a closer look at stacked sphere configurations as examples of fractal geometry. By stacked sphere configurations we will loosely mean (ensembles of) simplicial geometries which in the infinite-volume limit exhibit some properties of (ensembles of) branched polymers. These stacked sphere configurations already appeared in Euclidean dynamical triangulations, where they are the dominant configurations in the so-called elongated phase. In this elongated phase the subdivision moves, the $(1, D + 1)$ -moves dominate, with D the dimension of the triangulation. With a $(1, D + 1)$ -move we mean a move where one simplex is cut out and substituted by a complex of $D + 1$ simplices by subdivision [6].

A stacked sphere configuration is a branched polymer-like triangulation. It is formed by glueing simplices in a polymer-like configuration. In stacked sphere configurations the subdivision moves dominate.

A subdivision move consists of subdividing a simplex into subsimplices. This is done by inserting a vertex in the simplex and connecting all the vertices of the simplex with this new vertex by edges (see figure 6).



Figure 6: *Subdivision of a single simplex (left) into subsimplices (right). A vertex is inserted into the single simplex and this vertex is connected by edges to the vertices of the simplex.*

The name stacked spheres comes from the fact that we stack simplices and simplicial complexes with a sphere topology onto each other.

5 The geometry of stacked spheres

We want to study the geometry of stacked spheres. To do this we first need the notion of a dual graph of a simplicial complex (see section 6 for more information about graphs)

Definition 5.1 (Dual Graph) *A dual graph is a graph that is constructed in the following way. A vertex is inserted inside every simplex. When two simplices are glued together along their faces, the vertices inside these two simplices are connected by an edge.*

The edges between two vertices are dual to the gluing surfaces. That explains why the graph is called dual. In a p -dimensional stacked sphere configuration, we use p -simplices. Because these p -simplices have $p+1$ faces, the highest degree in the dual graph of this stacked sphere configuration is p . See figure (7) for a two-dimensional example.

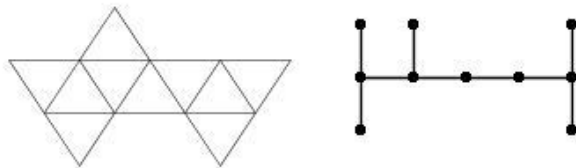


Figure 7: A simplicial complex and its dual graph

We can construct a mapping \mathcal{T} from stacked spheres to dual graphs. The mapping \mathcal{T} is not a bijection as we will illustrate in the following paragraphs [7]. We look at a p -dimensional stacked sphere configuration κ . Let the p -simplices of κ be ordered as $\sigma_i, i \in \{1, \dots, n\}$ with n the number of p -simplices in κ . Let τ_k be the face that σ_k has in common with another simplex σ_l where $\sigma_k, \sigma_l \in \{\sigma_i | i \in \{1, \dots, n\}\}$. Lets assume we cut the p -dimensional stacked sphere configuration κ along τ_k . σ_k has a copy of the boundary face τ_k and σ_l has a copy of the boundary face τ_k .

For $p = 2$ there are two different ways to glue these two copies of τ_k together. Label the two vertices of the first copy of τ_k as 1 and 2 and the two vertices of the second copy of τ_k as 3 and 4. We can glue the two copies of τ_k in the original way, to get the original stacked sphere configuration back, by identifying vertex 1 with vertex 3 and vertex 2 with vertex 4. It is also possible to glue the two copies of τ_k together by identifying vertex 1 with vertex 4 and vertex 2 with vertex 3. Here we rotate one of the copies of τ_k before we glue the copies together. Unless the stacked sphere configuration is symmetric, the two different glueings give us two different stacked sphere configurations. These

stacked sphere configurations have the same dual graph. See figure 8.

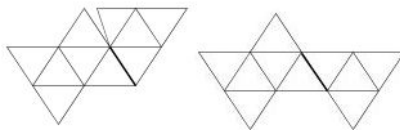


Figure 8: *Two simplicial complexes with the same dual graph. The fat line plays the same role as the τ_k which is explained in the text*

For $p > 2$ we have even more different ways to glue two copies of τ_k together, because there are more ways to rotate τ_k . All stacked sphere configurations we get in this way have the same dual graph. Hence, the mapping from stacked sphere configurations to dual graphs is not one-to-one.

We will study the spectrum of the Laplacian of these dual graphs (for more information about Laplacians, see 6.1). It is possible to derive certain geometric properties of the dual graphs from the spectrum of the Laplacian. And these geometric properties are related to similar geometric properties on the stacked sphere configurations.

To get an idea of the structures involved, we first look at a small subclass of stacked sphere configurations, the k -stage homogeneously subdivided simplices.

We saw in section 4 that we can perform subdivision moves on the simplices in the stacked sphere configurations. Take a single simplex and subdivide it. We call this a 1-stage homogeneously subdivided simplex. The dual graph of this once subdivided simplex is called a 1-stage dual Sierpinski gasket (more information about dual Sierpinski gaskets can be found in section 8). See figure 9.



Figure 9: *A 1-stage homogeneously subdivided simplex and its dual graph, a 1-stage regular dual Sierpinski gasket*

Homogeneously subdivide every subsimplex of the 1-stage homogeneously subdivided simplex. Then we get a 2-stage homogeneously subdivided simplex.

The dual graph of this subdivided simplex is called a 2-stage regular dual Sierpinski gasket. See figure 10.

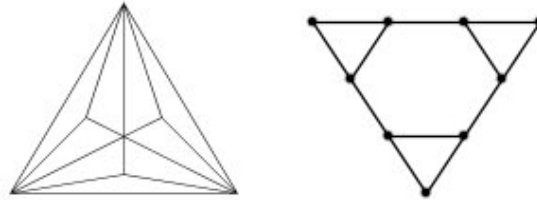


Figure 10: *A 2-stage homogeneously subdivided simplex and its dual graph, a 2-stage regular dual Sierpinski gasket*

Homogeneous subdivision of every subsimplex of a $(k-1)$ -stage homogeneously subdivided simplex gives us a k -stage homogeneously subdivided simplex. The dual graph of a k -stage homogeneously subdivided simplex is called a k -stage regular dual Sierpinski gasket.

A k -stage homogeneously subdivided simplex is just one example of a stacked sphere configuration. It is possible to calculate the spectrum of the dual graph of a k -stage homogeneously subdivided simplex in an analytical way. The spectrum of the 2-dimensional k -stage regular dual Sierpinski gasket is calculated in section 9.1. This spectrum is generalized to p -dimensional k -stage regular dual Sierpinski gaskets, $p \in \mathbb{N}$, in section 9.2.

These homogeneously subdivided simplices form a small subclass of stacked sphere configurations. We are also interested in more general stacked sphere configurations which have random, irregular structures. A step in this direction are the non-homogeneously subdivided simplices. These configurations are constructed by performing the subdivisions in a non-homogeneous way. The dual of such a configuration is called an irregular dual Sierpinski gasket. See figure 11 for an example.

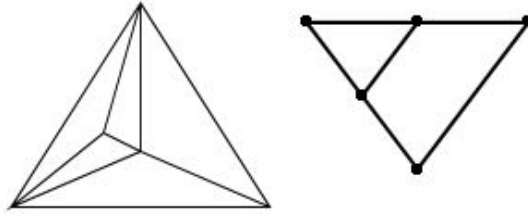


Figure 11: A *non-homogeneously subdivided simplex* and its dual graph, an *irregular dual Sierpinski gasket*.

In general, stacked sphere configurations are formed by glueing homogeneously and non-homogeneously subdivided simplices together along their boundary edges. A dual graph of a stacked sphere configuration is built up by regular and irregular dual Sierpinski gaskets which are connected with each other by coupling edges between their corner vertices. See figure 12 for an example.

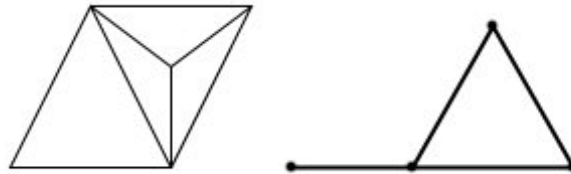


Figure 12: A *stacked sphere configuration* and its dual graph

We will also look at a special small class of stacked sphere configurations, the triangular stacked sphere configurations. A triangular stacked sphere configuration is a stacked sphere configuration of which the dual graph is built up from triangles which are connected to each other by coupling edges. See figure 13 for an example.

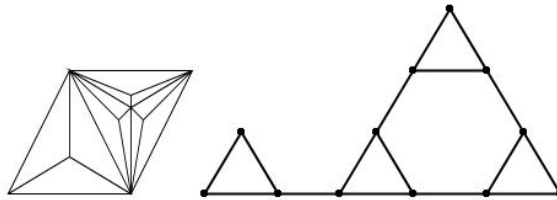


Figure 13: A *triangular stacked sphere configuration* and its dual graph

6 Graph Theory

We want to study the spectrum of the Laplacian of the dual graph of stacked sphere configurations. As we will see in section 9 the spectrum of the Laplacian of a dual Sierpinski gasket gives us information about the geometry of that Sierpinski gasket. And in sections 10 and 11 we see that the geometry of dual Sierpinski gaskets gives us some information about the geometry of general stacked sphere configurations. This section provides us a general overview of graph theory.

Definition 6.1 (Graph) *A graph X is defined as an ordered pair $(V(X), E(X))$. $V(X)$ is the vertex set and $E(X)$ is the edge set. Elements $e \in E(X)$ are unordered pairs of the elements of $V(X)$ with $e = xy, x \in V(X)$ and $y \in V(X)$.*

When $xy \in E(X)$ x is called adjacent to y . This is also denoted by $x \sim y$. A vertex is called incident to an edge when the vertex is one of the two vertices of the edge.

The degree $\delta(x)$ of a vertex x is the number of edges that are incident to that vertex. Valency is another word for degree and the use of these two words is inconsistent in the literature.

A path of length r from vertex x to vertex y in a graph is a sequence of $r+1$ distinct vertices starting from x and ending with y such that consecutive vertices are adjacent.

A subgraph of a graph X is a graph Y such that $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$. A subgraph Y of X is an induced subgraph if two vertices of $V(Y)$ are adjacent in Y if and only if they are adjacent in X . An induced subgraph that is maximal, with respect to connectedness, is called a connected component (also called a component). Two vertices are in the same component if and only if there is a path between those two vertices. If there is no path between those two vertices, the vertices are in different components.

A graph describes a binary relationship in which there is a set of objects, the vertex set, and vertices are connected to each other with an edge when they are adjacent.

6.1 Laplacian

It is possible to define linear operators on graphs. Let P be an operator working on graph X with n vertices. Then P can be seen as an operator mapping \mathcal{R}^n to \mathcal{R}^n . It can therefore be seen as a matrix. In the rest of the thesis $P(x, y)$ is defined as the component of matrix P on row x and column y

Definition 6.2 (Adjacency matrix) *The adjacency matrix A is given by*

$$A(x, y) = \begin{cases} 1, & \text{if } x \sim y \\ 0, & \text{otherwise.} \end{cases}$$

Definition 6.3 (Valency matrix) *The valency matrix D is given by*

$$D(x, y) = \begin{cases} \delta(x), & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

Definition 6.4 (Laplacian) *The Laplacian L is given by*

$$L(x, y) = A(x, y) - D(x, y) = \begin{cases} \delta(x), & \text{if } x = y \\ -1, & \text{if } x \sim y \\ 0, & \text{otherwise.} \end{cases}$$

This Laplacian is also called the combinatorial Laplacian. A variety of related Laplacians are used in the literature. Some authors prefer to write $L(x, y) = D(x, y) - A(x, y)$ but this only leads to a change of sign. Some authors refer to other definitions like the normalized Laplacian and Kigami Laplacian, as Laplacian. But the precise definition should always be clear from the context. In this thesis, we will use the combinatorial Laplacian.

Denote the eigenvalues of the Laplacian L by $\mu_1(L), \dots, \mu_n(L)$, such that

$$\mu_n(L) \leq \dots \leq \mu_2(L) \leq \mu_1(L).$$

The set of eigenvalues is called the spectrum. The spectrum gives some information about the geometrical nature of the graph. Some conditions on the spectrum are given by the following theorem [8].

Theorem 6.5 *The spectrum of the Laplacian of graph X satisfies the following conditions:*

- a) *All eigenvalues are real, the eigenvectors are mutually orthogonal and all eigenvalues $\mu_i(L) \leq 0$ for $1 \leq i \leq n$, with n the number of the vertices.*
- b) *$\mu_1(L) = 0$.*
- c) *The multiplicity of 0 is equal to the number of connected components and the eigenvectors belonging to non-zero eigenvalues are mutually orthogonal.*
- d) *The number of vertices is equal to the number of eigenvalues.*
- e) *The number of edges is equal to $-\frac{\text{Tr}(L)}{2}$.*

Proof a) The Laplacian is a symmetric matrix. All eigenvalues of symmetric matrices are real and the eigenvectors are mutually orthogonal. The Laplacian is a negative semi-definite matrix, hence all eigenvalues are non-positive

b) 0 is an eigenvalue of the Laplacian with eigenvector $(1, 1, \dots, 1)$. Because all eigenvalues are negative or equal to 0, $\mu_1(L) = 0$.

c) Assume the graph has m vertices and 2 components. Assume that the first

component has $k, k \leq n$ vertices and the second component has $n - k$ vertices. Let the first n components of a vector denote the n vertices of the first component and the last $m-n$ components of the same vector the $m-n$ vertices of the

second component. Then $(\overbrace{1, \dots, 1}^k, 0, \dots, 0)$ and $(0, \dots, 0, \overbrace{1, \dots, 1}^{n-k})$ are independent eigenvectors of the graph belonging to an eigenvalue 0. In an identical way it is easy to prove this for more components. d) The dimension of the space on which the linear operators on graphs work is equal to the number of vertices in a graph. It is also equal to the number of eigenvalues. Hence the number of vertices is equal to the number of eigenvalues

e) $tr(L) = tr(A - D) = -tr(D) = -\sum_{x \in V(X)} \delta(x)$. $\delta(x)$ is the number of edges that are incident to vertex x . Since every edge is incident to two vertices the number of edges is equal to $\frac{-\sum_{x \in V(X)} \delta(x)}{2} = -\frac{Tr(L)}{2}$ ■

Let \mathbf{u} be an eigenvector of L with eigenvalue μ . The spectrum of L can be found by solving the eigenvalue equation

$$L_n \mathbf{u} = \mu \mathbf{u} \quad (6)$$

In the theorem above it was concluded that for a graph with n vertices 0 is the highest eigenvalue, the normalized eigenvector belonging to the zero eigenvalue is $n^{-1/2} \overbrace{(1, \dots, 1)}^n$ and that the eigenvectors are mutually orthogonal.

Lemma 6.6 *For a gasket with n vertices, the following condition is true for an arbitrary eigenvector \mathbf{u} with non-zero eigenvalue*

$$\sum_{x_1 \dots x_n} \mathbf{u}(x_1, \dots, x_n) = 0 \quad (7)$$

Proof The dot product between \mathbf{u} and eigenvector \mathbf{v} with zero eigenvalue is given by

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \sum_{x_1 \dots x_n} \mathbf{u}(x_1, \dots, x_n) \mathbf{v}(x_1, \dots, x_n) \\ &= \sum_{x_1 \dots x_n} \mathbf{u}(x_1, \dots, x_n) n^{-1/2} \\ &= n^{-1/2} \sum_{x_1 \dots x_n} \mathbf{u}(x_1, \dots, x_n) \\ &= 0 \end{aligned}$$

Hence

$$\sum_{x_1 \dots x_n} \mathbf{u}(x_1, \dots, x_n) = 0 \quad \blacksquare$$

7 The spectrum of a regular lattice graph

For comparison with the spectrum of the dual Sierpinski gaskets, we first look at the spectrum of a lattice graph. A lattice graph is a graph that looks like a regular lattice. A lattice graph of size $k_1 \times k_2 \times \dots \times k_d$ is defined as a Cartesian product of one-dimensional lattice graphs of lengths k_1, k_2, \dots, k_d (see figure 14 for an example of a two-dimensional lattice graph). Let the k vertices of a one-dimensional lattice graph be labelled as $x_1, \dots, x_i, \dots, x_k$. Then a one-dimensional lattice graph is defined as a graph with $x_1, \dots, x_i, \dots, x_k$ as the vertex set and $x_i x_{i+1}, i \in \{1, \dots, k-1\}$ as the edge set.

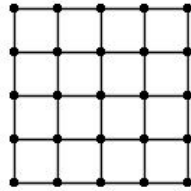


Figure 14: A two-dimensional lattice graph of size 5×5 .

x_1 and x_k are the two boundary vertices. When for a one-dimensional lattice graph the boundary vertices are not connected with each other by an edge, the lattice graph has free boundary conditions. When the vertices are connected with each other by an edge, the lattice graph has periodic boundary conditions (see figure 15).

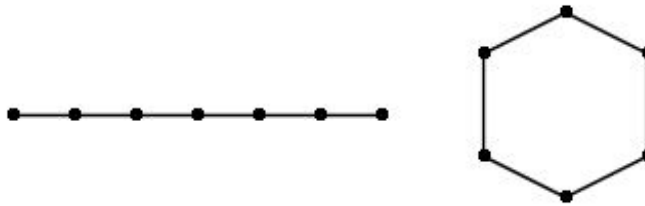


Figure 15: Two one-dimensional regular lattice graphs with six vertices. The graph on the left has free boundary conditions, the graph on the right has periodic boundary conditions.

The spectrum of the Laplacian on a one-dimensional lattice graph with k vertices is given by

$$\lambda_n = -2 + 2 \cos\left(\frac{n\pi}{k}\right), n = 0, 1, \dots, k-1 \quad (8)$$

$$(9)$$

in the case of free boundary conditions and by

$$\lambda_n = -2 + 2 \cos\left(\frac{2n\pi}{k}\right), n = 0, 1, \dots, k-1 \quad (10)$$

$$(11)$$

in the case of periodic boundary conditions [17]. Every eigenvalue is non-degenerate in the case of free boundary conditions. Every eigenvalue except 0 is doubly degenerate in the case of periodic boundary conditions and with k odd. Every eigenvalue except 0 and -4 is doubly degenerate in the case of periodic boundary conditions and with k even.

The spectrum of the Cartesian product of two graphs is equal to all possible sums of the eigenvalues of these two graphs [5]. This construction can then be iterated to a d -dimensional lattice graph. Let the lattice graph be of size $k_1 \times k_2 \times \dots \times k_d$. Then the spectrum of the Laplacian on this graph is given by

$$\lambda_{n_1, \dots, n_d} = -2d + 2 \sum_{i=1}^d \cos\left(\frac{n_i \pi}{k_i}\right), n_i = 0, 1, \dots, k_i - 1$$

in the case of free boundary conditions and by

$$\lambda_{n_1, \dots, n_d} = -2d + 2 \sum_{i=1}^d \cos\left(\frac{2n_i \pi}{k_i}\right), n_i = 0, 1, \dots, k_i - 1$$

in the case of periodic boundary conditions. When the lattice graph has periodic boundary conditions, it has the topology of a d -dimensional torus.

We see that the eigenvalues of the two-dimensional regular lattice graph are determined by all possible sums of the eigenvalues of the one-dimensional regular lattice graphs, whose Cartesian product is equal to the two-dimensional regular lattice graph. Assume that the two-dimensional lattice graph with periodic boundary conditions has size $k_1 \times k_2$. Let the eigenvalue μ_{k_1} of the one-dimensional lattice graph with k_1 vertices and periodic boundary conditions have multiplicity $g(\mu_{k_1})$ and the eigenvalue μ_{k_2} of the one-dimensional lattice graph with k_2 vertices and periodic boundary conditions have multiplicity $g(\mu_{k_2})$. Eigenvalue $\mu_{k_1} + \mu_{k_2}$ of the two-dimensional lattice graph with size $k_1 \times k_2$ and periodic boundary conditions has multiplicity $g(\mu_{k_1}) \times g(\mu_{k_2})$. Hence, the eigenvalue of $\mu_{k_1} + \mu_{k_2}$ has a multiplicity equal to 4 except in the

case that μ_{k1} or μ_{k2} are equal to 0 or -4 .

To compare the behaviour of the spectrum of the regular lattice graph with the spectrum of the dual Sierpinski gasket in greater detail, we now derive a transformation rule for the eigenvalues of 1-dimensional and 2-dimensional regular lattice graphs with periodic boundary conditions. The transformation rule for a one-dimensional lattice graph with periodic boundary conditions is given as follows: Let i be a parameter, such that $2^i k$ is the number of vertices on the one-dimensional lattice graph. Let $k \in \mathbb{N}$ be given. If we know the spectrum of the one-dimensional lattice graph with $2^i k$ vertices, we can calculate the spectrum of the one-dimensional lattice graph with $2^{i+1} k$ vertices by applying the transformation rule. The spectrum of the one-dimensional lattice graph with an arbitrary number of vertices can be calculated by equation (10).

Theorem 7.1 *Let μ_i be an eigenvalue of the one-dimensional lattice graph with $2^i k$ vertices and periodic boundary conditions. To this eigenvalue we can relate a corresponding eigenvalue μ_{i+1} which is an eigenvalue of the one-dimensional lattice graph with $2^{i+1} k$ vertices and periodic boundary conditions. These two eigenvalues are related by the following transformation rule*

$$\mu_i = \mu_{i+1}(\mu_{i+1} + 4). \quad (12)$$

Proof Let k be given. Take an arbitrary n , with $0 \leq n \leq 2^{i+1} k - 1$. Then

$$\begin{aligned} (-2 + 2 \cos(\frac{2n\pi}{2^{i+1}k})) (2 + 2 \cos(\frac{2n\pi}{2^{i+1}k})) &= -4 + 4 \cos^2(\frac{2n\pi}{2^{i+1}k}) \\ &= -4 \sin^2(\frac{2n\pi}{2^{i+1}k}) \\ &= -2 + 2 \cos(2 \frac{2n\pi}{2^{i+1}k}) \\ &= -2 + 2 \cos(\frac{2n\pi}{2^i k}). \end{aligned}$$

This is true for every n , with $0 \leq n \leq 2^{i+1} k - 1$. Hence

$$\mu_i = \mu_{i+1}(\mu_{i+1} + 4)$$

for given k . This is true for every $k \in \mathbb{N}$ ■

The inverse of equation (12) is given by

$$\mu_{i+1} = \frac{-4 \pm \sqrt{16 + 4\mu_i}}{2}. \quad (13)$$

Hence, the eigenvalues are bounded between -4 and 0 .

Theorem 12 can be generalized to two-dimensional regular lattice graphs with periodic boundary conditions. To do this we first define

$$\theta_i = \cos\left(\frac{2\pi n_2}{2^i k_2}\right) - 2, \quad n_2 = 0, 1, \dots, 2^i k_2 - 1$$

with k_2 a given parameter.

Let i be the parameter, such that $2^i k_1 \times 2^i k_2$ is the size of the two-dimensional lattice graph with periodic boundary conditions. If we know the spectrum of a two-dimensional lattice graph with size $2^i k_1 \times 2^i k_2$ and periodic boundary conditions, we can calculate the spectrum of the two-dimensional lattice graph with size $2^{i+1} k_1 \times 2^{i+1} k_2$ and periodic boundary conditions by applying the transformation rule.

Theorem 7.2 *Let μ_i be an eigenvalue of the two-dimensional lattice graph with size $2^i k_1 \times 2^i k_2$. To this eigenvalue we can relate a corresponding eigenvalue μ_{i+1} which is an eigenvalue of the two-dimensional lattice graph with size $2^{i+1} k_1 \times 2^{i+1} k_2$ and periodic boundary conditions. These two eigenvalues are related by the following transformation rule*

$$\mu_i = \mu_{i+1}(\mu_{i+1} - 4\theta_i) + 2(\theta_i + 1)(2\theta_i - 1), \quad (14)$$

$$\theta_i = 2\theta_{i+1}(\theta_{i+1} + 4) + 5. \quad (15)$$

Proof See [4] for a proof. ■

θ_i is bounded between -3 and -1 . The map for θ_i maps $[-3, -1]$ onto itself. For given $\theta_i \in [-3, -1]$ we define $a = 2\theta_i - 2$ and $b = 2\theta_i + 2$. Then the map for μ_i maps $[a, b]$ onto itself.

We will compare the spectral properties of the lattice graph and the dual graph of a stacked sphere configuration in chapter (9.4).

It is possible to find the spectra of other lattice graphs. A triangular lattice graph is a two-dimensional lattice graph of size $k_1 \times k_2$ with one extra diagonal edge in every square of the graph, such that the orientation of the diagonal edge is the same in every square (see figure 16). The spectrum of a triangular lattice graph with size $k_1 \times k_2$ and with periodic boundary conditions is given by [9]

$$\lambda_{n,m} = -6 + 2 \cos\left(\frac{2n\pi}{k_1}\right) + 2 \cos\left(\frac{2m\pi}{k_2}\right) + 2 \cos\left(2\pi\left(\frac{n}{k_1} + \frac{m}{k_2}\right)\right), n = 0, 1, \dots, k_1 - 1, m = 0, 1, \dots, k_2 - 1.$$

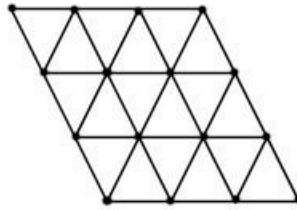


Figure 16: A triangular lattice graph of size 4×4 .

8 Sierpinski gaskets

Sierpinski gaskets are fractal graphs. There are two different kinds of Sierpinski gaskets, namely the vertex-coupled and dual Sierpinski gaskets.

A 2-dimensional vertex-coupled Sierpinski gasket is inductively defined in the following way (see figure 17).

- Let \widehat{V}^1 be the graph defined by $V(\widehat{V}^1) = \{a, b, c\}$ and $E(\widehat{V}^1) = \{ab, bc, ac\}$. So \widehat{V}^1 is a graph with 3 vertices and 1 loop. Graph \widehat{V}^1 is called the 1-stage vertex-coupled Sierpinski gasket.
- Let $\widehat{V}_k^i, k \in \{1, 2, 3\}, i \in \mathbb{N}, i > 0$ be 3 copies of \widehat{V}^i . i denotes the stage of the gaskets here. Let a_k^i, b_k^i, c_k^i be the 3 corner vertices of \widehat{V}_k^i . Then we define \widehat{V}^{i+1} by $V(\widehat{V}^{i+1}) = \bigcup_{k \in \{1, 2, 3\}} V(\widehat{V}_k^i)$ and $E(\widehat{V}^{i+1}) = \bigcup_{k \in \{1, 2, 3\}} E(\widehat{V}_k^i)$ with $a_1^i = a_2^i, b_2^i = b_3^i$ and $c_3^i = c_1^i$. Graph \widehat{V}^{i+1} is called the $(i + 1)$ -stage vertex-coupled Sierpinski gasket.

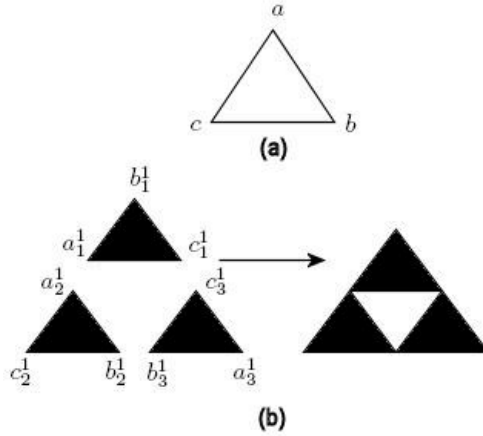


Figure 17: (a) An 1-stage vertex-coupled Sierpinski gasket. (b) The creation of a 2-stage vertex-coupled Sierpinski gasket out of 1-stage vertex-coupled Sierpinski gaskets.

An $(i + 1)$ -stage vertex-coupled Sierpinski gasket is built by taking three i -stage vertex-coupled Sierpinski gaskets and letting the corner vertices coincide. In the literature, when one talks about a Sierpinski gasket, one usually refers to a vertex-coupled Sierpinski gasket.

A 2-dimensional dual Sierpinski gasket is inductively defined as follows. See figure 18.

- Let \tilde{V}^1 be the graph defined by $V(\tilde{V}^1) = \{a, b, c\}$ and $E(\tilde{V}^1) = \{ab, bc, ac\}$. Hence \tilde{V}^1 is a graph with 3 vertices and 1 loop. Graph \tilde{V}^1 is called a 1-stage dual Sierpinski gasket.
- Let $\tilde{V}_k^i, k \in \{1, 2, 3\}, i \in \mathbb{N}$ be 3 copies of \tilde{V}^i . i denotes the stage of the gaskets. Let a_k^i, b_k^i, c_k^i be the 3 corner vertices of \tilde{V}_k^i . Then we define \tilde{V}^{i+1} by $V(\tilde{V}^{i+1}) = \bigcup_{k \in \{1, 2, 3\}} V(\tilde{V}_k^i)$ and $E(\tilde{V}^{i+1}) = \bigcup_{k \in \{1, 2, 3\}} E(\tilde{V}_k^i) \cup \{a_1^i a_2^i, b_2^i b_3^i, c_3^i c_1^i\}$. Graph \tilde{V}^{i+1} is called a $(i + 1)$ -stage dual Sierpinski gasket.

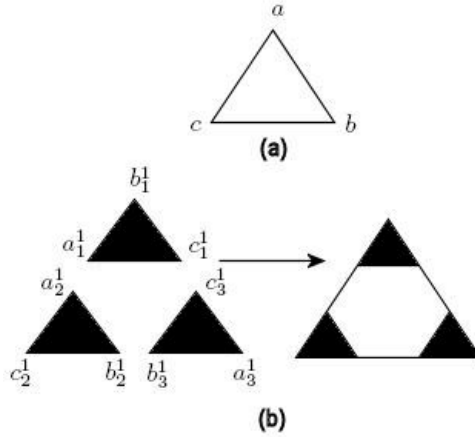


Figure 18: (a) An 1-stage dual Sierpinski gasket. (b) The creation of a 2-stage dual Sierpinski gasket out of 1-stage dual Sierpinski gaskets

An $(i + 1)$ -stage dual Sierpinski gasket is built by taking three i -stage dual Sierpinski gaskets and letting the corner vertices of each i -stage Sierpinski gasket be connected by edges. This gives us a dual Sierpinski gasket. The dual Sierpinski gasket can also be constructed from a vertex-coupled Sierpinski gasket. This can be done in the following way:

- Orient the whole gasket in such a way that it is pointing upwards.
- With a triangle in a graph we mean a subgraph that consists of 3 vertices that are connected to each other in a triangle.

- Put a vertex in every upwards pointing triangle (called a cell) in the gasket and connect these vertices by edges, when the cells are adjacent to each other.

Hence, there is a mapping from the triangles of a i -stage vertex-coupled Sierpinski gasket to the vertices of a $(i - 1)$ -stage dual Sierpinski gasket, $i \in \mathbb{N}$. This explains why a dual Sierpinski gasket is called dual.

A p -dimensional vertex-coupled Sierpinski gasket is inductively defined in the following way:

- Let \widehat{V}^1 be the graph defined by $V(\widehat{V}^1) = \{a_m | m \in \{1, \dots, p + 1\}\}$ and $E(\widehat{V}^1) = \{a_m a_n | m \neq n, m, n \in \{1, \dots, p + 1\}\}$. Hence \widehat{V}^1 is a graph with $p + 1$ vertices which are connected to each other by edges. Graph \widehat{V}^1 is called the 1-stage vertex-coupled Sierpinski gasket.
- Let $\widehat{V}_k^i, k \in \{1, \dots, p + 1\}, i \in \mathbb{N}$ be $p + 1$ copies of \widehat{V}^i . Then we define \widehat{V}^{i+1} by $V(\widehat{V}^{i+1}) = \bigcup_{k \in \{1, \dots, p+1\}} V(\widehat{V}_k^i)$ and $E(\widehat{V}^{i+1}) = \bigcup_{k \in \{1, \dots, p\}} E(\widehat{V}_k^i)$. The \widehat{V}_m^i are pairwise connected by coinciding different corner vertices for every pair. Graph \widehat{V}^{i+1} is called the $(i + 1)$ -stage vertex-coupled Sierpinski gasket.

We can get a p -dimensional dual Sierpinski gasket as follows:

- Orient the whole gasket in such a way that it is pointing upwards
- With a p -dimensional pyramid in a graph we mean a subgraph that consists of $p + 1$ vertices that are connected to each other in a p -dimensional triangle.
- Put a vertex in every upwards pointing pyramid (called a cell) in the gasket and connect these vertices by edges, when the cells are adjacent to each other.

This gives us a p -dimensional dual Sierpinski gasket.

8.1 Labelling of regular dual Sierpinski gaskets

To simplify the description of a dual Sierpinski gasket, we introduce a certain labelling scheme. In 2 dimensions a labelling scheme for the cells of a vertex-coupled Sierpinski gasket is inductively given as follows. See figure 19

- With a triangle in a graph we mean a subgraph that consists of 3 vertices that are connected to each other in a triangle.
- At stage $n = 1$ orient the triangle in such a way that it is pointing upwards. At every stage every upside triangle gets a label.
- At stage $n = 2$ label the upper triangle (1), the lower left triangle (2) and the lower right triangle (3).
- At stage $n = k, k > 2$ the triangles are labelled by the following labels $(x_1, x_2, \dots, x_{k-1})$, where $x_i \in \mathcal{F}$, $\mathcal{F} = \{1, \dots, 3\}$ and $1 \leq i \leq k - 1$. Assume that an arbitrary triangle Δ has label $(x_1, x_2, \dots, x_{k-1})$. At stage $n = k + 1$ the 3 new triangles in Δ are labelled as follows: the upper triangle in Δ gets label $(x_1, x_2, \dots, x_k, 1)$, the lower left triangle in Δ gets label $(x_1, x_2, \dots, x_k, 2)$ and the lower right triangle in Δ gets label $(x_1, x_2, \dots, x_k, 3)$

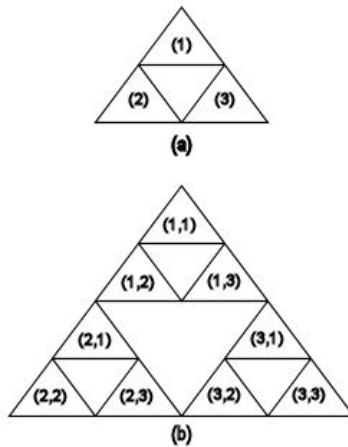


Figure 19: (a) Labelling of a 2-stage vertex-coupled Sierpinski gasket. (b) Labelling of a 3-stage vertex-coupled Sierpinski gasket

This labelling scheme can also be used for dual Sierpinski gaskets. We saw in chapter 8 that there is a mapping from the triangles of a vertex-coupled Sierpinski gasket to the vertices of a dual Sierpinski gasket. The labeling scheme for the dual Sierpinski gasket is as follows. See figure 20

- With a triangle in a graph we mean a subgraph that consists of 3 vertices that are connected to each other in a triangle.
- At stage $n = 1$ orient the triangle in such a way that it is pointing upwards. The upper vertex is labeled (1), the lower left vertex is labeled (2) and the lower right vertex is labeled (3) .
- At stage $n = k, k > 1$ the vertices are labelled by the following labels (x_1, x_2, \dots, x_k) , where $x_i \in \mathcal{F}$, $\mathcal{F} = \{1, \dots, 3\}$ and $1 \leq i \leq k$. Assume that an arbitrary vertex A has a label (x_1, x_2, \dots, x_k) . At stage $n = k + 1$ vertex A will be a vertex of a triangle with two new vertices. Call this triangle Δ . Then the vertices in Δ are labelled in the following way: the upper vertex in Δ gets label $(x_1, x_2, \dots, x_k, 1)$, the lower left vertex in Δ gets label $(x_1, x_2, \dots, x_k, 2)$ and the lower right vertex in Δ gets label $(x_1, x_2, \dots, x_k, 3)$

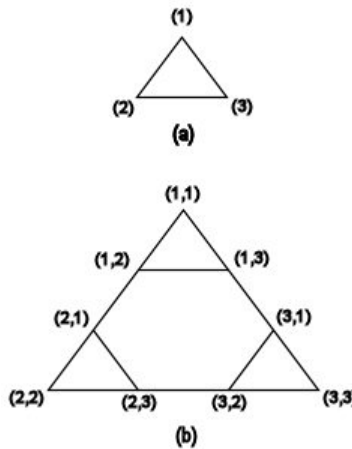


Figure 20: (a) Labelling of a 1-stage dual Sierpinski gasket. (b) Labelling of a 2-stage dual Sierpinski gasket

This labelling scheme can easily be generalized to higher-dimensional vertex-coupled Sierpinski gaskets. To do so, the former labelling procedure is changed in the following way for p dimensions:

- With a p -dimensional pyramid in a graph we mean a subgraph that consists of $p + 1$ vertices that are connected to each other in a p -dimensional pyramid.
- At stage $n = 1$ orient the p -dimensional pyramid in such a way that it is pointing upwards.
- At stage $n = 2$ give the p -dimensional pyramids that are pointing upwards different labels $(i), i \in \{1, \dots, p + 1\}$. The orientation for the labels is important for later stages.
- At stage $n = k, k > 2$ the p -dimensional pyramids are labelled by the following labels $(x_1, x_2, \dots, x_{k-1})$, where $x_i \in \mathcal{F}, \mathcal{F} = \{1, \dots, p + 1\}$ and $1 \leq i \leq k - 1$. Assume that an arbitrary p -dimensional pyramid Δ has label $(x_1, x_2, \dots, x_{k-1})$. At stage $n = k$ the $p + 1$ p -dimensional pyramids in Δ get different labels $(x_1, x_2, \dots, x_{k-1}, i), i \in \mathcal{F}$ where the orientation for the labels is the same as for stage $n = 2$.

This labelling scheme can also be used for dual Sierpinski gaskets. Let a cell with label (x_1, x_2, \dots, x_k) in a k -stage vertex-coupled Sierpinski gasket be mapped to a vertex in the $(k - 1)$ -stage dual Sierpinski gasket. This vertex gets the same label

The labels of the vertices that are adjacent to a vertex with label (x_1, x_2, \dots, x_n) are given as follows:

- When the vertex is labelled as

$$\overbrace{(i, \dots, i, i)}^{n-1}, i \in \{1, \dots, p\},$$

the vertex is a corner vertex. In p dimensions it has p neighbouring vertices. The labels of these neighbouring vertices are given by

$$\overbrace{(i, \dots, i, m)}^{n-1}, m \in \{1, \dots, p\}, m \neq i.$$

- When the vertex is not a corner vertex, it is labelled as

$$(x_1, x_2, \dots, x_{n-k-1}, j, \overbrace{i, \dots, i}^k), \quad 2 \geq k \geq n-1, i, j \in \{1, \dots, p\}.$$

The neighbouring vertices in the same pyramid are labelled as

$$(x_1, x_2, \dots, x_{n-k-1}, j, \overbrace{i, \dots, i}^{k-1}, m), \quad 2 \geq k \geq n-1, i, j, m \in \{1, \dots, p\}, m \neq i.$$

The neighbouring vertex that is in another pyramid is labelled as

$$(x_1, x_2, \dots, x_{n-k-1}, i, \overbrace{j, \dots, j}^k), \quad 2 \geq k \geq n-1, i, j \in \{1, \dots, p\}.$$

Components of vectors on graphs can also be defined by these labelling. For example take a vector \mathbf{u} , then the component of this vectors on vertex (x_1, x_2, \dots, x_n) is equal to $\mathbf{u}(x_1, x_2, \dots, x_n)$.

9 Spectra of dual Sierpinski gaskets

In this section we will calculate the spectra of dual Sierpinski gaskets and derive some geometrical results which also apply on more general stacked sphere configurations.

In subsection 9.1 we will calculate the spectra of two-dimensional regular dual Sierpinski gaskets. We first look at the spectra of the 1-stage and 2-stage dual Sierpinski gaskets. From the eigenstate structure of the $\mu = -3$ and $\mu = -5$ eigenstates we are able to calculate the multiplicities of eigenvalues $\mu = -3$ and $\mu = -5$. We will see that the $\mu = -3$ eigenstates are symmetric eigenstates and the $\mu = -5$ eigenstates are anti-symmetric eigenstates.

We will then calculate the spectra of the n -stage dual Sierpinski gasket, with $n \geq 3$. It turns out that the system of eigenvalue equations of the n -stage dual Sierpinski gasket can be related to the system of eigenvalue equations of the $(n - 1)$ -stage dual Sierpinski gasket. This gives us a recursion relation between the eigenvalues of a $(n - 1)$ -stage dual Sierpinski gasket and the eigenvalues of a n -stage dual Sierpinski gasket. This recursion relation is given by

$$\mu_{n-1} = \mu_n(\mu_n + 5)$$

with μ_{n-1} an eigenvalue of the $(n - 1)$ -stage dual Sierpinski gasket and μ_n a corresponding eigenvalue of the n -stage dual Sierpinski gasket. We will also calculate the multiplicities of the eigenvalues of these gaskets.

In subsection 9.2 we will generalize these results to p -dimensional regular dual Sierpinski gaskets, with $p \geq 3$. The spectra of these dual Sierpinski gaskets have a similar structure as the spectra of the two-dimensional dual Sierpinski gaskets. We also find a recursion relation between the eigenvalues of a $(n - 1)$ -stage dual Sierpinski gasket and the eigenvalues of a n -stage dual Sierpinski gasket. This recursion relation is given by

$$\mu_{n-1} = \mu_n(\mu_n + p + 3)$$

with μ_{n-1} an eigenvalue of the $(n - 1)$ -stage dual Sierpinski gasket and μ_n a corresponding eigenvalue of the n -stage dual Sierpinski gasket.

In subsection 9.3 we will derive some results from the spectra of two-dimensional regular dual Sierpinski gaskets. We will see that the eigenvalues are bounded between 0 and -5 . These bounds turn out to be dependent on the dimension, because for p -dimensional regular dual Sierpinski gaskets the eigenvalues are bounded between 0 and $-p - 3$.

We will define edge-triangle cycles and give a proof that the multiplicity of $\mu = -5$ is equal to the number of edge-triangle cycles. We will also generalize this to edge- i -gasket cycles. The multiplicity of $\mu = \eta_{i-1}(\sigma_1, \dots, \sigma_{i-1})$ is equal

to the number of edge- i -gasket cycles. It turns out that these results are also valid for more general stacked sphere configurations, as we will see in section 10 and section 11.

In subsection 9.4 we will compare the spectra of the dual Sierpinski gaskets with the spectra of regular two-dimensional lattice graphs. For the regular two-dimensional lattice graphs and the dual Sierpinski gaskets we can derive similar transformation rules. The spectra of these graphs are different. The spectrum of the regular dual Sierpinski gasket is self-similar. The spectrum of the regular two-dimensional lattice graph is not self-similar, because the multiplicities of the eigenvalues are equal to 2 or 4 except for the eigenvalues $\mu = 0$ and $\mu = -4$.

9.1 Spectrum of 2-dimensional regular dual Sierpinski gaskets

Assume a label of a vertex in a triangle of a n -stage gasket is given by $x = (x_1, \dots, x_n)$, $x_i \in \{1, 2, 3\}$, $i \in \{1, \dots, n\}$. Then the label of the neighbouring vertex that is encountered when the triangle is followed clockwise is then given by x' and the label of the vertex that is encountered counterclockwise x'' . When the vertex has a neighbouring vertex in another triangle, the label of that neighbouring vertex is given by \bar{x} .

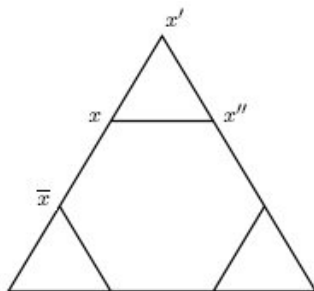


Figure 21: Labelling of a vertex and its neighbouring vertices.

With $u(x)$ we will mean the vector component of eigenvector u on vertex x . The calculation of the spectrum of a 2-dimensional homogeneous dual Sierpinski gasket is inspired by the work of Cosenza and Kapral [4].

For stage $n = 1$ equation (6) is given by

$$(\mu + 2)u(x) = u(x') + u(x'') \quad (16)$$

for an arbitrary vertex x in the 1-stage gasket.

For $\mu = 0$, it is found that $u = 3^{-1/2}(1, 1, 1)$. Condition (7) combined with equation (16) gives the other eigenvalue $\mu = -3$ with multiplicity 2. Hence $\mu = 0$ is an eigenvalue belonging to a homogeneous eigenstate and $\mu = -3$ is an eigenvalue with multiplicity 2.

For stage $n = 2$ equation (6) is equal to a system of equations

$$(\mu + 2)u(x) = u(x') + u(x''), \text{ if } x = (i, i), i \in \{1, 2, 3\} \quad (17)$$

$$(\mu + 3)u(x) = u(x') + u(x'') + u(\bar{x}), \text{ otherwise.} \quad (18)$$

Define now

$$c(1) = u(1, 1) + u(2, 2) + u(3, 3) \quad (19)$$

$$c(2) = u(1, 2) + u(2, 3) + u(3, 1) \quad (20)$$

$$c(3) = u(1, 3) + u(2, 1) + u(3, 2) \quad (21)$$

Combining (17) up to (21) we get

$$(\mu + 2)c(1) = c(2) + c(3)$$

$$(\mu + 3)c(2) = c(1) + 2c(3)$$

$$(\mu + 3)c(3) = c(1) + 2c(2)$$

This system of equations has non-trivial solutions if the determinant of the coefficients is equal to zero. Hence

$$\mu(\mu + 3)(\mu + 5) = 0. \quad (22)$$

The solutions of (22) are $\mu = 0$, $\mu = -3$ and $\mu = -5$.

The solution $\mu = 0$ corresponds to the homogenous eigenstate, with eigenvector equal to $3^{-1}(\overbrace{1, \dots, 1}^9)$

When we insert solution $\mu = -3$ in (17) and (18) we get

$$u(x) + u(x') + u(x'') = 0, \text{ if } x = (i, i), i \in \{1, 2, 3\} \quad (23)$$

$$u(x') + u(x'') + u(\bar{x}) = 0, \text{ otherwise.} \quad (24)$$

From (7), (23) and (24) we conclude that for $\mu = -3$

$$u(x) + u(x') + u(x'') = 0 \quad (25)$$

$$u(x) = u(\bar{x}) \quad (26)$$

We see clearly the coupling between neighbouring triangles in (25) and (26). The $\mu = -3$ mode is also called a symmetric mode, because of the symmetry of the eigenstate.

Let $g_n(\mu)$ be the multiplicity of eigenvalue μ in a n-stage gasket. In a (n+1)-stage gasket we have 3 n-stage gaskets. If they were not coupled, we would have

$$g_{n+1}(-3) = 3g_n(-3) \quad (27)$$

But the n-stage gaskets are coupled through $u(x) = u(\bar{x})$. The multiplicity is equal to the minimum number of variables that is needed to specify a vector

that obeys (25) and (26). Every coupling determines one variable. Hence (27) changes to

$$g_{n+1}(-3) = 3g_n(-3) - 3. \quad (28)$$

Combining this with the fact that $g_1(-3) = 2$ we get

$$g_n(-3) = \frac{3^{n-1} + 3}{2} \quad (29)$$

for the multiplicity of - 3.

When we insert solution $\mu = -5$ in (17) and (18) we get

$$3u(x) + u(x') + u(x'') = 0, \text{ if } x = (i, i), i \in \{1, 2, 3\} \quad (30)$$

$$2u(x) + u(x') + u(x'') + u(\bar{x}) = 0, \text{ otherwise.} \quad (31)$$

From (7),(30) and (31) we conclude that for $\mu = -5$

$$u(x) + u(x') + u(x'') = 0 \quad (32)$$

$$u(x) = -u(\bar{x}) \quad (33)$$

Just as with $\mu = -3$, we see clearly the coupling between neighbouring triangles in (32) and (33). The $\mu = -5$ mode is also called an asymmetric mode, because of the antisymmetry of the eigenstate.

In a $(n+1)$ -stage gasket we have 3 n-stage gaskets. If they were not coupled, we would have

$$g_{n+1}(-5) = 3g_n(-5) \quad (34)$$

But the n-stage gaskets are coupled through $u(x) = -u(\bar{x})$. From (30) and (32) we conclude that $u(x) = 0$ when $x = \overbrace{(i, \dots, i)}^n, i \in \{1, 2, 3\}$ is a corner vertex. For a n-stage gasket we therefore have

$$u(\overbrace{(i, \dots, i)}^n) + u(\overbrace{(j, \dots, j)}^n) + u(\overbrace{(k, \dots, k)}^n) = 0.$$

where $i \neq j \neq k, i, j, k \in \{1, 2, 3\}$.

In a $(n + 1)$ -stage gasket, such a n-stage gasket will satisfy $u(m \overbrace{i, \dots, i}^n) + u(m \overbrace{j, \dots, j}^n) + u(m \overbrace{k, \dots, k}^n) = 0, m \in \{1, 2, 3\}$. Together with the coupling between the n-stage gaskets, we see that the coupling is determined by 1 variable.

Hence (34) changes to

$$g_{n+1}(-5) = 3g_n(-5) + 1.$$

Combining this with the fact that $g_1(-5) = 0$ we get

$$g_n(-5) = \frac{3^{n-1} - 1}{2} \quad (35)$$

for the multiplicity of 5.

We now go back to 2-stage gaskets. Assume that all the $c(i), i \in \{1, 2, 3\}$ are equal to zero. Then (19) up to (21) has trivial solutions. In that case we can rewrite (17) and (18) to

$$(\mu + 2)(u(2, 2) + u(3, 3)) = (u(2, 1) + u(3, 1)) + (u(2, 3) + u(3, 2)) \quad (36)$$

$$(\mu + 2)(u(2, 3) + u(3, 2)) = (u(2, 1) + u(3, 1)) + (u(2, 2) + u(3, 3)) \quad (37)$$

$$(\mu + 4)(u(2, 1) + u(3, 1)) = (u(2, 2) + u(3, 3)) \quad (38)$$

This system of equations has nontrivial solutions when the determinant is equal to zero. Hence

$$(\mu + 3)(\mu^2 + 5\mu + 3) = 0$$

The solutions of this equation are given by $\mu = -3$ and $\mu = \frac{-5 \pm \sqrt{13}}{2}$. Assume that $(u(2, 1) + u(3, 1)), (u(2, 2) + u(3, 3))$ and $(u(2, 3) + u(3, 2))$ are equal to zero. Then (36) up to (38) has trivial solutions. In that case we can rewrite (36) up to (38) to

$$(\mu + 2)u(3, 3) = u(3, 1) + u(3, 2)$$

$$(\mu + 2)u(3, 2) = u(3, 1) + u(3, 3)$$

$$(\mu + 4)u(3, 1) = u(3, 3)$$

This system of equations has the same coefficients as (36) up to (38). Hence we get the same equation for the eigenvalues. And therefore, we get the same eigenvalues.

Hence, for a 2-stage gasket we have the following spectrum: $\mu = 0$ with multiplicity 1, $\mu = -3$ with multiplicity 3, $\mu = -5$ with multiplicity 1, $\mu = \frac{-5 + \sqrt{13}}{2}$ with multiplicity 2 and $\mu = \frac{-5 - \sqrt{13}}{2}$ with multiplicity 2.

For n-stage gaskets with $n \geq 3$ the spectrum can be found in the following way. Look at a $(n - 1)$ -stage gasket. We take a vertex with label (x_1, \dots, x_{n-1}) and call it x . Just as before, we define the vertices in the same triangle as x' and x'' . The neighbouring vertex in another neighbouring triangle is \bar{x}

Define

$$u_n(x, i) = u_n(x_1, \dots, x_{n-1}, i)$$

for $i \in 1, 2, 3$, and

$$c_n(x) = (u_n(x, 1) + u_n(x, 2) + u_n(x, 3)). \quad (39)$$

This is the sum of the vector components on the vertices in the same triangle.

Assume none of the (x, i) , with $i \in \{1, 2, 3\}$, is a corner vertex. In the following calculations $k, l, m \in \{1, 2, 3\}$, but the precise value of k, l, m is not important. Then (6) and (39) gives

$$\begin{aligned} \mu(\mu + 5)c_n(x) &= \mu(\mu + 5)(u_n(x, 1) + u_n(x, 2) + u_n(x, 3)) \\ &= \mu(\mu + 5)u_n(x, 1) + \mu(\mu + 5)u_n(x, 2) + \mu(\mu + 5)u_n(x, 3) \\ &= \mu(2u_n(x, 1) + u_n(x, 2) + u_n(x, 3) + u_n(x', k)) \\ &\quad + \mu(u_n(x, 1) + 2u_n(x, 2) + u_n(x, 3) + u_n(x'', l)) \\ &\quad + \mu(u_n(x, 1) + u_n(x, 2) + 2u_n(x, 3) + u_n(\bar{x}, m)) \\ &= -3(u_n(x, 1) + u_n(x, 2) + u_n(x, 3)) \\ &\quad + (u_n(x', 1) + u_n(x', 2) + u_n(x', 3)) \\ &\quad + (u_n(x'', 1) + u_n(x'', 2) + u_n(x'', 3)) \\ &\quad + (u_n(\bar{x}, 1) + u_n(\bar{x}, 2) + u_n(\bar{x}, 3)) \\ &= -3c_n(x) + c_n(x') + c_n(x'') + c_n(\bar{x}) \end{aligned}$$

Assume one of the (x, i) is a corner vertex. We choose $(x, 1)$, but the same holds for the cases $i = 2$ or $i = 3$. Then (6) and (39) gives

$$\begin{aligned} \mu(\mu + 5)c_n(x) &= \mu(\mu + 5)(u_n(x, 1) + u_n(x, 2) + u_n(x, 3)) \\ &= \mu(\mu + 5)u_n(x, 1) + \mu(\mu + 5)u_n(x, 2) + \mu(\mu + 5)u_n(x, 3) \\ &= \mu(3u_n(x, 1) + u_n(x, 2) + u_n(x, 3)) \\ &\quad + \mu(u_n(x, 1) + 2u_n(x, 2) + u_n(x, 3) + u_n(x', k)) \\ &\quad + \mu(u_n(x, 1) + u_n(x, 2) + 2u_n(x, 3) + u_n(x'', l)) \\ &= -2(u_n(x, 1) + u_n(x, 2) + u_n(x, 3)) \\ &\quad + (u_n(x', 1) + u_n(x', 2) + u_n(x', 3)) \\ &\quad + (u_n(x'', 1) + u_n(x'', 2) + u_n(x'', 3)) \\ &= -2c_n(x) + c_n(x') + c_n(x'') \end{aligned}$$

Set $\nu = \mu(\mu + 5)$ and $u(x) = c_n(x)$. Then the previous equations transform to

$$\nu u(x) = -3u(x) + u(x') + u(x'') + u(\bar{x}) \quad (40)$$

when the (x, i) are not corner vertices and

$$\nu u(x) = -2u(x) + u(x') + u(x'') \quad (41)$$

when one of the (x, i) is a corner vertex. We see that it is possible to transform the eigenvalue equation of a n -stage gasket to the eigenvalue equation of a $(n-1)$ -stage gasket in this way, with eigenvalue ν instead of μ and eigenvector components $c_n(x)$ instead of $u_n(x)$.

The spectrum of a n -stage gasket consists of the spectrum of a $(n-1)$ -stage gasket and new eigenvalues corresponding to the n -stage gasket, obtained by the relation

$$\mu_{n-1} = \mu_n(\mu_n + 5)$$

where μ_{n-1} is an eigenvalue of the $(n-1)$ -stage gasket and μ_n the new eigenvalues of the n -stage gasket.

The inverse of this equation is given by

$$\mu_n = \frac{-5 \pm \sqrt{25 + 4\mu_{n-1}}}{2} \quad (42)$$

This is a recursion equation. We define $\mu_n = \mu_n(\sigma_1, \dots, \sigma_n)$ with $\sigma_i, i \in \{1, \dots, n-1\}$ and $\sigma_i \in \{+, -\}$. Then the sign of σ_i is defined by the sign that is taken in the following equation

$$\mu_{i+1}(\sigma_1, \dots, \sigma_i) = \frac{-5 \pm \sqrt{25 + 4\mu_i(\sigma_1, \dots, \sigma_{i-1})}}{2} \quad (43)$$

We define $\mu_i = \chi_i$ if $\mu_2 = -3$ and $\mu_i = \eta_i$ if $\mu_2 = -5$. Hence we get

$$\chi_{i+1}(\sigma_1, \dots, \sigma_i) = \frac{-5 \pm \sqrt{25 + 4\chi_i(\sigma_1, \dots, \sigma_{i-1})}}{2} \quad (44)$$

$$\eta_{i+1}(\sigma_1, \dots, \sigma_i) = \frac{-5 \pm \sqrt{25 + 4\eta_i(\sigma_1, \dots, \sigma_{i-1})}}{2} \quad (45)$$

Then using the fact that we can transform a n -stage gasket to a $(n-1)$ -stage gasket and equations (29) and (35) gives for the multiplicity of $\chi_i(\sigma_1, \dots, \sigma_{i-1})$ in a n -stage gasket the following formula

$$g_n(\chi_i(\sigma_1, \dots, \sigma_{i-1})) = \frac{3^{n-i-1} + 3}{2} \quad (46)$$

and for the multiplicity of $\eta_i(\sigma_1, \dots, \sigma_{i-1})$ in a n -stage gasket the following formula

$$g_n(\eta_i(\sigma_1, \dots, \sigma_{i-1})) = \frac{3^{n-i-1} - 1}{2}. \quad (47)$$

There are 2^i possible η_i 's and 2^i possible χ_i 's, each possible eigenvalue is defined by the vector $(\sigma_1, \dots, \sigma_i)$.

The total number of eigenvalues is given by

$$\begin{aligned}
& \sum_{i=0}^{n-1} 2^i g_n(\chi_i(\sigma_1, \dots, \sigma_{i-1})) + \sum_{i=0}^{n-2} 2^i g_n(\eta_i(\sigma_1, \dots, \sigma_{i-1})) + 1 \\
&= \sum_{i=0}^{n-1} 2^i \left(\frac{3^{n-i-1} + 3}{2} \right) + \sum_{i=0}^{n-2} 2^i \left(\frac{3^{n-i-1} - 1}{2} \right) + 1 \\
&= 2^n + 3^{n-1} \sum_{i=0}^{n-2} 2^i 3^{-i} + \sum_{i=0}^{n-2} 2^i + 1 \\
&= 2^n + 3^{n-1} \frac{1 - (\frac{2}{3})^{n-1}}{1 - \frac{2}{3}} + \frac{1 - 2^{n-1}}{1 - 2} + 1 \\
&= 2 \times 2^{n-1} + 3^n - 3 \times 2^{n-1} - 1 + 2^{n-1} + 1 \\
&= 3^n
\end{aligned}$$

as is expected.

Hence, the spectrum of a n-stage gasket consists of

μ	$g_n(\mu)$
n=1	
0	1
-3	2
n=2	
0	1
-3	3
-5	1
$\frac{-5+\sqrt{13}}{2}$	2
$\frac{-5-\sqrt{13}}{2}$	2
$n \geq 3$	
0	1
-3	$\frac{3^{n-1}+3}{2}$
-5	$\frac{3^{n-1}-1}{2}$
$\chi_i(\sigma_1, \dots, \sigma_i)$	$\frac{3^{n-i-1}+3}{2}$
$\eta_i(\sigma_1, \dots, \sigma_i)$	$\frac{3^{n-i-1}-1}{2}$
$\frac{-5+\sqrt{25-4\chi_i(\sigma_1, \dots, \sigma_i)}}{2}$	2
$\frac{-5-\sqrt{25-4\chi_i(\sigma_1, \dots, \sigma_i)}}{2}$	2

with $i \in \{1, \dots, n-1\}$ and $\sigma \in \{+, -\}$.

9.2 Spectrum of p-dimensional regular dual Sierpinski gaskets

A generalization of the spectrum of a 2-dimensional Sierpinski gasket to a p-dimensional Sierpinski gasket is given in the following way. In the case of a p-dimensional gasket, the corner vertices are adjacent to p other vertices and the non-corner vertices are adjacent to p+1 other vertices. Because of this, the eigenvalue equation (6) is changed to

$$(\mu + p)u(x) = \sum_{x' \sim x} u(x'), \text{ if } x \text{ is a corner vertex} \quad (48)$$

$$(\mu + p + 1)u(x) = \sum_{x' \sim x} u(x') + u(\bar{x}), \text{ otherwise.} \quad (49)$$

Here, the x' are the vertices in the same simplex as x and they are adjacent with x . \bar{x} is a vertex adjacent to x but in another simplex.

Just like as in the case of a 2-dimensional gasket, the eigenstates with non-zero and different eigenvalues are mutually orthogonal. This gives us the following equation (see (7)):

$$\sum_{x \in A} u(x) = 0 \quad (50)$$

where A is the set of vertices in one simplex.

For a 1-stage gasket, the eigenvalue equation (48) is given by

$$(\mu + p)u(x) = \sum_{x' \sim x} u(x') \quad (51)$$

where x' are the vertices that are adjacent to x .

From (50) and (51) we see that the spectrum of a 1-stage gasket consists of 0 with multiplicity 1 and $-p - 1$ with multiplicity p . The eigenstate belonging to 0 is $u = (p + 1)^{-1/2} \overbrace{(1, \dots, 1)}^{p+1}$

For a 2-stage gasket we can follow the same procedure as in the case of a 2-dimensional 2-stage gasket. We first define

$$c(1) = \sum_{i=1}^{p+1} u(i, i) \quad (52)$$

$$c(2) = \sum_{i=1}^{p+1} u(i, i + 1(\text{mod}(p + 1))) \quad (53)$$

...

$$c(j) = \sum_{i=1}^{p+1} u(i, i + j - 1(\text{mod}(p + 1))) \quad (54)$$

...

$$c(p + 1) = \sum_{i=1}^{p+1} u(i, i + n - 1(\text{mod}(p + 1))) \quad (55)$$

with $2 \leq j \leq p + 1$.

Combining (48), (49) and (52) up to (55) we get the following system of equations for even p

$$\begin{aligned} (\mu + p)c(1) &= c(2) + c(3) + \dots + c(p + 1) \\ (\mu + p + 1)c(2) &= c(1) + c(3) + \dots + 2c(p + 1) \\ &\dots \\ (\mu + p + 1)c(j) &= \sum_{i=1}^{p+1} c(i) + c(p - j + 3) - c(j) \\ &\dots \\ (\mu + p + 1)c(p + 1) &= c(1) + 2c(2) + \dots + c(p) \end{aligned}$$

and the following system for odd p

$$\begin{aligned}
(\mu + p)c(1) &= c(2) + c(3) + \dots + c(p+1) \\
(\mu + p + 1)c(2) &= c(1) + c(3) + \dots + 2c(p+1) \\
&\dots \\
(\mu + p + 1)c(k) &= \sum_{i=1}^{p+1} c(i) + c(p-k+3) - c(k) \\
&\dots \\
(\mu + p + 1)c\left(\frac{p+1}{2}\right) &= \sum_{i=1}^{p+1} c(i) - c\left(\frac{p+1}{2}\right) \\
&\dots \\
(\mu + p + 1)c(l) &= \sum_{i=1}^{p+1} c(i) + c(p-l+3) - c(l) \\
&\dots \\
(\mu + p + 1)c(p+1) &= c(1) + 2c(2) + \dots + c(p)
\end{aligned}$$

with $2 \leq j \leq p+1$, $2 \leq k < \frac{p+1}{2}$ and $\frac{p+1}{2} < l \leq p+1$.

This system of equations has non-trivial solutions if the determinant of the coefficients is equal to zero. Hence, for even p we get

$$\mu(\mu + p + 1)^{\frac{p}{2}}(\mu + p + 3)^{\frac{p}{2}} = 0 \quad (56)$$

and for odd p

$$\mu(\mu + p + 1)^{\frac{p+1}{2}}(\mu + p + 3)^{\frac{p-1}{2}} = 0 \quad (57)$$

The solutions of these equations are given by $\mu = 0$ with multiplicity 1, $\mu = -p - 1$ with multiplicity $\frac{p}{2}$ for even p and $\frac{p+1}{2}$ for odd p and $\mu = -p - 3$ with multiplicity $\frac{p}{2}$ for even p and $\frac{p-1}{2}$ for odd p.

From (48), (49) and (50) we get for $\mu = -p - 1$

$$\sum_{x' \in A} u(x') = 0 \quad (58)$$

$$u(x) = u(\bar{x}) \quad (59)$$

and for $\mu = -p - 3$

$$\sum_{x' \in A} u(x') = 0 \quad (60)$$

$$u(x) = -u(\bar{x}) \quad (61)$$

with A the set of vertices of one simplex. This is comparable with the equations we found for $\mu = 3$ and $\mu = 5$ for the 2-dimensional gasket.

In a $(n+1)$ -stage gasket we have $p+1$ n -stage gaskets. If these were not coupled, then we would have for the multiplicity of $\mu = -p - 1$ and $\mu = -p - 3$

$$g_{n+1}(-p - 1) = (p + 1)g_n(-p - 1) \quad (62)$$

$$g_{n+1}(-p - 3) = (p + 1)g_n(-p - 3). \quad (63)$$

But the gaskets are coupled. For $\mu = -p - 1$ the n -stage gaskets are coupled through $u(x) = u(\bar{x})$. The multiplicity is equal to the minimum number of variables that is needed to specify a vector that obeys (58) and (59). Every coupling determines 1 variable. The $p + 1$ n -stage gaskets are coupled through $\sum_{i=1}^p i = \frac{(p+1)p}{2}$ couplings. Hence we get for the multiplicity of $\mu = -p - 1$

$$g_{n+1}(-p - 1) = (p + 1)g_n(-p - 1) - \frac{(p + 1)p}{2} \quad (64)$$

Combining this with the fact that $g_1(-p - 1) = p$ we get for the multiplicity of $\mu = -p - 1$

$$g_n(-p - 1) = \frac{(p - 1)(1 + p)^{n-1} + (1 + p)}{2} \quad (65)$$

For $\mu = -p - 3$ the n -gaskets are coupled through $u(x) = -u(\bar{x})$. From (48), (49) and (60) we conclude that $u(x) = 0$ when x is a corner vertex. For a n -stage gasket we therefore have

$$\sum_{i \in \{1, \dots, p+1\}} u(\overbrace{i, \dots, i}^n) = 0. \quad (66)$$

In a $(n+1)$ -stage gasket, such a n -stage gasket will satisfy

$$\sum_{i \in \{1, \dots, p+1\}} u(m, \overbrace{i, \dots, i}^n) = 0.$$

Together with the coupling between the n -gaskets, we see that the coupling is determined by $\frac{p(p-1)}{2}$ variables. Hence we get the following equation for the multiplicity

$$g_{n+1}(-p - 3) = (p + 1)g_n(-p - 3) + \frac{p(p - 1)}{2}. \quad (67)$$

Combining this with the fact that $g_1(-p - 3) = 0$ we get for the multiplicity of $\mu = -p - 3$

$$g_n(-p - 3) = (p - 1) \frac{(1 + p)^{n-1} - 1}{2} \quad (68)$$

As was the case with the 2-dimensional gaskets, the spectrum of n-stage gaskets with $n \geq 3$ can be found in the following way. Look at a pyramid in a $(n - 1)$ -stage gasket. We take a vertex there with label (x_1, \dots, x_{n-1}) and call it x . The other vertices in the same pyramid are called $x', x' \in (A - \{x\})$. The neighbouring vertex in another neighbouring pyramid is \bar{x}

Define

$$u_n(x, i) = u_n(x_1, \dots, x_{n-1}, i) \quad (69)$$

for $i \in 1, \dots, p + 1$ and

$$c_n(x) = \sum_{i=1}^{p+1} (u_n(x, i)). \quad (70)$$

This is the sum of the vector components on the vertices in the same pyramid.

Assume none of the (x, i) , with $i \in \{1, \dots, p + 1\}$, is a corner vertex. Then (48), (49) and (70) gives

$$\mu(\mu + p + 3)c_n(x) = \mu(\mu + p + 3) \sum_{i=1}^{p+1} (u_n(x, i)) \quad (71)$$

$$= -(p + 1)c_n(x) + \sum_{(A-\{x\})} c_n(x') + c_n(\bar{x}) \quad (72)$$

Assume one of the (x, i) is a corner vertex. We choose $(x, 1)$ but the same holds for (x, i) , $i \in \{2, \dots, p + 1\}$. Then (48), (49) and (70) gives

$$\mu(\mu + p + 3)c_n(x) = \mu(\mu + p + 3) \sum_{i=1}^{p+1} (u_n(x, i)) \quad (73)$$

$$= -pc_n(x) + \sum_{(A-\{x\})} c_n(x') \quad (74)$$

Set $\nu = \mu(\mu + p + 3)$. Then the previous equations are transformed to

$$\nu u_n(x) = -(p + 1)u_n(x) + \sum_{(A-\{x\})} u_n(x') + u_n(\bar{x})$$

when none of the (x, i) are corner vertices and

$$\nu u_n(x) = -pu_n(x) + \sum_{(A-\{x\})} u_n(x')$$

when one of the (x, i) is a corner vertex. Then we see from (71) and (73) that it is possible to transform the eigenvalue equation of a n-stage gasket to

the eigenvalue equation of a (n-1)-stage gasket in this way, with eigenvalue ν instead of μ and eigenvector components $c_n(x)$ instead of $u_n(x)$.

The spectrum of a n-stage gasket consists of the spectrum of a (n-1)-stage gasket and new eigenvalues corresponding to the n-stage gasket, obtained by the relation

$$\mu_{n-1} = \mu_n(\mu_n + p + 3)$$

where μ_{n-1} is an eigenvalue of a (n-1)-stage gasket and μ_n a new eigenvalue of the n-stage gasket.

The inverse of this equation is given by

$$\mu_n = \frac{-(p+3) \pm \sqrt{(p+3)^2 + 4\mu_{n-1}}}{2} \quad (75)$$

This is a recursion equation. We define $\mu_n = \mu_n(\sigma_1, \dots, \sigma_n)$ with $\sigma_i, i \in \{1, \dots, n-1\}$ and $\sigma_i \in \{+, -\}$. Then the sign of σ_i is defined by the sign that is taken in the following equation

$$\mu_{i+1}(\sigma_1, \dots, \sigma_i) = \frac{-(p+3) \pm \sqrt{(p+3)^2 + 4\mu_i(\sigma_1, \dots, \sigma_{i-1})}}{2} \quad (76)$$

We define $\mu_i = \chi_i$ if $\mu_2 = -p - 1$ and $\mu_i = \eta_i$ if $\mu_2 = -p - 3$. Then combining (65), (71) and (73) gives for the multiplicity of $\chi_i(\sigma_1, \dots, \sigma_{i-1})$ in a n-gasket the following formula

$$g_n(\chi_i(\sigma_1, \dots, \sigma_{i-1})) = \frac{(p-1)(1+p)^{n-i-1} + (1+p)}{2} \quad (77)$$

And combining (68), (71) and (73) gives for the multiplicity of $\eta_i(\sigma_1, \dots, \sigma_{i-1})$ in a n-gasket the following formula

$$g_n(\eta_i(\sigma_1, \dots, \sigma_{i-1})) = (p-1) \frac{(1+p)^{n-i-1} - 1}{2} \quad (78)$$

There are 2^i possible η_i 's and 2^i possible χ_i 's, each possible eigenvalue is defined by the vector $(\sigma_1, \dots, \sigma_i)$.

The total number of eigenvalues is given by

$$\begin{aligned}
& \sum_{i=0}^{n-1} 2^i g_n(\chi_i(\sigma_1, \dots, \sigma_{i-1})) + \sum_{i=0}^{n-2} 2^i g_n(\eta_i(\sigma_1, \dots, \sigma_{i-1})) + 1 \\
&= \sum_{i=0}^{n-1} 2^i \left(\frac{(p-1)(1+p)^{n-i-1} + (1+p)}{2} \right) \\
&+ \sum_{i=0}^{n-2} 2^i \left((p-1) \frac{(1+p)^{n-i-1} - 1}{2} \right) + 1 \\
&= 2^{n-1} \times p + \sum_{i=0}^{n-2} 2^i (p-1)(1+p)^{n-i-1} + \sum_{i=0}^{n-2} 2^i + 1 \\
&= p \times 2^{n-1} + (p-1)(p+1)^{n-1} \frac{1 - (\frac{2}{1+p})^{n-1}}{1 - \frac{2}{1+p}} + \frac{1 - 2^{n-1}}{1-2} + 1 \\
&= p \times 2^{n-1} + (1+p)^n - (1+p) \times 2^{n-1} - 1 + 2^{n-1} + 1 \\
&= (1+p)^n
\end{aligned}$$

as is expected.

Hence, for a p-dimensional n-gasket the spectrum consists of

μ	$g_n(\mu)$
n=1	
0	1
-(p+1)	p
n=2	
0	1
-(p+1)	$\frac{p(1+p)}{2}$
-(p+3)	$\frac{(p-1)p}{2}$
$\frac{-(p+3) + \sqrt{(p+3)^2 + 4(p+1)}}{2}$	p
$\frac{-(p+3) - \sqrt{(p+3)^2 + 4(p+1)}}{2}$	p
n ≥ 3	
0	1
-(p+1)	$\frac{(p-1)(1+p)^{n-1} + (1+p)}{2}$
-(p+3)	$(p-1) \frac{(1+p)^{n-1} - 1}{2}$
$\chi_i(\sigma_1, \dots, \sigma_i)$	$\frac{(p-1)(1+p)^{n-i-1} + (1+p)}{2}$
$\eta_i(\sigma_1, \dots, \sigma_i)$	$(p-1) \frac{(1+p)^{n-i-1} - 1}{2}$
$\frac{-(p+3) + \sqrt{(p+3)^2 - 4\chi_i(\sigma_1, \dots, \sigma_i)}}{2}$	p
$\frac{-(p+3) - \sqrt{(p+3)^2 - 4\chi_i(\sigma_1, \dots, \sigma_i)}}{2}$	p

with $i \in \{1, \dots, n-1\}$ and $\sigma \in \{+, -\}$.

9.3 Results from the spectrum of a 2-dimensional regular dual Sierpinski gasket

We saw in chapter (9.1) that the spectrum of a n -stage regular dual Sierpinski gasket is given by the spectrum of a $(n - 1)$ -stage gasket and new eigenvalues corresponding to the n -stage gasket, obtained by the relation

$$\mu_n = \frac{-5 \pm \sqrt{25 + 4\mu_{n-1}}}{2}$$

where μ_{n-1} is an eigenvalue of the $(n - 1)$ -stage gasket and μ_n the new eigenvalues of the n -stage gasket.

Theorem 9.1 *For every $n \in \mathbb{N}$, every eigenvalue of the n -stage gasket is bounded by 0 and -5 . We have $-5 \leq \mu_n \leq 0$ for every μ_n in the spectrum of the n -stage gasket.*

Proof From theorem 6.5 we conclude that $\mu_n \leq 0$ for every μ_n in the spectrum of the n -stage gasket. We know that $\mu_1 \in 0, -3$. Assume that $0 \geq \mu_{i-1} \geq -5$ with $i \in \mathbb{N}$. Then $5 \geq \sqrt{25 + 4\mu_{i-1}} \geq 0$. Hence

$$0 \geq \frac{-5 \pm \sqrt{25 + 4\mu_{i-1}}}{2} \geq -5.$$

Hence $0 \geq \mu_i \geq -5$. This is true for every $i \in \mathbb{N}$. This proves that $0 \geq \mu_n \geq -5$ for every $n \in \mathbb{N}$. ■

This can be generalized to higher dimensions.

Corollary 9.2 *The eigenvalues of a p -dimensional regular dual Sierpinski gasket are bounded by 0 and $-(p + 3)$*

Proof From theorem 6.5 we conclude that $\mu_n \leq 0$ for every μ_n in the spectrum of the n -stage gasket. We know that $\mu_1 \in 0, -(p + 1)$. Assume that $0 \geq \mu_{i-1} \geq -(p + 3)$ with $i \in \mathbb{N}$. Then by (75) we conclude that $0 \geq \mu_i \geq -(p + 3)$. This is true for every i . Hence $0 \geq \mu_n \geq -(p + 3)$ for every $n \in \mathbb{N}$. ■

We can derive some geometrical information about the eigenstates. For $\mu = -3$ we have

$$u(x) + u(x') + u(x'') = 0 \tag{79}$$

$$u(x) = u(\bar{x}) \tag{80}$$

with x a label of an arbitrary vertex in a triangle. For $\mu = -5$ we have

$$u(x) + u(x') + u(x'') = 0 \tag{81}$$

$$u(x) = -u(\bar{x}) \tag{82}$$

and in a n -stage gasket $u(i, \dots, i) = 0, i \in \{1, 2, 3\}$ with (i, \dots, i) a corner vertex of the n -stage gasket.

Definition 9.3 (Edge-triangle cycle) *An edge-triangle cycle is a cycle that is built up from edges, which alternatively belong to a triangle and to a coupling between triangles.*

See figure 22 for an example of an edge-triangle cycle.

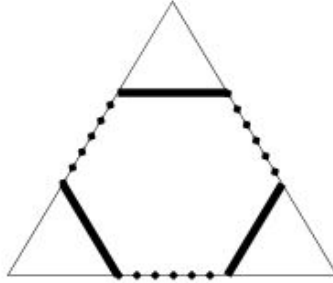


Figure 22: An edge-triangle cycle. The fat edges are the edges that belong to a triangle and the dotted edges are edges that belong to a coupling between triangles.

Proposition 9.4 *Every eigenstate that belongs to eigenvalue $\mu = -5$ is cyclic in the sense that the vertices with non-zero eigenstate-values form an edge-triangle cycle.*

Proof A 1-stage gasket does not have an eigenvalue $\mu = -5$

By equations (30) and (32) eigenstates that belong to eigenvalue $\mu = -5$ have zero eigenstate-values at the corner vertices. This means that every vertex with a non-zero eigenvalue has two neighbouring vertices in the same triangle and one neighbouring vertex in a neighbouring triangle. Assume that the eigenstate is not cyclic. Then there are vertices with non-zero eigenstate-values where only one of the neighbouring vertices has a non-zero eigenstate-value.

This can not be true because equation (81) states that at least one of the neighbouring vertices in the same triangle has a non-zero eigenstate-value and equation (82) states that the neighbouring vertex that is in another triangle has a non-zero eigenstate value. Hence, there are no vertices with exactly one neighbouring vertex with non-zero eigenstate-value.

Hence, every eigenstate is cyclic or built up of more than one cycle. If the eigenstate is built up of more than one cycle, it can be split up into independent edge-triangle cycles. Hence, every eigenstate belonging to eigenvalue $\mu = -5$ is cyclic. ■

For the 1-stage gasket we have one independent edge-triangle cycle. For the 2-stage gasket we have four independent edge-triangle cycles. The multiplicity of $\mu = -5$ counts the number of edge-triangle cycles.

We have the following generalization of the edge-triangle cycles.

Definition 9.5 (Edge- i -gasket cycle) *An edge- i -gasket cycle is a cycle which alternatively consist of a i -stage gasket and a coupling edge between the i -stage gaskets.*

When we take $i = 1$ we get an edge-triangle cycle back. See figure 23 for an example of a edge-2-gasket cycle

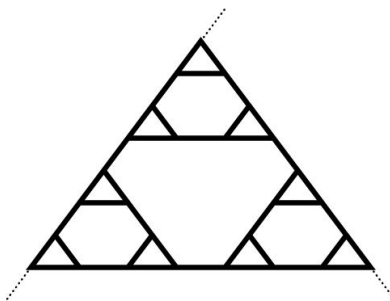


Figure 23: An example of a edge-2-gasket cycle. The dotted edges do not belong to the cycle but are coupling edges to the rest of the graph. The gaskets are connected to each other by coupling edges and form a cycle.

We saw that in chapter (9.1) that we can transform the system of eigenvalue equations of a i -stage gasket to the system of eigenvalue equations of a $(i - 1)$ -stage gasket. By this transformation rule, $\eta_1(\sigma_1)$ is transformed to -5 and 2-gaskets are transformed to triangles. Because of this we see that the eigenstates belonging to eigenvalue $\mu = \eta_1(\sigma_1)$ are edge-2-gasket cycles. In general the eigenstates belonging to eigenvalue $\mu = \eta_i(\sigma_1, \dots, \sigma_i), i \in \mathbb{N}$ are edge- $(i + 1)$ -gasket cycles.

9.4 Comparison of the spectral properties of the dual Sierpinski gasket, the one-dimensional lattice graph and the two-dimensional lattice graph

We will compare the spectral properties of the dual graph of a stacked sphere configuration and the regular lattice graph in this subsection.

There are spectral similarities between the regular dual Sierpinski gasket and the lattice graph. We saw in section (7) that if we know the spectrum of a one-dimensional lattice graph with $2^i k$ vertices and periodic boundary conditions we can calculate the spectrum of a one-dimensional lattice graph with $2^{i+1} k$ vertices and periodic boundary conditions by the transformation rule

$$\mu_i = \mu_{i+1}(\mu_{i+1} + 4) \quad (83)$$

with μ_i an eigenvalue of the lattice graph with $2^i k$ vertices and μ_{i+1} a corresponding eigenvalue of the lattice graph with $2^{i+1} k$ vertices.

We also saw in section (7) that if we know the spectrum of a two-dimensional lattice of size $2^i k_1 \times 2^i k_2$ and with periodic boundary conditions we can calculate the spectrum of a two-dimensional lattice of size $2^{i+1} k_1 \times 2^{i+1} k_2$ and with periodic boundary conditions by the transformation rule

$$\mu_i = \mu_{i+1}(\mu_{i+1} - 4\theta_i) + 2(\theta_i + 1)(2\theta_i - 1), \quad (84)$$

$$\theta_i = 2\theta_{i+1}(\theta_{i+1} + 4) + 5. \quad (85)$$

with μ_i an eigenvalue of the lattice graph of size $2^i k_1 \times 2^i k_2$, μ_{i+1} an eigenvalue of the lattice graph of size $2^{i+1} k_1 \times 2^{i+1} k_2$ and

$$\theta_i = \cos\left(\frac{2\pi n_2}{2^i k_2}\right) - 2, \quad n_2 = 0, 1, \dots, 2^i k_2 - 1.$$

In a similar way we found a transformation rule for transforming the spectrum of a i -stage dual gasket to the spectrum of a $(i + 1)$ -stage dual gasket. This rule is given by

$$\mu_i = \mu_{i+1}(\mu_{i+1} + 5) \quad (86)$$

with μ_i an eigenvalue of the i -stage dual gasket and μ_{i+1} an eigenvalue of the $(i + 1)$ -stage dual gasket.

The number of vertices triples when we go from a regular i -stage dual gasket to a regular $(i + 1)$ -stage dual gasket. Because equation (86) is only a quadratic polynomial, we get some extra eigenvalues that cannot be calculated from the transformation rule (86) directly. As we saw in chapter (9.1) these other eigenvalues can be derived by looking at the geometrical properties of the gasket itself.

In contrast with the dual Sierpinski gaskets, the number of vertices in a one-dimensional lattice merely doubles when we go from a one-dimensional lattice with $2^i k$ vertices to a one-dimensional lattice with $2^{i+1} k$ vertices. The transformation rule (83) is a quadratic polynomial, hence every eigenvalue can be calculated from the transformation rule (83) directly.

The number of vertices in a two-dimensional lattice quadruples when we go from a two-dimensional lattice of size $2^i k_1 \times 2^i k_2$ to a two-dimensional lattice of size $2^{i+1} k_1 \times 2^{i+1} k_2$. The transformation rule (84) is a quadratic polynomial, but the transformation rule for θ_i (equation 85) is also quadratic. Therefore, each eigenvalue μ_i of the two-dimensional lattice of size $2^i k_1 \times 2^i k_2$ is related to four eigenvalues μ_{i+1} of the two-dimensional lattice of size $2^{i+1} k_1 \times 2^{i+1} k_2$ and every eigenvalue can be calculated from the transformation rules directly.

The difference between the spectra also becomes apparent when we plot the multiplicity of the eigenvalue μ against eigenvalue μ itself. For the regular dual Sierpinski gasket, we observe qualitatively that the plot is self-similar. By self-similar we mean that after enlarging a part of the plot, it looks the same as the complete plot. See figure 24 for a plot of a 6-stage gasket.

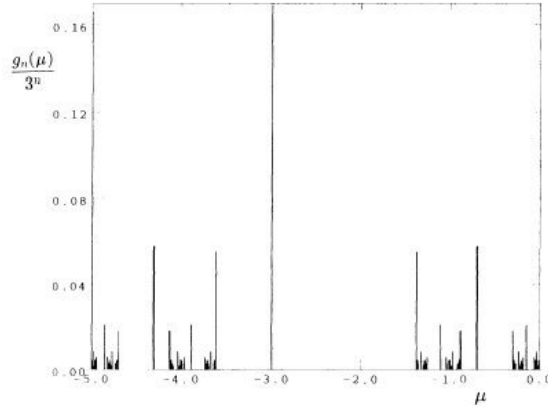


Figure 24: A plot of a 6-stage gasket. Here the multiplicity of eigenvalue μ is plotted against μ . The multiplicity is normalized against the total number of vertices. The plot is from [4].

The self-similarity of this system becomes apparent when we look at the system of eigenvalue equations. We saw in subsection 9.1 that we could transform the system of eigenvalue equations of a i -stage gasket to the system of eigenvalue equations of a $i + 1$ -stage gasket. Hence, we can transform the system of eigenvalue equations of a part of the system to the system of eigenvalue equations of the whole system.

When we look at the multiplicities of the eigenvalues we see that

$$\frac{g_n(\chi_i)}{g_n(\chi_{i+1})} = \frac{3^{n-i-2} + 1}{3^{n-i-3} + 1} \sim 3, n \gg 1 \quad (87)$$

$$\frac{g_n(\eta_i)}{g_n(\eta_{i+1})} = \frac{3^{n-i-1} - 1}{3^{n-i-2} - 1} \sim 3, n \gg 1 \quad (88)$$

with χ_i and η_i the eigenvalues of the dual Sierpinski gasket, defined by equation (43) in section 9.1. The multiplicities scale with a factor equal to 3 for large n .

The self-similarity becomes more apparent when we plot the cumulative multiplicity of the eigenvalues μ against μ (see figure 25 for a plot of a 6-stage gasket).

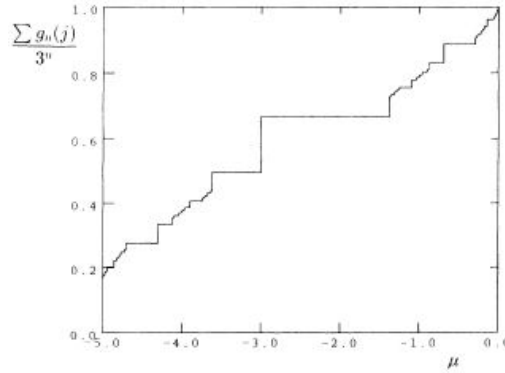


Figure 25: A plot of a 6-stage gasket. Here the cumulative multiplicity of eigenvalues μ is plotted against μ . This multiplicity is normalized against the total number of vertices. The plot is from [4].

For the two-dimensional lattice graph the plot of the spectrum looks very different (see figure 26 for a plot of the multiplicity of the eigenvalues μ against μ of the regular lattice graph with $k_1 = 4$, $k_2 = 4$ and $i = 3$).

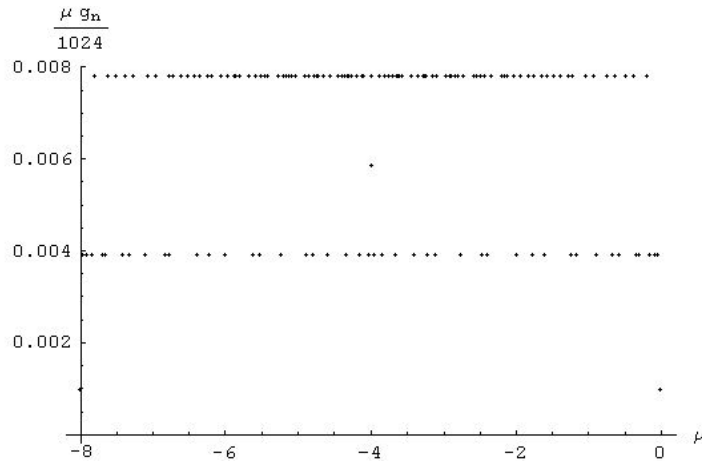


Figure 26: A plot of a two-dimensional lattice graph with periodic boundary conditions. The total number of vertices of this two-dimensional lattice graph is equal to 1024. The multiplicity of eigenvalues μ is plotted against μ . This multiplicity is normalized against the total number of vertices.

There is a big difference between those spectral plots. We already saw that not every eigenvalue of the dual Sierpinski gasket can be calculated by its transformation rule directly, because at every stage the number of vertices is tripled. These eigenvalues can be calculated by looking at the properties of the eigenstates of the dual Sierpinski gasket. At every stage we get more eigenstates belonging to eigenvalues $\mu = -3$ and $\mu = -5$. We see in equations (87) and (88) that the multiplicity of these eigenvalues grows with a scale factor equal to 3 every time when we go from a i -stage gasket to a $(i + 1)$ -stage gasket.

In the case of the lattice graph every eigenvalue can be calculated by the transformation rule. Combining this with the fact that the multiplicities of the eigenvalues are equal to two or four, except in the case of $\mu = 0$ and $\mu = -4$, we see that the spectral plot is built up by two constant lines.

10 The spectrum of a regular dual Sierpinski gasket and its application to the spectra of the dual graph of triangular stacked sphere configurations

A triangular stacked sphere configuration is a special class of stacked sphere configurations whose dual graph consists of triangles that are connected to each other by coupling edges. See figure 27 for an example. In this section we will derive some results on the spectra of those configurations. We will see that we can relate the spectrum of the dual graph of a triangular stacked sphere configuration to the spectrum of the so-called consolidated graph of this dual graph. The consolidated graph of the dual graph of a triangular stacked sphere configuration is again a dual graph of a stacked sphere configuration. We will also see that we can use the transformation rule that we found for dual Sierpinski gaskets on general stacked sphere configurations. And we can get information about the number of cycles and triangles of the dual graph from the spectrum. The problem is that the consolidated graph of the dual graph is a random structure and it seems that there is no analytical method to completely relate the spectrum of this consolidated graph with graphs whose spectrum is already known.

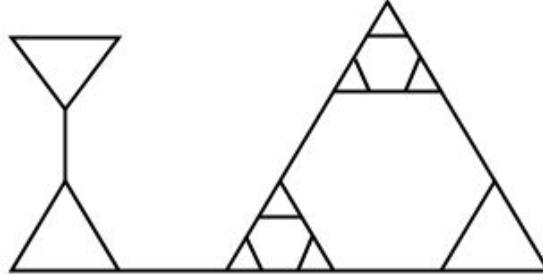


Figure 27: *An example of a triangular stacked sphere configuration. We see that it consists of triangles that are connected to each other by coupling edges*

From chapter (9.3) we conclude that the multiplicity of $\mu = -5$ in a dual Sierpinski gasket is equal to the number of edge-triangle cycles. We also saw that the multiplicity of $\mu = \eta_i(\sigma_1, \dots, \sigma_i), i \in \mathbb{N}$ in a dual Sierpinski gasket is equal to the number of edge- i -gasket cycles.

For a triangular stacked sphere configuration we have the following system of eigenvalues

$$(\mu + 2)u(x) = u(x') + u(x''), \text{ if } x \text{ is a two-valent vertex} \quad (89)$$

$$(\mu + 3)u(x) = u(x') + u(x'') + u(\bar{x}), \text{ if } x \text{ is a three-valent vertex} \quad (90)$$

with the meaning of x', x'' and \bar{x} the same as in chapter (9.1)

If we insert $\mu = -3$ in equations (89) and (90) we get

$$u(x) + u(x') + u(x'') = 0 \quad (91)$$

$$u(x) = u(\bar{x}) \quad (92)$$

with x an arbitrary vertex.

If we insert $\mu = -5$ in equations (89) and (90) we get

$$u(x) + u(x') + u(x'') = 0 \quad (93)$$

$$u(x) = -u(\bar{x}) \quad (94)$$

with x an arbitrary vertex.

These systems of equations are exactly the same as in the case of a dual sierpinski gasket.

Theorem 10.1 *For triangular stacked sphere configurations the multiplicity of $\mu = -5$ is equal to the number of edge-triangle cycles.*

Proof This is proven in the same way as theorem (9.4) is proven. In this case it is possible that a triangle contains 2 corner vertices. The eigenstate-values of these corner vertices for eigenstates belonging to eigenvalue $\mu = -5$ are equal to zero. Because of equation (93) the eigenstate-value of the other vertex in the same triangle is equal to zero and because of equation (94) the eigenstate-value of the vertex that is neighbouring to this vertex is also equal to zero. Hence, for eigenstates belonging to eigenvalue $\mu = -5$, the vertices with non-zero eigenstate-values have two neighbouring vertices with non-zero eigenstate-values. ■

Proposition 10.2 *For a triangular stacked sphere configuration we have $g(-3) + g(-5) - 1 = T$, with $g(y)$ the multiplicity of eigenvalue y and T the number of triangles in the triangular stacked sphere configuration*

Proof If we look at the dual of a single subdivided simplex, the number of independent variables that is needed to specify a vector that obeys (91) and (92) is two. Hence the multiplicity of $\mu = -3$ is two. If we add another triangle

to this triangle by adding a coupling edge between these 2 triangles, we need to define an extra variable to specify a vector that obeys (91) and (92). If we add triangles in this way, by connecting them by coupling edges, we need to specify for every triangle an extra variable to specify a vector that obeys (91) and (92). Hence, if there were no edge-triangle cycles we would have that $g(-3) - 1 = T$, with T the number of triangles.

When the graph has one edge-triangle cycle, we need one variable less to specify a vector that obeys (91) and (92). When the graph has more edge-triangle cycles, every cycle decreases the number of variables that is need to specify a vector that obeys (91) and (92) by one. From theorem (9.4) we see that the number of edge-triangle cycles is given by the multiplicity of $\mu = -5$. Hence, we have $g(-3) + g(-5) - 1 = T$, with T the number of triangles. ■

By the transformation rule we also see that an eigenstate which belongs to eigenvalue $\eta_i(\sigma)$ contains an edge- i -gasket cycle.

Definition 10.3 (Blow up of a vertex) *By blowing up a vertex we mean that we change the vertex into a triangle. This is done by deleting the vertex from the graph and connecting the edges, that were connected to this vertex, to the triangle in such a way that at most one edge is connected to each vertex of the triangle.*

Definition 10.4 (Consolidation of a triangle) *By consolidating a triangle we mean that we change a triangle into a vertex. This is done by deleting the edges of the triangle and coinciding the vertices of the triangle into one vertex.*

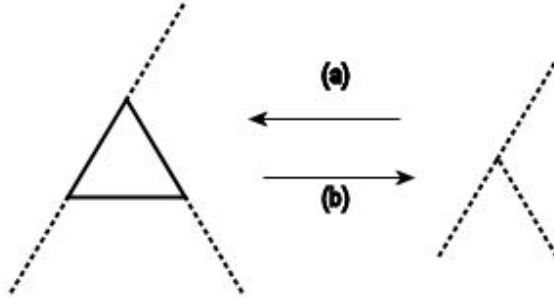


Figure 28: (a) Blow up of a 3-valent vertex into a triangle and (b) consolidation of a triangle into a 3-valent vertex. The dotted edges are coupling edges to the rest of the graph.

Theorem 10.5 *If we know the eigenvalues of the dual graph of a general stacked sphere configuration, we can calculate the eigenvalues of the graph that is obtained by blowing up every vertex of this dual graph. Assume that the eigenvalues of the dual graph are given by μ , then the eigenvalues of the blown up graph are given by $0, -3, -5$ and those eigenvalues ν such that $\mu = \nu(\nu + 5)$.*

Proof Let $\mu = \nu(\nu + 5)$. Let u be an eigenvector of the blown up graph of the dual graph and v an eigenvector of the dual graph. Let $c(x) = u(x, 1) + u(x, 2) + u(x, 3)$ be the sum of the eigenvector components of the vertices in a triangle (call this triangle A) in the blown up graph. These vertices are labelled by $(x, 1), (x, 2)$ and $(x, 3)$. Let the label of the vertex that we get when we consolidate this triangle be given by x . Let the eigenvector component of this vertex be given by $v(x)$. We use the same notation for the labels of the neighbouring vertices as in chapter (9.1). Then the eigenvalue equations of the blow up graph can be transformed into the eigenvalue equations of the dual graph.

Lets assume that triangle A is connected by 3 coupling edges. We then know from chapter (9.1) that $\nu(\nu + 5)c(x) = -3c(x) + c(x') + c(x'') + c(\bar{x})$ can be transformed into $\mu v(x) = -3v(x) + v(x') + v(x'') + v(\bar{x})$.

Lets assume that triangle A is connected by 2 coupling edges. We then know from chapter (9.1) that $\nu(\nu + 5)c(x) = -2c(x) + c(x') + c(x'')$ can be transformed into $\mu v(x) = -2v(x) + v(x') + v(x'')$.

Now look at a triangle that is connected to another triangle. Let the sum of the eigenvector components of this triangle be given by $c(x) = u(x, 1) + u(x, 2) + u(x, 3)$ with $(x, 1), (x, 2)$ and $(x, 3)$ the labels of the three vertices. Let $c(x')$ be the sum of the eigenvector components of the neighbouring triangle. Then $\nu(\nu + 5)c_n(x) = -c_n(x) + c_n(x')$. This can be transformed in the same way into $\mu v(x) = -v(x) + v(x')$.

Therefore, we see that the we can transform the eigenvalue equations of the blown up graph into the eigenvalue equations of the dual graph. This is done by setting $\mu = \nu(\nu + 5)$ and $c(x) = v(x)$, with μ the eigenvalues of the dual graph and ν the eigenvalues of the blown up graph. ■

Hence, if we know the eigenvalues of the dual graph, we also know the eigenvalues of the blown up graph. By this theorem we also know the eigenvalues of the graph which is obtained by blowing the graph up an arbitrary times.

Let the set of eigenvalues of the dual graph of a triangular stacked sphere configuration that one wants to blow up be S_{old} and let the set of the eigenvalues of the blown up graph be S_{new} . Let $\delta \in S_{old}$ be an arbitrary eigenvalue unequal to 0. Let $\theta(\sigma) \in S_{new}$ be the eigenvalue such that $\delta = \theta(\theta + 5)$. Then

$$\theta(\sigma) = \frac{-5 \pm \sqrt{25 - 4\delta}}{2}$$

with $\sigma \in \{+, -\}$, depending on which sign is taken in the equation. Let $g_{old}(y)$ be the multiplicity of eigenvalue y for the graph that one wants to blow up and let $g_{new}(y)$ be the multiplicity of eigenvalue y for the blown up graph. Then, from theorem (10.5) we get

$$g_{new}(\theta(+)) = g_{new}(\theta(-)) = g_{old}(\delta).$$

We also have

$$g_{new}(0) = g_{old}(0).$$

The blow up of a triangle gives one edge-triangle cycle. From proposition (10.2) we get that the number of triangles is equal to $g(-3) + g(-5) - 1$. The edge-triangle cycles in the graph that one wants to blow up, still remain edge-triangle cycles in the blown up graph. Hence

$$\begin{aligned} g_{new}(-5) &= g_{old}(-5) + T_{old} \\ &= 2g_{old}(-5) + g_{old}(-3) - 1 \end{aligned}$$

with T_{old} the number of triangles in the graph that one wants to blow up.

For triangular stacked sphere configurations we also have

$$\begin{aligned} g_{new}(-3) &= T_{new} + 1 - g_{new}(-5) \\ &= 3T_{old} + 1 - 2g_{old}(-5) + g_{old}(-3) - 1 \\ &= 3(g_{old}(-5) + g_{old}(-3) - 1) + 1 - 2g_{old}(-5) - g_{old}(-3) + 1 \\ &= g_{old}(-5) + 2g_{old}(-3) - 1 \end{aligned}$$

with T_{new} the number of triangles in the blown up graph and $T_{new} = 3T_{old}$.

We have expressed the multiplicities of the eigenvalues of the blown up graph in the multiplicities of the eigenvalues of the graph that one wants to blow up.

The consolidated graph of a dual graph of a triangular stacked sphere configuration is in general not completely build up by triangles which are connected to each other by coupling edges. We can calculate the spectrum of the consolidated graph in the following way.

Let the eigenvalues of the dual graph of a triangular stacked sphere configuration that are unequal to 0 be θ . For those $\theta(\sigma), \sigma \in \{+, -\}$ we have a δ in the spectrum of the consolidated graph such that $\delta = \theta(\theta + 5)$ and $g_{new}(\theta(+)) = g_{new}(\theta(-)) = g_{old}(\delta)$. Here $g_{new}(y)$ is the multiplicity of eigenvalue y in the non-consolidated graph and $g_{old}(y)$ is the multiplicity of eigenvalue y in the consolidated graph. The definition of $\theta(+)$ and $\theta(-)$ is the same as before.

Also, we have

$$g_{old}(0) = g_{new}(0).$$

The problem now is that we want to know the eigenvalues δ of the consolidated graph of the dual graph of the triangular stacked sphere configurations. If we know these eigenvalues we know the complete spectrum of the dual graph of a triangular stacked sphere configuration. The problem with the consolidated graph of the dual graph is that this graph is a random structure and it is not well known whether we can relate the spectrum of this consolidated graph with a graph with a well known spectrum. Hence we do not know whether the spectra of these consolidated graphs can be calculated in an analytical way.

We found out that we can relate the spectra of the dual graphs of triangular stacked sphere configurations with the spectra of the consolidated graphs of these dual graphs. We can also relate the spectrum of the dual graph of a general stacked sphere configuration with the spectrum of the blown up graph of this dual graph. Furthermore, we found information about the number of cycles and triangles in the dual graphs of these stacked sphere configurations. To know the complete spectrum of the dual graph of a triangular stacked sphere configuration, we need to know the spectrum of the consolidated graph of the dual graph. These consolidated graphs are random structures and it is not well known whether we can calculate the spectra of these graphs in an analytical way.

11 Geometrical properties of general stacked sphere configurations

We now look at the geometrical properties of the dual graphs of general stacked sphere configurations. We will see that we can derive some information about the number of cycles and triangles in those stacked sphere configurations.

Just as in the case of the dual graph of a triangular stacked sphere configuration, an eigenstate of which the eigenstate components form an edge-triangle cycle has an eigenvalue $\mu = -5$. In general an eigenstate of which the eigenstate components form an edge- i -gasket cycle has an eigenvalue $\mu = \eta_i(\sigma)$.

Proposition 11.1 *The eigenvalues δ of the dual graph of a stacked sphere configuration are bounded by*

$$-6.25 \leq \delta \leq 0$$

Proof We know by theorem (6.5) that $\delta \leq 0$. The spectrum of the blown up graph of the dual graph can be calculated from the spectrum of the dual graph by equation

$$\theta(\sigma) = \frac{-5 \pm \sqrt{25 - 4\delta}}{2}$$

with δ an eigenvalue of the dual graph and $\theta(\sigma)$, $\sigma \in \{+, -\}$ an eigenvalue of the blown up graph of the dual graph. If $25 - 4\delta < 0$, then θ becomes imaginary. But the Laplacian is symmetric, hence the eigenvalues have to be imaginary. So we have a contradiction here. Therefore $25 - 4\delta \geq 0$ for every δ in the spectrum of the dual graph. Hence $\delta \geq -6.25$. ■

A path of triangles consists of triangles which are connected to each other by coupling edges and form a path. Let I be the set of triangles. Let this set be partitioned by $\{1, \dots, n\}$. Then a triangle belongs to $I(j)$, $j \in \{1, \dots, n\}$ if there is a path of triangles between this triangle and all the other triangles in $I(j)$. See figure 29 for an example.

If we look at all the triangles in $I(j)$, with $j \in \{1, \dots, n\}$ arbitrary, then we have the following property

Proposition 11.2 *For all the triangles in $I(j)$, with $j \in \{1, \dots, n\}$ arbitrary, we have that*

$$G(-3) + G(-5) - 1 + C(j) = T(j)$$

with $C(j)$ the number of coupling edges with which these triangles are connected to the rest of the graph, $T(j)$ the number of triangles in $I(j)$ and $G(y)$ is the number of eigenstates that belong to an eigenvalue y with non-zero eigenstate values in $I(j)$ and zero eigenstate values outside $I(j)$.

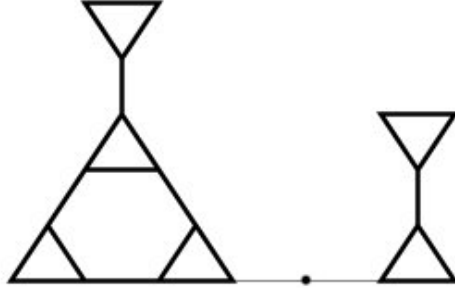


Figure 29: An example of a dual graph of a general stacked sphere configuration. The fat edges form two sets of the partitioned set of triangles. The triangles in the left set are connected to each other by paths of triangles, just like the triangles in the right set. But the triangles in the left set are not connected to the triangles in the right set, because of the two edges between those sets.

Proof We saw in proposition (10.2) that for a graph which consists of T triangles that are connected to each other by coupling edges, we had the following equation

$$g(-3) + g(-5) - 1 = T.$$

If the subgraph built up by the triangles in $I(j)$ was not connected by coupling edges to the rest of the graph, we would have an equivalent equation

$$G(-3) + G(-5) - 1 = T(j).$$

Every coupling edge decreases the number of variables that is needed to specify a vector that obeys 91 and 92 by one. Therefore the equation is changed to

$$G(-3) + G(-5) - 1 + C(j) = T(j).$$

If $T(j) - C(j) + 1 \leq 0$ then $G(-3) = 0$ and $G(-5) = 0$. ■

For the total graph we have

$$G(-3) + G(-5) - 1 + C = T$$

with $C = \sum_{i=0}^n C(i)$ and $T = \sum_{i=0}^n T(i)$.

We have found in this section that we can derive information about the number of triangles and edge-triangle cycles in the general stacked sphere configurations. Moreover, we found a relation between the multiplicity of the $\mu = -3$ eigenstates and the $\mu = -5$ eigenstates. The problem with these general stacked sphere configurations is that they have a random structure. It is difficult to know a priori whether the spectrum of the dual graph of such a general stacked sphere configuration can be related to a graph whose spectrum is known. Therefore, the solution for the complete spectrum of the dual graph of a general stacked sphere configuration remains an unknown quantity.

12 Discussion

We are able to calculate the spectrum of the dual Sierpinski gaskets, which is a small subclass of the dual graphs of stacked sphere configurations. The eigenvalues of the eigenstates of the dual Sierpinski gaskets are given by -3 , -5 , ξ_i and η_i (see section 9.1, equations (44) and (45)). The eigenstates of the n -stage gasket belonging to these eigenvalues are localized, except for the eigenstates belonging to eigenvalues ξ_{n-1} and η_{n-1} . Therefore we can pinpoint some eigenstates of the dual graphs of general stacked sphere configurations with eigenvalues equal to -3 , -5 , ξ_i and η_i .

We are also able to calculate the spectrum of the dual graph of triangular stacked sphere configurations from the consolidated graph of the dual graph of the triangular stacked sphere configuration.

The problem is that general stacked sphere configurations as those appearing in quantum gravity have a random structure. General branched polymers are examples of dual graphs of stacked sphere configurations. Those are random structures and little seems to be known about whether the spectrum of the particular class of branched polymers can be calculated analytically. The dual graphs of the stacked sphere configurations are built up by regular and irregular dual Sierpinski gaskets which are connected with each other by coupling edges between their corner vertices.

One idea we pursued to get a handle on the spectrum was to look for a local transformation move that describes the spectrum of the graph in terms of the spectrum of the same graph, but with one vertex blown up or one triangle consolidated. The problem is that such a transformation rule depends on the complete graph and is therefore different for every stacked sphere configuration, rendering this kind of local recursion rule not particularly useful.

Other people have also worked on random fractal graphs. Many of those people come close to the problem we want to solve, but it seems that none of them could solve the spectra of the dual graphs of the random stacked sphere configurations in an analytical way.

J. Jordan [12] describes the spectrum of random fractal graphs and gives some numerical results on irregular dual Sierpinski gaskets. He does not give an analytical description of the spectrum of irregular fractal graphs.

R. Strichartz [16] proves that the spectrum of random fractals of non-dual Sierpinski gaskets can be related to a so-called cell-graph. The problem is that if we want to know the complete spectrum of the random fractal of non-dual Sierpinski gaskets, we also need to know the spectrum of the cell-graph itself.

The dual Sierpinski gaskets have some spectral similarity with lattice graphs,

as we saw in chapter 9.4. It would be interesting to try to exploit this fact somewhat more by looking at non-homogeneous lattice graphs and how their spectrum is carried over to the spectrum of Sierpinski gaskets.

This thesis provides a starting point for probing the geometry of space-time at small length scales. We managed to completely calculate the spectra of the dual graphs of a small subclass of the stacked sphere configurations, the homogeneously subdivided simplices. From these spectra we got some interesting results on the spectra of general stacked sphere configurations. We are also able to calculate the number of cycles and triangles in these general stacked sphere configurations. It is our hope that these preliminary investigations lead to further research about the space-time structure at small length-scales which will help us to better understand the intricate structure of space-time in the near future.

References

- [1] J. Ambjørn, M. Carfora, and A. Marzuoli. *The Geometry of Dynamical Triangulations*. Springer Verlag, 1997.
- [2] J. Ambjørn, J. Jurkiewicz, and R. Loll. Reconstructing the universe. *Phys.Rev. D*, 72:064014, 2005.
- [3] J. Ambjørn, J. Jurkiewicz, and R. Loll. The universe from scratch. *Contemp.Phys.*, 47:103, 2006.
- [4] M.G. Cosenza and R. Kapral. Coupled maps on fractal lattices. *Phys. Rev. A.*, 46:1850, 1992.
- [5] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Math. J.*, 23:298, 1973.
- [6] G. Gionti. The geometry of the elongated phase in 4-d simplicial quantum gravity. *arXiv:hep-lat/9711018*, 1997.
- [7] G. Gionti. *Discrete Approaches Towards the Definition of a Quantum Theory of Gravity*. PhD thesis, 1998.
- [8] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer Verlag, 2001.
- [9] J.v.d. Heuvel and S. Pejic. Using laplacian eigenvalues and eigenvectors in the analysis of frequency assignment problems. *Annals of Operations Research*, 107:349, 2001.
- [10] C.J. Isham. Prima facie questions in quantum gravity. *arXiv:gr-qc/9310031*, 1993.
- [11] C.J. Isham. Structural issues in quantum gravity. *arXiv:gr-qc/9510063*, 1995.
- [12] J. Jordan. *Renormalisation of random hierarchical systems*. PhD thesis, 2003.
- [13] E. Kiritsis. Introduction to superstring theory. *arXiv:hep-th/9709062*, 1997.
- [14] W.A. Miller. The hilbert action in regge calculus. *Class. Quant. Grav.*, 14, 1997.
- [15] T. Regge. General relativity without coordinates. *Nuovo Cimento*, 19:558, 1961.
- [16] R. Strichartz. Fractafolds based on the sierpinski gasket and their spectra. *Trans. Am. Math. Soc.*, 355:4019, 2003.
- [17] W.-J. Tzeng and F.Y. Wu. Spanning trees on hypercubic lattices and non-orientable surfaces. *Applied Mathematics Letters*, 13:19, 2000.