Non-perturbative effects in type IIA string theory

Hugo Looyestijn

Under supervision of dr. S. Vandoren.
Institute for theoretical physics, Utrecht.
Abstract

We consider membrane instanton corrections to the universal hypermultiplet, which arise on compactification of type IIA string theory on a Calabi-Yau. Supersymmetry requires the geometry of this multiplet to be a four-dimensional quaternionic-Kähler manifold. If we only consider one of the two membrane instantons and five-brane instantons are absent, this metric has two commuting isometries. We use two descriptions of four-dimensional quaternionic-Kähler spaces with two commuting isometries based on the Toda and the Calderbank-Pedersen equations, and we show how solutions of one equation can be transformed to the other equation. This is used in the analysis of earlier constructed solutions and the Eisenstein series. We finally investigate the scalar potential and show how this potential can obtain meta-stable de Sitter vacua.
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1 Introduction

In the 17th century, Newton laid a basic foundation for our physical understanding of the world. Many aspects of our universe could be described and explained by Newton’s work. When Maxwell formulated his equations to describe electric and magnetic fields, it seemed that the main part of physics was completed. It lead Lord Kelvin in 1900, in an address to the British Association for the Advancement of Science, to say: “There is nothing new to be discovered in physics now. All that remains is more and more precise measurement”. The new physics which appeared after that statement have been huge, and we are all but sure what the fundamental physical theory is. There are two major improvements to classical mechanics found, but they are difficult to combine, and they therefore might not be the final answer.

Starting in 1905, the theory of general relativity was formulated, first by Einstein. The theory of general relativity describes the effects of gravity with a high precision. Amongst the effects are non-intuitive ideas such as Lorentz contraction (objects appear smaller if one sees them while travelling with high velocity) and time dilatation (clocks run slower if one travels with high velocity). Although counter-intuitive, they have been shown correct in experiment, and modern-day GPS satellites use them to make their signals more precise. Interpretation of general relativity states that space and time should be combined together; instead of a world with 3 spacial dimensions and one time dimensions, we live in a world with 4 space-time dimensions.

On the other hand, at the end of 1900, by the famous solution of Planck for the problem now known as the ultra-violet catastrophe, quantum mechanics was started. As a theory describing effects on small scales and of individual particles, it has had a major impact on our understanding of the world. Many experimentally known facts of chemistry, such as the periodic table, could now be put on firm theoretical basis. The development of modern technology, such as lasers and computer chips comes from quantum theory. The quantum theory that describes the known particles in our world, the Standard Model, is the most successful theory ever: some theoretical predictions and experimental verifications match up to an astonishing 8 digits.

There is however a striking problem: there is no theory that successful combines both theories. General relativity is not a quantum theory of gravitation. The underlying principles of both theories are also quite different: general relativity is based on geometrical concepts, quantum theory has no clear geometrical interpretation. Furthermore, in situations where gravitation has huge effects, for instance very close to a black hole or at the Big Bang, we have no theory to work with. The search for a theory of everything is still on.

One of the candidates for this theory of everything is string theory. This theory uses strings as building blocks, instead of particles. Avoiding the use of point-like particles, string theory escapes many of the problems that appear when one tries to combine general relativity with quantum mechanics. There has been much analysis on string theory, but it is not the theory of everything – yet.

One intriguing phenomenon is that string theory, as opposed to the theories above, requires a certain number of dimensions. There is nothing in general
relativity or quantum mechanics that does require the number of dimensions. The presently known string theories use 10 or 11 dimensions. Because our everyday experiments tell us that we live in 4 space-time dimensions, we need to get rid of some dimensions. The proposed solution is to make some dimensions very small, by winding them around a circle, so they are not observable anymore. An often used example is a garden hose. This is a three-dimensional object, but if one sees it from a distance, it looks one-dimensional. If we perform further measurements, we might probe the structure of the garden hose by sending particles at it. If we only use balls that are slightly smaller than the diameter of the ball, and send them through the hose, our experiments would suggest that it really is one-dimensional. It would take much smaller particles to observe the inner structure of the hose.

This process of removing dimensions is called compactification. The exact procedure of compactifications is difficult. We know how to do the main part of this compactification, which is (slightly confusing) known as the classical part. In certain cases, we also know how the small fluctuations around this main part look like. These fluctuations are called perturbative effects. In the small part of string theory considered in this thesis, we know what all the perturbative effects are. Even this information does not provide us with the total recipe for compactification, because there is information which cannot be reached by considering fluctuations. This can be illustrated with the fact that knowing the Taylor expansion of a function around a certain point does not describes the behavior of the total function.

The effects that cannot be described by fluctuations are called non-perturbative effects. In this thesis, we consider instanton effects. Instantons are solutions to the Euclidean theory with finite action. The Euclidean theory can be obtained by changing the time variable \( t \) from the Minkowski theory to \(-i\tau\), where \( \tau \) is the Euclidean time variable. Instantons can be used in quantum field theory to study tunnelling effects. In string theory, they can drastically change the physical theory. One of the possible applications is the cosmological constant problem. Present observational data suggests that this famous constant is small, but non-zero. Inclusion of instanton effects might provide for cosmological models with a small cosmological constant.

Because considering fluctuations around the classical solution does not help us describing the non-perturbative effects, we need to find other methods of describing those effects. To continue the analogy with the Taylor expansion: although we do not know the entire function, we might know some things about the function: it is perhaps monotonically increasing, or it is periodic. We can then look at all monotonically increasing functions, and try to determine which of these functions have the right properties.

In this thesis we follow a similar procedure. After the compactification from 10 to 6 dimension, we end up with a theory in four dimensions. The theory now contains many states which are very heavy, so we only look at the low-energy part of the theory. We then have a four-dimensional, low-energy effective action. This is a supersymmetric theory which contains gravity, and because it is obtained by compactification of type IIA string theory, it is called type IIA supergravity. Supersymmetry is a symmetry of many modern physical theories, and it interchanges bosonic (integer spin) and fermionic (half-integer spin)
particles. There is no physical evidence of supersymmetry yet, but it might be found when the Large Hadron Collider is ready, in 2007. The theory we consider has $N = 2$ supersymmetry.

Besides gravity, there are other fields. In this thesis, we focus on a small subset of these fields, called the universal hypermultiplet. This hypermultiplet contains the dilaton, and hence receives $g_s$ corrections. The bosonic part of the universal hypermultiplet contains four real scalars. These four fields form a non-linear sigma model, and can therefore be described by the geometry of a four-dimensional space. The main goal of this thesis is therefore a determination of this four-dimensional space, because we then know how the universal hypermultiplet works.

There are properties of this four-dimensional space which allow us to classify them. It turns out that supersymmetry requires the four-dimensional space to be a very special space, called a quaternionic-Kähler space. Another fortunate thing is that these spaces have been classified by mathematicians.

We study two of those classifications, which are called the Przanowski-Tod and the Calderbank-Pedersen metrics. They describe the metric in terms of a partial differential equation. When these four-dimensional spaces admit the same symmetries, both metrics describe the same space, and so there must be a relation between those spaces. In this thesis we find the exact relation between both classifications.

Using this relation, we can study some solutions to the differential equation in both classifications, and look at the relations between those solutions. One of the studied solutions is a solution to the Przanowski-Tod equation, found by Davidse et al. Another solution, to the Calderbank-Pedersen metric, is the Eisenstein series. This series has been used before in non-perturbative string theory, so it is a very interesting solution. We will look at properties of those solutions, and how they might be interesting for cosmological models. Hopefully, by considering these solutions, we get to know more about the problem of compactification.

In section 2, we review the mathematical tools that are needed for this work. In section 3 we describe the physical status quo on this subject. After a short introduction to instantons in quantum field theories, we look at some general features of string theory. In section 4 and 5 we show the new physics that has emerged from this thesis. Section 4 describes the relations between the Toda and Calderbank-Pedersen metrics. Section 5 uses this new relation to study more properties of the geometry of the universal hypermultiplet. Finally, in section 6, the conclusions and suggestions for further research are formulated.
2 Mathematics

This section contains a review of the mathematical tools needed in this thesis. We will have a look at manifolds, geometry, connections, complex manifolds, Kähler manifolds, homology, cohomology and Calabi-Yau manifolds. These are the basic ingredients of string theory. We continue with more specialistic topics such as the symplectic groups, hyper- and quaternionic-Kähler manifolds, Weyl tensors and Einstein manifolds. Finally we discuss important classes of metrics for the quaternionic-Kähler manifolds, the Przanowski-Tod and Calderbank-Pedersen metrics.

2.1 Manifolds

We start with the general definition of a manifold.

Definition 1. A topological \( m \)-manifold is a Hausdorff space, such that each point has an environment which is homeomorphic to an open subset of \( \mathbb{R}^m \).

To get to the definition of a smooth manifold, we need the concept of charts. A chart can tell you how to read the manifold in question.

Definition 2. Let \( M \) be a topological \( m \)-manifold. A chart is a pair \( (U, \kappa) \), such that \( U \) is open in \( M \) and \( \kappa \) is a homeomorphism from \( U \) onto an open subset of \( \mathbb{R}^m \). If \( (U, \kappa) \) and \( (U', \kappa') \) are two different charts, the homeomorphism \( \kappa' \kappa^{-1} : \kappa(U \cap U') \to \kappa'(U \cap U') \), between open subsets of \( \mathbb{R}^m \), is called a transition map. A set of charts such that their domains cover \( M \) is called a atlas.

Until now, we did not speak about differentiability. A topological \( m \)-manifold does not have the tools, for instance the structure of a vector space, to speak about differentiability on the manifold. We therefore turn our attention to \( \mathbb{R}^m \), because we know what differentiability is on \( \mathbb{R}^m \). The adjective smooth will always mean \( C^\infty \) differentiable.

Definition 3. Let \( M \) be a topological \( m \)-manifold. A smooth atlas \( \mathcal{A} \) for \( M \) is an atlas \( \mathcal{A} \) such that all possible transition maps are smooth.

If \( \kappa \) and \( \lambda \) are charts from a smooth atlas, then both \( \kappa \lambda^{-1} \) and \( \lambda \kappa^{-1} \) are smooth, so the transition maps are diffeomorphisms.

Suppose \( M \) is a topological \( m \)-manifold, and \( \mathcal{A} \) an atlas for \( M \). Even though we can read each part of \( M \) with a chart, it is still possible to add additional charts to \( \mathcal{A} \). A caveat is that the new transition maps are not diffeomorphisms. The next lemma shows that this is the only obstruction:

Lemma 4. Let \( \mathcal{A} \) be a smooth atlas for \( M \). Let \( \hat{\mathcal{A}} \) be the collection of charts, such that all the possible transition maps between charts from \( \mathcal{A} \) and \( \hat{\mathcal{A}} \) are diffeomorphisms. Then \( \hat{\mathcal{A}} \) is an atlas which contains \( \mathcal{A} \). The atlas \( \hat{\mathcal{A}} \) is called a maximal atlas for \( M \): each smooth atlas that contains \( \mathcal{A} \) is contained in \( \hat{\mathcal{A}} \).

Definition 5. A smooth manifold is a pair \( (M, \mathcal{A}) \), where \( M \) is a topological \( m \)-manifold and \( \mathcal{A} \) is a maximal smooth atlas for \( M \). As usually in mathematics,
we usually denote the pair \((M, \mathcal{A})\) just with \(M\) and say \(M\) is a smooth manifold.

\[\square\]

From lemma 4 we see that any smooth atlas can be used to make a topological \(m\)-manifold \(M\) into a smooth manifold.

**Convention 6.** From now on, all manifolds are considered smooth, paracompact, Hausdorff and usually connected, unless stated otherwise.

**Definition 7.** Let \(M\) be a manifold of dimension \(n\). A pseudo-Riemannian metric of signature \((p, q)\) is a 2-form \(g\) on \(M\) such that \(g_x\) is non-degenerate on each point \(x \in M\), and has signature \((p, q)\). The pair \((M, g)\) is called a **pseudo-Riemannian manifold**. In the special where \(q = 0\) we call the metric a **Riemannian metric**, and the pair \((M, g)\) an **Riemannian manifold**.

In the remainder of this section, \((M, g)\) is a \(m\)-dimensional Riemannian manifold with a metric connection \(\nabla\).

### 2.1.1 Fiber bundles

We now turn our attention to fiber bundles, and in particular to the **orthonormal frame bundle** \(\text{OF}(M, g)\) of a Riemannian manifold \((M, g)\) of dimension \(4m = n\). Let \(x \in M\). The fiber above \(x\) is the set

\[\text{OF}_x = \{(x, f)| f a g_x\text{ orthonormal basis in }T_xM\}\]

and the projection \(\pi\) is just projection on the first component. This defines a fiber bundle, as can be easily checked. If \(f\) is an element of the fiber above \(x\), we also call \(f\) a **frame** above \(x\), hence the name frame bundle. We can also think about \(f = (f_1, f_2, \ldots, f_n)\) as a linear mapping \(R^n \to T_xM\), which assigns to the standard basis element \(e_i\) of \(R^n\) the element \(f_i\) of \(T_xM\).

What is special about this bundle, is that there exist a natural action of the orthonormal group \(O(n)\) on each fiber. We see an element \(C\) of \(O(n)\) as a map \(R \to R\), and element \(f\) of \(\text{OF}_x\) as a map \(R^n \to T_xM\). We can compose these to form the action \(\phi\) of \(O(n)\) on \(\text{OF}_x\):

\[\phi(C): f \to f \circ C\]

The action of this group is transitive and free. This which means that, after choosing a fixed element \(f_0 \in \text{OF}_x\), we can obtain a different frame by applying precisely one orthonormal transformation. So, after choosing \(f_0\), we have a diffeomorphism of \(O(n)\) with \(\text{OF}_x\).

### 2.1.2 Holonomy

**Definition 8.** Let \(p \in M\) and consider a closed loop \(\gamma\) at \(p\). If we parallel transport a tangent vector \(X \in T_p M\) around \(\gamma\), we obtain a new tangent vector \(X'\) in \(T_p M\). In general, this new tangent vector is different from the original one. Each curve \(\gamma\) induces a lineair transformation \(P_\gamma: T_p M \to T_p M\). The set of all these transformations is easily seen to be a group and is called the **holonomy group** of \(M\) at \(p\), and is denoted \(\text{Hol}(p)\).  

\[\square\]
If $M$ is arcwise-connected, the holonomy groups $\text{Hol}(p)$ and $\text{Hol}(q)$ are isomorphic, for each $p, q \in M$ so we can speak about the holonomy group of $M$.

The holonomy group at $x \in M$ is a subgroup of $O(T_x M, g_x)$. We can use the isomorphism above to be able to see the holonomy group as a subgroup of $O(\mathbb{R}^m)$. If $M$ is orientable, the holonomy group is even a subgroup of $SO(m)$.

### 2.1.3 Isometries and Killing vectors

**Definition 9.** A diffeomorphism $f : M \rightarrow M$ is called an **isometry** if it preserves the Riemannian metric:

$$f^* g_{f(p)} = g_p, \quad \forall p \in M. \quad (1)$$

The set of all isometries form a group, and are in fact a Lie group with a smooth action on $M$.

If $x$ and $y$ are the local coordinates of $p$ and $f(p)$ respectively, we can write equation (1) in local coordinates as

$$\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha \beta}(f(p)) = g_{\mu \nu}(p).$$

**Definition 10.** A vector field $X$ on $M$ is called a **Killing vector field** if the (local) one-parameter group of diffeomorphisms generated by $X$ consists of (local) isometries.

**Proposition 11.** For a vector field $X$, the following are equivalent:

- a. $X$ is a Killing vector field
- b. The Lie derivative of $g$ by $X$ vanishes, i.e. $L_X g = 0$.

This follows immediately by recalling that the Lie derivative in the direction $X$ can be written as

$$L_X g = \frac{d}{dt} (\phi^t_\ast g)|_{t=0}.$$ 

In local coordinates, the condition $L_X g = 0$ can be written in the following equivalent forms:

$$X^\xi \partial_\xi g_{\mu \nu} + (\partial_\mu X^\xi) g_{\xi \nu} + (\partial_\nu X^\xi) g_{\xi \mu} = 0,$$

$$(\nabla_\mu X)_\nu + (\nabla_\nu X)_\mu = \partial_\mu X_\nu + \partial_\nu X_\mu - 2\Gamma^\lambda_{\mu \nu} X_\lambda = 0.$$

### 2.2 Complex analysis

Before we turn from the real manifolds to the complex manifolds, we first introduce some concepts from complex analysis.

**Definition 12.** Let $f$ be a function from $\mathbb{C}^n$ to $\mathbb{C}$. The function $f$ is called **analytic** in $p \in \mathbb{C}^n$ if there is an open subset $U$, such that $p \in U$ and the
function $f$ can be represented by a power series, uniform converging on $U$. The function $f$ is called **holomorphic** if it is complex differentiable.

We usually identify $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, by writing $z = x + iy$. A complex function $f$ is usually written in terms of real functions $u, v$ as $f(x + iy) = u(x + iy) + iv(x, y)$.

**Theorem 13.** Let $f$ be a function from $\mathbb{C}^n$ to $\mathbb{C}$, and let $p \in \mathbb{C}^n$. The following three statements are equivalent.

a. $f$ is analytic in $p$.

b. $f$ is holomorphic in $p$.

c. If we decompose $f$ into $u, v$ then $u$ and $v$ satisfy (componentwise) the **Cauchy-Riemann equations** at $p$:

$$
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
$$

### 2.2.1 Complex manifolds

The generalization to complex manifolds is now straightforward. We replace $\mathbb{R}^n$ by $\mathbb{C}^n$ and ‘smooth’ by ‘analytic’ in the definitions for charts, atlases and manifolds and we find:

**Definition 14.** Let $M$ be a topological $m$-manifold. A complex chart is a map from $M$ onto an open subset of $\mathbb{C}^n$. A complex analytic atlas $A$ is an atlas for $M$, such that all transition maps are analytic. A analytic complex manifold is the pair $(M, A)$ or simply just $M$.

From now on, all complex manifolds are analytic, unless stated otherwise. If $\kappa$ is a complex chart, then we can view $\kappa$ as a real chart by identifying $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. We then find that each complex manifold is a smooth real manifold. The converse is not true: there exist smooth real manifolds that cannot be given a complex atlas.

On a complex manifold, we can consider the complexified tangent space $T_pX^\mathbb{C}$. The space $T_pX^\mathbb{C}$ is essentially the same as $T_pX$, but we now allow for complex coefficients when making combinations of tangent vectors. Suppose we have a complex manifold of real dimension $2d = n$, then we have the following basis of $T_pX$:

$$
T_pX : \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}
$$

For the complexified tangent space, we now take particular linear combinations to make a basis for $T_pX^\mathbb{C}$:

$$
\left\{ \frac{\partial}{\partial z^1} + i \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial z^d} + i \frac{\partial}{\partial \bar{z}^d}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d}, i \frac{\partial}{\partial x^1} + \frac{\partial}{\partial \bar{z}^1}, \ldots, -i \frac{\partial}{\partial x^d} - \frac{\partial}{\partial \bar{z}^d} \right\}.
$$

This is usually written in terms of complex coordinates

$$
T_pX^\mathbb{C} : \left\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^d}, \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^d} \right\}.
$$
We can now introduce a complex structure \( J \) on this tangent space, which multiplies each tangent vector with a factor \( i \), and hence \( J^2 = -1 \).

In general, a linear map \( X \) on a tangent space is called an **almost complex structure** if \( X^2 = -1 \). The question is now: does this map \( X \) come from a complex manifold (so can the tangent space be made the tangent space of a complex manifold, and the map \( X \) is just the map \( J \))? This is not always the case; when it is the case the almost complex structure is said to be integrable.

The almost-complex structure \( J \) allows a decomposition of each tangent space, corresponding to the eigenvalues under \( J_x \):

\[
T_x M^\mathbb{C} = T_x M^+ \bigoplus T_x M^-.
\]

Each real tangent vector \( X \in T_x M \) can accordingly be decomposed as

\[
X = U + \overline{U}, \quad \text{where} \quad U = \frac{1}{2}(X - iJ X), \quad \overline{U} = \frac{1}{2}(X + iJ X).
\]

This provides an isomorphism between \( T_x M \) and \( T_x M^+ \), which is \( \mathbb{C} \)-linear.

**Definition 15.** Let \( M \) be a complex manifold. A **Hermitian metric** on \( M \) is a Riemannian metric \( g \) such that

\[
g(J_x X, J_x Y) = g(X, Y), \quad \forall X, Y \in T_x M, \forall x \in M.
\]

Equivalently, we call the metric \( g \) \( J \)-invariant.

We find a **Hermitian scalar product** \( h \) on each \( T_x M^\mathbb{C} \) by defining

\[
h(U, V) = g(U, \overline{V}), \quad U, V \in T_x M^\mathbb{C}, x \in M.
\]

If we restrict \( h \) to the space \( T_x M^+ \), we can find, using the decomposition (2) a Hermitian scalar product \( H \) on \( T_x M \). We find

\[
H(X, Y) = \frac{1}{2} \left( g(X, Y) - ig(JX, Y) \right).
\]

**Definition 16.** Let \( M \) be a complex manifold with Riemannian metric \( g \). The **Kähler form** \( \omega \) of a Hermitian metric \( h \) is the 2-form defined by

\[
\omega(X, Y) = g(JX, Y).
\]

From (3) we see that the Kähler form is, apart from a factor \(-\frac{1}{2}\), equal to the imaginary part of the Hermitian scalar product \( H \). The real part of \( H \) corresponds to the original Hermitian metric \( g \).

**2.2.2 Kähler manifolds**

**Definition 17.** A **Kähler manifold** is a Hermitian manifold \((M, g)\) whose Kähler form \( \omega \) is closed: \( d\omega = 0 \), and the metric \( g \) is called the **Kähler metric** of \( M \).
Definition 18. A compact Kähler manifold with vanishing first Chern class is called a Calabi-Yau manifold.

A Calabi-Yau manifold can also be characterized as a Kähler manifold which admit a Ricci flat metric. This was conjectured by Calabi and proven by Yau, hence the name Calabi-Yau manifold.

2.3 Homology and cohomology

Given a $k$-form $\omega \in \Omega^k$, we can form a differential form of degree $k+1$, in the following way: if $\omega = \sum f_i \, dx^i$, we define

$$d\omega = \sum_{j=1}^n \frac{\partial f_i}{\partial x^j} \, dx^j \wedge dx^i,$$

and for a general $k$-form we extend this by linearity. This operation is called exterior derivation. The exterior derivative therefore gives a mapping

$$d : \Omega^p(M) \to \Omega^{p+1}(M),$$

with the property that $d^2 = 0$, due to the asymmetry of the wedge product.

If $\omega$ is a $p$-form, we call $\omega$ closed if $d\omega = 0$, and $\omega$ is called exact if there is a $(p-1)$-form $f$ such that $df = \omega$. We see that every exact form is closed, but the converse needs not to be true. We define the de Rham cohomology group of order $p$ to be the quotient:

$$H^p_{dR} = \frac{\text{closed } p\text{-forms}}{\text{exact } p\text{-forms}} = \frac{\ker d_p}{\text{img } d_{p-1}}.$$

Dual to the concept of cohomology is the homology. An standard $p$-simplex is the convex set $\Delta_p \subset \mathbb{R}^p$ generated by the $p+1$ points

$$P_0 = (0, \ldots, 0), P_1 = (1, 0, \ldots, 0), P_p = (0, \ldots, 0, 1).$$

We write $\Delta_p = (P_0, P_1, \ldots, P_p)$. A singular $p$-simplex is a mapping $\Delta_p \to M$ of a simplex into a manifold. We define a chain as a formal sum of simplices, with coefficients in $\mathbb{R}$. Adding two chains by adding the formal sums give rise to a group, the (singular) $p$-chain group.

The boundary $\partial \Delta_p$ of $\Delta_p$ is the $(p-1)$-chain

$$\partial \Delta_p := \sum_k (-1)^k (P_0, \ldots, \hat{P}_k, \ldots, P_p),$$

where the hat denotes omission of that particular simplex. This already looks a lot like the definition of the exterior derivative. Again we have $\partial^2 = 0$. We define a $p$-cycle as a $p$-chain $z_p$ such that $\partial z_p = 0$. The vector subspace of all cycles is denoted $Z_p$. A $p$-boundary $\beta_p$ is a $p$-chain which is a boundary of some $(p+1)$ chain, i.e. $\beta_p = \partial z_{p+1}$. The vector subspace of all boundaries is denoted...
Because $B_p \subset Z_p$, we can make the quotient group, which is called the (singular) homology group:

$$H_p := \frac{Z_p}{B_p}$$

The Betti numbers are defined as the dimensions of the homology groups, so

$$b_i := \dim H_p.$$ 

We define the cohomology groups as the duals of the homology groups:

$$H^p := H_p^*.$$ 

The notation already suggests

**Theorem 19** (De Rham’s theorem). *If $M$ is a compact, oriented, smooth manifold, then the groups $H^p$ are isomorphic to the groups $H^p_{dR}$.***

This also implies that the Betti numbers are the dimensions of the cohomology groups: $b_p = \dim H^p_{dR}$. So the Betti numbers measure the maximal number of independent closed $p$-forms on $M$, which do not have a combination which is exact.

On a complex manifold, we can refine the de Rham cohomology by the use of the Dolbeault operators. We now declare a $(p,q)$ form $\omega$ to be $\bar{\partial}$ closed if $\bar{\partial}\omega = 0$, and exact if there is a $(p,q-1)$ form $f$ such that $\bar{\partial}f = \omega$. The $(p,q)$ Dolbeault cohomology groups is now

$$H^{r,s}_{\text{Dol}} = \frac{\ker \bar{\partial}_p}{\im \bar{\partial}_{p-1}},$$

and the Hodge number $h^{p,q}$ is defined as the dimension of $H^{r,s}_{\text{Dol}}$. It also follows that

$$b_k = \sum_{p+q=k} h^{p,q}.$$ 

### 2.3.1 Poincaré duality

Another famous result relates the cohomology and homology groups. If $M$ is compact and oriented, we have the isomorphism

$$\left( H^k \right)^* = H_k \simeq H^{n-k}.$$ 

Applying this twice shows $\left( H^k \right)^{**} \simeq H^k$, which shows that each homology group has a finite dimension. As the dimension of the dual group is the same as the dimension of the group, we find that the Betti numbers satisfy

$$b_k = b_{n-k}.$$ 

It can also be shown that the Hodge numbers satisfy

$$h^{p,q} = h^{n-p,n-q}.$$
2.3.2 Calabi-Yau manifolds

The theorems of the previous sections all hold for Calabi-Yau manifolds, so we have a lot of relations between Hodge numbers. Due to Kählerity, we have \( h^{p,q} = h^{q,p} \), and due to Poincaré we have \( h^{p,q} = h^{n-p,n-q} \).

Furthermore, we also have \( h^{0,0} = 1 \) and \( h^{1,0} = 0 \) on a Calabi-Yau. Hence, the Hodge-diamond of a Calabi-Yau has a major reduction:

\[
\begin{array}{cccc}
  h^{3,0} & h^{2,0} & h^{1,0} & 1 \\
  h^{3,1} & h^{2,1} & h^{1,1} & h^{0,1} \\
  h^{3,2} & h^{2,2} & h^{1,2} & h^{0,2} \\
  h^{3,3} & h^{2,3} & h^{1,3} & h^{0,3} \\
\end{array}
\]

Notice that \( h^{1,1} \geq 1 \), because the Kähler form itself is a non-zero, closed \((1,1)\) form, and thus determines a non-zero element in \( H^{1,1}_{\text{Dol}} \).

An important quantity will be the number of 3-cycles. From the diagram above, we can read off that this equals

\[
b_3 = 1 + h^{1,2} + h^{1,2} + 1 = 2(1 + h^{1,2}).
\]

2.4 Quaternions and the Symplectic groups

In this section we introduce the symplectic groups, and discuss some of their properties.

We can embed \( \mathbb{C} \) into \( \mathbb{R}^2 \) by sending \( z = x + iy \in \mathbb{C} \) to the vector \((x, y)^t\). Because this mapping is a linear isomorphism, we obtain an isomorphism \( \mathbb{C} \simeq \mathbb{R}^2 \), and so in general \( \mathbb{R}^{2n} \simeq \mathbb{C}^n \).

Likewise, we can send an element \( z = a + bi + cj + dk \in \mathbb{H} \) to the element \((a + bi, c + di)^t\) in \( \mathbb{C}^2 \). This mapping is also an linear isomorphism, and hence we obtain the identifications

\[
\mathbb{H}^n \simeq \mathbb{C}^{2n} \simeq \mathbb{R}^{4n}.
\]

Furthermore, if we equip \( \mathbb{H}^n, \mathbb{C}^{2n}, \mathbb{R}^{4n} \) with their default norms, these mappings preserve the norms. Using these mappings, we can view an element of \( \text{GL}(n, \mathbb{H}) \) as an element of \( \text{GL}(2n, \mathbb{C}) \) or \( \text{GL}(4n, \mathbb{R}) \).

**Definition 20.** The symplectic group \( \text{Sp}(n) \) is the subgroup of \( \text{GL}(n, \mathbb{H}) \) which preserve the standard hermitian form on \( \mathbb{H}^n \):

\[
\langle x, y \rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n.
\]

So \( \text{Sp}(n) \) is in fact just \( \text{U}(n, \mathbb{H}) \). Using the observations above, we can also view \( \text{Sp}(n) \) as a subgroup of \( \text{GL}(2n, \mathbb{C}), \text{SU}(2n) \) or as a subgroup of \( \text{GL}(4n, \mathbb{R}) \).
can also characterize the group Sp($m$) as a subgroup of $O(4m)$ by
\[ \text{Sp}(m) = \{ A \in O(4m) \mid A(Ix) = IA(x), A(Jx) = JA(x) \} \]
where $I$ and $J$ are the mappings in $\mathbb{R}^{4m}$ corresponding to multiplication by $i$ and $j$ in $\mathbb{H}^n$. Notice that an $A \in \text{Sp}(m)$ also satisfies $A(Kx) = KA(x)$.

Let $M$ be a real manifold of dimension $n = 4m$, and let $x \in M$. Each tangent space $T_x M$ is isomorphic to $\mathbb{H}^n$, so also isomorphic to $\mathbb{R}^n$. If $e_i, 1 \leq j \leq m$, is an orthonormal $\mathbb{H}$-frame in $T_x M$, we have an orthonormal $\mathbb{R}$-frame
\[ e_1, I_x(e_1), J_x(e_1), K_x(e_1), e_2, I_x(e_2), \ldots, J_x(e_m), K_x(e_m), \]
which provide for $4m = n$ independent real vectors. This mapping from a quaternionic to a real basis provides a fiber bundle embedding
\[ \text{Sp}(m) \bigg\downarrow \bigg\downarrow O(n) \]
where the structure groups are written next to each bundle.

### 2.4.1 Hyperkähler and quaternionic-Kähler manifolds

**Definition 21.** A Riemannian manifold $(M, g)$ is called hyperkählerian if its holonomy group is contained in $\text{Sp}(n)$.

**Proposition 22.** A Riemannian manifold $M(M, g)$ is hyperkählerian if and only if there exists on $M$ two complex structures $I$ and $J$ such that $IJ = -JI$.

**Proof.** First we show that the containment of the holonomy group in $\text{Sp}(n)$ allows us to define global complex structures $I$ and $J$, that obey $IJ = -JI$. It is clear that we can define those locally, for instance in the domain of a chart. So let $x_0 \in M$, and define structures $I_{x_0}$ and $J_{x_0}$. Let $x \in M$ be arbitrary. Let $\gamma$ be a path connecting $x_0$ and $x$. We then have the induced mapping
\[ P_\gamma : T_{x_0} M \to T_x M, \]
and we can define $I_x = P_\gamma I_{x_0} P_{\gamma}^{-1}$ and likewise for $J$. This is well defined: suppose $\gamma'$ is another path connecting $x_0$ and $x$. Then $I_x = I_x'$ precisely when $I_{x_0}$ is invariant under the holonomy of $\gamma'^{-1} \circ \gamma$. This is a closed loop at $x_0$, so we find that the existence of the global structures is implied by the requirement that $\text{Hol}(M) \subset \text{Sp}(n)$.

We now have the global structures $I, J$ and $K := IJ$, and we will show that they are covariantly constant. We look at the subbundle $\mathbf{H}F$ of $O(n)$. Let $x \in M$ and suppose $e$ is a frame at $x$. This frame is of the form
\[ e = (e_1, Ie_1, Je_1, Ke_1, e_2, \ldots, Ke_n). \]
If we apply an element of the holonomy group, we get a new frame, $e'$, of the form

$$e' = (e_1', (Ie_1)', (Je_1)', (Ke_1)', e_2', \cdots, (Ke_n')).$$

This is precisely an element of $HF$ if it is of the form

$$e' = (e_1', I(e_1'), J(e_1'), K(e_1'), e_2', \cdots, K(e_n')).$$

As the frame $e$ was arbitrary, each element of $T_xM$ with length 1 can be the element $e_1$. So we find that we should have

$$\text{Hol}(I(v)) = I(\text{Hol}(v)),$$

and likewise for $J$ and $K$, for all $v$: we already established this for unit length, but the equations are linear. This again is equivalent to

$$\text{Hol} \circ I \circ \text{Hol}^{-1} = I,$$

and likewise for $J$ and $K$, which mean $I, J$ and $K$ are invariant under parallel transport. Finally, this is equivalent to the statement that $I, J$ and $K$ are covariantly constant. □

**Definition 23.** A $4n$-dimensional $4n$–dimensional Riemannian manifold, with $n > 1$ is called **quaternion-Kähler** if its holonomy group is contained in $Sp(n) \cdot Sp(1)$. The dot is the direct product of the two groups modulo $\mathbb{Z}/2$.

If $n = 1$, then the definition would give us that quaternion-Kähler manifolds are simply the oriented Riemannian manifolds, since $Sp(1) \cdot Sp(1) = SO(4)$. This is not the case that interest us, so we look for an alternative way of characterizing these manifolds. Besse [1], for instance, calls 4-dimensional oriented Riemannian manifolds quaternion-Kähler, but we will find an inequivalent definition for 4-dimensional quaternion-Kähler manifolds.

**Proposition 24.** Let $n \geq 2$. A Riemannian manifold $(M, g)$ is quaternion-Kähler if and only if there exists a covering of $M$ by open sets $U_i$, and, for each $i$, two almost complex structures $I$ and $J$ on $U_i$ such that

a. $g$ is Hermitian for $I$ and $J$ on $U_i$,

b. $IJ = -JI$,

c. The Levi-Civita derivatives of $I$ and $J$ are linear combinations of $I$, $J$ and $K = IJ$,

d. for any $x$ in $U_i \cap U_j$, the vector space of endomorphisms of $T_xM$ generated by $I, J$ and $K$ is the same for $i$ and $j$.

**Proof.** See for instance Besse [1]. □

Properties b and d state the existence of a subbundle $S$ of rank 3, which is locally spanned by a complex of almost complex structures $H = (J_α)$, $J_α^2 = -1$ and $J_1J_2 = -J_2J_1 = J_3$. This is an almost quaternionic structure; by property c it is a quaternionic structure. Finally, property a turns this into a quaternionic-Kähler structure.
2.5 Self-dual Weyl tensor

To find the analogue of quaternionic-Kähler manifolds in dimension 4, we first need to look at the Weyl tensor. We denote the Ricci tensor as \( r \), and the scalar curvature, which is the trace of the Ricci tensor, as \( s \).

We first present a decomposition of the Riemann-tensor. This decomposition is induced by a decomposition of the bundle where the Riemann-tensor takes values, which is not irreducible. If we decompose this bundle into irreducible components, we find the so-called Ricci decomposition of the Riemann tensor. For an elegant, coordinate-free description we define

**Definition 25.** The Kulkarni-Nomizu product of two symmetric 2-tensors \( h \) and \( k \) is the 4-tensor

\[
(h \otimes k)(x,y,z,t) = h(x,y)k(y,t) + h(y,t)k(x,z) - h(x,t)k(y,z) - h(y,z)k(x,t)
\]

Then we have the following

\[
R = \frac{s}{2n(n-1)} g \otimes g - \frac{1}{n-2} z \otimes g + W
\]

where we have denoted \( r - \frac{s}{n} g \) by \( z \), which is just the traceless part of the Ricci tensor \( r \). The tensor \( W \) is called the Weyl tensor. So, in a sense, the Weyl tensor \( W \) appears after dividing the Riemann tensor \( R \) enough by the metric tensor \( g \). It is also clear that the Weyl tensor is the conformally invariant part of the Riemann tensor.

The Hodge map \( * \) on an oriented Riemannian manifold is the unique vector-bundle isomorphism

\[
*: \bigwedge^p T^*M \to \bigwedge^{n-p} T^*M,
\]

which satisfies for each \( \alpha, \beta \in \bigwedge^p M \)

\[
\alpha \wedge * \beta = g(\alpha, \beta)\omega_g,
\]

where \( \omega_g \) is the volume form of \( M \). We now consider \( n = 4 \). The Hodge map induces an endomorphism of \( \bigwedge^2 T^*M \) in the special case \( p = 2 \), also denoted by \( * \):

\[
*: \bigwedge^2 T^*M \to \bigwedge^2 T^*M,
\]

such that \( *^2 \) is the identity on \( T^*M \). The eigenvalues of this map are therefore \( \pm 1 \), and we can split the space \( \bigwedge^2 := \bigwedge^2 T^*M \) and an element \( \alpha \in \bigwedge^2 \) as

\[
\bigwedge^2 = \bigwedge^+ \bigoplus \bigwedge^-,
\]

\[
\alpha = \alpha^+ + \alpha^-.
\]

Notice that as \( \bigwedge^2 \) is a 6-dimensional space, each eigenspace is 3-dimensional.

This terminology comes from the following. A pseudo-Riemannian manifold $(M, g)$ is called conformally flat if locally there is a $C^\infty$ function $f$ such that $(M, e^{2f}g)$ is flat. The factor 2 is of course just convention. If $n \geq 0$, a $n$-dimensional pseudo-Riemannian manifold is conformally flat if the Weyl tensor vanishes, so a vanishing anti-self-dual Weyl tensor means that we are “halfway on our way to conformal flatness”.

Definition 27. A Riemannian manifold $M$ is called an Einstein manifold if the Ricci scalar is proportional to the curvature: $r = \lambda g$ for some constant $\lambda$. By taking a trace this implies $r = s n g$.

From now on, we will define a quaternionic-Kähler manifold of dimension 4 to be an Einstein, self-dual manifold.

2.6 Local coordinates description

2.6.1 Vielbeins

Let $\tau^*_M$ be the cotangent bundle of a Riemannian manifold $(M, g)$ of dimension $n$. Locally we can find $n$ differential one-forms, such that they form an orthonormal basis with respect to $g$. A set $e$ of such sections is called a vielbein or a frame. Let $e = (e^1, ..., e^n)$ be such a set, then we can decompose each $e^a$ as $e^a = e^a_{\mu} dx^\mu$ for functions $e^a_{\mu}$. As $g(e^a, e^b) = \eta^{ab}$, we find that $e^a_{\mu} e^b_{\nu} g^{\mu\nu} = \eta^{ab}$, which implies $g^{\mu\nu} = e^a_{\mu} e^b_{\nu} \eta^{ab}$. The functions $e^a_{\mu}$ are the inverses of $e^a_{\mu}$.

Both indices $a$ and $\mu$ run from 1 to $n$. In physical language, the so-called flat index $a$ takes values in the orthogonal group, and $\mu$, being the one-form index, is called the curved index. This means that under a transformation of the manifold $M$, the functions $e^a_{\mu}$ transform just as the components $\omega_{\mu}$ of a differential one-form $\omega = \omega_{\mu} dx^\mu$, and not as the components $x^a_{\mu}$ of a general $(1,1)$ form $x = x^a_{\mu} dx^\mu \partial_\mu$.

We now turn our attention to quaternionic-like manifolds. Using a set of vielbeins, we can express the quaternionic structure in terms of these. Let $q^X$ with $X = 1, \ldots, 4m$ be local coordinates. The components of the vielbein would be $f^X_i$. For reasons that will appear later, we will use a vielbein $f^X_A$, where $i = 1, 2$ and $A = 1, \ldots, 2m$.

The vielbein, and the inverse vielbein $f^X_i$ can be used to define the quaternionic structure:

$$\vec{J}_X^Y := -if^X_A \vec{\sigma}_i \cdot f^Y_A,$$

where $\vec{\sigma}$ are the usual Pauli matrices. They satisfy the quaternionic algebra.
2.6.2 Quaternionic spaces

Instead of a characterization in terms of the holonomy groups, the physical literature often uses one in local coordinates, in terms of the Nijenhuis tensor or the connection. This is not directly used in this thesis, but we present it here for completeness.

We define the ‘diagonal’ Nijenhuis tensor by

\[ N_{XY}^Z = \frac{1}{6} J|_X W. (\partial_{|W|} J_Y^Z - \partial_Y J_W^Z), \]

and we require for a quaternionic manifold

\[ (1 - 2r) N_{XY}^Z = - J|_X Z. N_{Y|V} W. J_W^V. \]  \hspace{1cm} (4)

We use the Nijenhuis tensor to define the so-called SU(2) curvature \( \omega^{\text{Op}}_X \) as

\[ (1 - 2r) \omega^{\text{Op}}_X = N_{XY}^Z J_Z^Y. \]

The condition (4) has a solution, called the Oproiu connection:

\[
\begin{align*}
\Gamma^{\text{Op}}_{XY} Z & := \Gamma^{\text{Ob}}_{XY} Z - J|_X Z \cdot \bar{\omega}_Y^{\text{Op}}, \\
\Gamma^{\text{Ob}}_{XY} Z & := - \frac{1}{6} \left( \partial_{|X|} J_Y^Z + J|_X U \times \partial_{|U|} J_Y^W \right) \cdot J_W^Z
\end{align*}
\]  \hspace{1cm} (5)

where \( \Gamma^{\text{Ob}}_{XY} Z \) is the Obata connection, which is the solution if \( \bar{\omega}_X = 0 \).

Using the Oproiu connection and the definition of the SU(2) curvature, we can define the so-called GL(\( m, \mathbb{H} \)) curvature \( \omega^{B}_X A \) as:

\[ \omega^{B}_X A = \frac{1}{2} f^B_i (\partial_X f^Y_i + \Gamma_{XZ}^Y f^Z_i - \omega^{i}_X A f^Y_j) + \omega^{j}_X A f^Y_i. \]

The Oproiu connection now implies that the quaternionic structure is covariantly constant:

\[ D_X f^B_i := \partial_X f^B_i - \Gamma_{XY}^Z f^A_Z + \omega^{i}_X A f^Y_i + \omega^{j}_X B f^B_j = 0. \]  \hspace{1cm} (6)

Conversely, any connections \( \Gamma_{XY}^Z \) and \( \bar{\omega}_X \) that satisfy equation (6), imply (4). Furthermore, they must be related to the connections defined in (5) by a \( \xi \)-transformation.

Just like the Hermitian connection on a Kähler manifold coincides with the Levi-Civita connection, we now see that the Obata and Levi-Civita coincide on a hyper-Kähler manifold. The Oproiu and Levi-Civita connection on a quaternionic-Kähler manifold need not to be the same, but they are related by a so-called \( \xi \)-transformation.

2.7 Metrics

The physical description in the next section will be a geometrical description. To describe physical properties, we need to describe four-dimensional quaternionic-Kähler spaces. There are different classes of metrics derived by mathematicians. These metrics were introduced to the physical world by Ketov [2].
2.7.1 Przanowski metrics

In this section I will briefly outline the derivation of the Toda equation by Przanowski [3]. Every self-dual Euclidean Einstein space $M$ with a cosmological constant $\Lambda \neq 0$ is locally a Hermitian manifold. This means that for each point $p \in M$ there is an open neighborhood $U$ and local complex coordinates $z^1, z^2$, such that the metric on $U$ takes the form

$$ds^2 = g_{\alpha \bar{\beta}}(dz^\alpha \otimes dz^{\bar{\beta}} + dz^{\bar{\beta}} \otimes dz^\alpha),$$

(7)

where the indices $\alpha, \beta$ take values 1, 2, and $g_{\alpha \bar{\beta}} = g_{\bar{\beta} \alpha}$. Furthermore, one can choose the coordinates $z^1, z^2$ such that the tensor $g_{\alpha \bar{\beta}}$ is determined in terms of a scalar function:

$$g_{\alpha \bar{\beta}} = \left(\frac{3}{\Lambda}\right)\left(u_{\alpha \bar{\beta}} - 2\delta^2_{\alpha} \delta^2_{\bar{\beta}} e^{-u}\right),$$

where $u = u(z^\alpha, z^{\bar{\alpha}})$ satisfies the nonlinear partial differential equation

$$u_{11}u_{22} - u_{12}u_{21} - (2u_{11} + u_1)e^{-u} = 0.$$  

(8)

The subscripts 1, 2 etc. indicate differentiation with respect to the coordinates $z^1, z^2$. This notation will be used throughout this thesis.

Further analysis, which we do not repeat here, shows that there are two distinct cases, depending on the direction of the Killing vector, labelled (A) and (B), such that each self-dual Einstein manifold with $\Lambda \neq 0$ and a Killing vector field falls in precisely one of these cases.

In the case (A), the only simplification the article by Przanowski offers is the statement that the solution $u$ to (8) has a solution of the form $u = u(z^1, z^{\bar{1}}, z^2 + z^{\bar{2}})$.

2.7.2 Przanowski-Tod metric

We continue with case (B). In this case, the function $u$ takes the form

$$u = u(z^1 + z^{\bar{1}}, z^2, z^{\bar{2}}),$$

and should satisfies the system of differential forms

$$du - pdw - zdz^2 - \bar{q}dz^{\bar{2}} = 0,$$

$$dp \wedge dq \wedge dz^2 - e^{-u}(2dp \wedge dz^2 \wedge dz^{\bar{2}} + p^2dw \wedge dz^2 \wedge dz^{\bar{2}}) = 0,$$

(9)

where we have defined $w = z^1 + z^{\bar{1}}$. The problem of solving equation (8) is equivalent to the problem of finding three-dimensional integral varieties of equation (9) with $dw \wedge dz^2 \wedge dz^{\bar{2}} \neq 0$. We can put this system into an easier form by performing the contact transformation

$$(w, z^2, z^{\bar{2}}, u, p, q, \bar{q}) \mapsto (v, z^2, z^{\bar{2}}, h, r, s, \bar{s})$$

$$v = p^{-1}, \quad h = w - up^{-1}, \quad r = -u$$

$$s = -qp^{-1}, \quad \bar{s} = -\bar{q}p^{-1}.$$
This gives the system

\[ dh - rdv - sdz^2 - \bar{s}dz^2 = 0 \]
\[ dv \wedge d\bar{s} \wedge dz^2 + d(v^2 e^r) \wedge dz^2 \wedge d\bar{z}^2 = 0. \]

If we take \( v, z^2, \bar{z}^2 \) as the three independent coordinates, and substitute \( h = h(v, z^2, \bar{z}^2), r = r(v, z^2, \bar{z}^2), s = s(v, z^2, \bar{z}^2), \bar{s} = \bar{s}(v, z^2, \bar{z}^2) \), we find, from the first line

\[ h_v dv + h_2 dz^2 + h_2 d\bar{z}^2 - rdv - s dz^2 - \bar{s}d\bar{z}^2 = 0. \]

This then shows

\[ h_v = r, \quad h_2 = -s, \quad h_2 = -\bar{s}, \quad \bar{s}_2 + (v^2 e^r)_v = 0, \]

which combines to \( h_{22} + (v^2 e^{h_v})_v = 0 \). Finally, let \( \hat{g} = \hat{g}(v, z^2, \bar{z}^2) \) be a real function such that \( \hat{g}_v = h + v(\ln v^2 - 2) \) and define \( F = \hat{g}_{vv} \), then we arrive at the three-dimensional \( SU(\infty) \) Toda-equation for the function \( F \):

\[ F_{22} + (e^F)_{vv} = 0. \]

We replace the complex coordinate \( z^2 \) by real coordinates \( u \) and \( v \). In the local coordinates \( (r, u, v, t) \), the Przanowski-Tod (PT) metric now reads

\[ g_{\text{PT}} = \frac{1}{r^2} \left[ f dr^2 + f e^h (du^2 + dv^2) + f^{-1}(dt + \Theta)^2 \right]. \tag{10} \]

The Killing vector is translation in the coordinate \( t \). The metric is determined in terms of the potential \( h \), which is subject to the Toda equation:

\[ (\partial^2_u + \partial^2_v)h + \partial^2_r e^h = 0. \tag{11} \]

The function \( f = f(r, u, v) \) is related to the potential \( h \) by

\[ f = -\frac{3}{2\Lambda} (2 - r \partial_r h), \tag{12} \]

and the one-form \( \Theta(r, u, v) = \Theta_r dr + \Theta_u du + \Theta_v dv \) is a solution to

\[ d\Theta = (\partial_u f dv - \partial_v f du) \wedge dr + \partial_r (f e^h) du \wedge dv. \]

For some time, only this case was used in string theory. It was realized only later by string theorists that the story above is not the complete story, as the original article by Przanowski made the distinction between two different classes of metrics, of which the above metric is only one.

The original article by Przanowski makes a small caveat. At the time that this thesis was written, it seems that this obstruction does not occur; furthermore, case (A) might also be transformed into situation (B).
2.7.3 Calderbank-Pedersen metric

If the manifold $M$ admits two commuting Killing vector fields, the metric can be specified to the Calderbank-Pedersen (CP) metric [4]:

\[
\tilde{g}_{CP} = \frac{F^2}{4F^2} - 4\rho^2(F^2 + F_\eta^2)\,d\rho^2 + d\eta^2 + \frac{((F - 2\rho F_\rho)\alpha - 2\rho F_\eta^2\beta)^2 + (-2\rho F_\eta^2\alpha + (F + 2\rho F_\rho)\beta)^2}{F^2(F^2 - 4\rho^2(F^2 + F_\eta^2))},
\]

where $F(\rho, \eta)$ is a solution to the linear differential equation

\[
F_{\rho\rho} + F_{\eta\eta} = \frac{3F}{4\rho^2},
\]

and uses the one-forms $\alpha = \sqrt{\rho}d\phi$ and $\beta = \rho^{-1/2}(d\psi + \eta d\phi)$. In a more compact notation we introduce matrices

\[
Q = \begin{pmatrix}
\frac{1}{2}F - \rho F_\rho & -\rho F_\eta \\
-\rho F_\eta & \frac{1}{2}F + \rho F_\rho
\end{pmatrix}, \quad N = \frac{1}{\rho} \begin{pmatrix} 0 & 1 \\ \rho & \eta \end{pmatrix},
\]

and we write (with a change of a factor 2 with respect to (13), for reasons that will become clear later)

\[
g_{CP} = 2\tilde{g}_{CP} = -\frac{2\det Q}{F^2\rho^2}(d\rho^2 + d\eta^2) + M^{IJ}(\rho, \eta)d\phi_Id\phi_J,
\]

with the matrix $M$

\[
M^{IJ} = -\frac{2}{F^2\det Q}(NQ^2N^t)^{IJ}.
\]

If $\det Q > 0$, $g_{CP}$ is a selfdual Einstein metric of positive scalar curvature; if $\det Q < 0$, the metric $-g_{CP}$ is a selfdual Einstein metric of negative scalar curvature $-6$. The statement is that any metric on $M$ arises locally in this way.
3 Physics

This section reviews some physical aspects. After a short introduction to solitons and instantons in field theories, we turn to string theory. We will look at some aspects of the classical string, the different string theories, the low-energy spectrum of the type II theories, the GSO projection, compactifications, branes, supersymmetry, multiplets, isometries and dualities. After this general theory we get to more specialized subjects, such as the universal hypermultiplet and the possible quantum corrections it receives. We furthermore discuss some of the other work that has been done on this subject.

3.1 Solitons and instantons in field theories

To give an introduction to instantons in string theory, we discuss some aspects of instantons and solitons in field theories. A very good textbook on this subject is Rajaramans book [5] and there is a small section on solitons and instantons in Ryder [6].

Solitons arise in the study of non-linear partial differential equations. If a solution to these partial differential equations only depends on the difference of two of the coordinates used in the partial differential equation, then the solution will keep its exact shape. A familiar example are solutions of the free wave equation, which are of the form \( f(x \pm vt) \). We call these solutions solitary waves.

Solitons are special kind of solitary waves. A formal definition would require substantial mathematics; for our purposes a soliton

a. is a solitary wave.

b. is localized, so that it decays (or approaches a constant) at infinity.

c. can interact with other solitons, but emerges (asymptotically) separate (up to irrelevant properties, e.g. phase shifts).

These properties are tempting to identify solitons with particles; they are localized and can interact, but can escape as a single particle. Although classical solutions, solitons are important for the quantum theory as one can find new particle states by quantizing these solitons.

Although these are nice properties for solutions to have, they are also very stringent. Especially the third property is very difficult to check: we need to know all the possible soliton solutions, and let them interact and check if they asymptotically emerge. As a consequence, there are not many systems with solitons known, although they do have solitary wave solutions. Another complication is that in the physical literature, the difference between solitary waves and solitons has become blurred, and in many cases localized solutions are called solitons, although only the solitary wave character has been demonstrated. From this point on, I will also follow this practice, and only speak about solitons.

Instantons are localized, fine-action classical solutions of the Euclideanized field theory. In the quantum theory, they form an alternative description of tunnelling phenomena.
There furthermore is a fundamental correspondence between solitons and instantons. In many cases, instantons of a $d$-dimensional model are the same as stationary solitons of the $d+1$-dimensional model. We can see this in the following way: suppose that the $d+1$-dimensional model has the Lagrangian

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} |\nabla \phi|^2 - V(\phi)$$

and hence the energy density

$$\mathcal{H} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} |\nabla \phi|^2 + V(\phi)$$

We perform a Wick-rotation on the action by changing time to $t = -i\tau$, and we find

$$S_E = \int d\tau dx \left( \frac{1}{2} \frac{d^2 \phi}{d\tau^2} - \frac{1}{2} |\nabla \phi|^2 - V(\phi) \right) = \int d\tau dx \mathcal{L}_E.$$ 

If we take a static solution, with $\dot{\phi} = 0$, we therefore find

$$\mathcal{L}_E = -\mathcal{H}, \text{ for static solutions.}$$

If the static solution has a finite energy, it has a finite Euclidean action. Therefore, static solitons in $d+1$ dimensions are instantons in $d$ dimensions. Let us now look at examples why solitons and instantons are useful.

### 3.1.1 The sine-Gordon kink

Everyone’s favorite introductions to solitons is the **sine-Gordon equation**. This equation is given by

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{b^2} \sin(b\phi) = 0,$$

and describes a scalar field in one space and one time dimension. A soliton is given by

$$\phi(x, t) = \frac{4}{b} \arctan \left[ \pm (1 - v^2)^{-1/2} (x - vt)/\sqrt{b} \right]. \quad (16)$$

where $v$ is the velocity. This moving solution can also be obtained by Lorentz-boosting the solution with $v = 0$. This is really a soliton in the sense of the strict definition above.

Furthermore, there is an infinite number of constant solutions

$$\phi(x, t) = \frac{2\pi n}{b}, \quad n \in \mathbb{Z}. \quad (17)$$

If this were a classical problem, we therefore would have an infinite degenerate ground state.
A possible Lagrangian density, and the corresponding potential and energy density for the sine-Gordon equation is

\[ \mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - V(\phi) \]

\[ V(\phi) = \frac{1}{b^2} [1 - \cos(b\phi)] \]

\[ \mathcal{H} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + V(\phi). \]

This shows that the classical solutions (17), which are points where \( V(\phi) = 0 \), have zero energy. If we use the time-independent solution (16) with \( v = 0 \), we find an energy of \( \frac{8}{\pi^2} \). We therefore have a solution with an energy which is inversely proportional to the coupling constant \( b \).

The stability of this solution indicates the existence of a conservation law. If we define

\[ J^\mu = \frac{b}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi \]

then we have \( \partial_\mu J^\mu = 0 \). This current is not a Noether current: it does not follow from a symmetry of the Lagrangian. This is also indicated by the face that it divergencelessness does not require the equations of motion. The corresponding charge is

\[ Q = \int_{-\infty}^{\infty} J^0 dx = \frac{b}{2\pi} [\phi(\infty) - \phi(-\infty)] = N , \]

where \( N \) is the difference between the numbers of the ground states which the soliton crosses. The solution (16) with \( v = 0 \) goes from \( \phi = 0 \) to \( \phi = 2\pi/b \) and therefore has \( N = 1 \). It can be seen as a tunneling process from the classical ground state at \( \phi = 0 \) to the ground state at \( \phi = 2\pi/b \). In a simple model, we look at a infinite string with pegs, as in figure 3.1.1. Each of these pegs has a small coupling to the neighbor, and there also is gravity acting downwards. The ground state would the state where every peg hangs vertically. The situation in the figure, however, is also a stable state, and it cannot be deformed continuously to the ground state. Let us now look at the quantum picture.

The potential energy of this system is one example of a large class of systems with a periodic potential, i.e. a potential satisfying \( V(q) = V(q + 2\pi) \) for all \( q \), with minima at \( q = 2\pi N \). This is a well-known systems, especially in solid-state physics, as it models the behavior of electrons in a one-dimensional lattice. See for instance the book by Kittel[7]. Without tunnelling, there would be an infinite number of ground states with energy \( E = \frac{1}{2} \hbar \omega \), where \( \omega \) is the oscillation frequency in a minima. Tunnelling effects will lift this degeneracy and cause the
energy eigenfunctions to be linear combinations of the original ground states \( u_0 \):

\[
\phi_\theta(q) = \sum_{N=-\infty}^{N=\infty} e^{iN\theta} u_0(q - 2\pi N),
\]

where \( \theta \) parameterizes the states in the band. The coefficients ensure that the states are eigenfunctions of the unitary operator which shifts \( q \to q + 2\pi \). The energy of such a state is approximately given by

\[
E_\theta \simeq \frac{1}{2} \hbar \omega - \alpha \cos \theta,
\]

where \( \alpha \) is a constant. These results, and more, can be obtained by instanton calculations. The complete procedure, including replacing zero-mode coordinates by collective coordinates and summing over all instantons and anti-instantons is described in Rajaraman’s book \[5\].

### 3.1.2 Instantons in Yang-Mills theory

The next example deals with SU(2) gauge theory. We consider the Euclideanised version of four-dimensional space-time \( E^4 \). This four-dimensional space has coordinates \((x_1, x_2, x_3, x_4)\) with \( x_4 = ix_0 \). We have the Euclidean field tensor \( F^\mu_\nu \), where \( \alpha \) is the SU(2) index, which is defined in the same way as the Minkowskian version:

\[
F^\mu_\nu = \partial^\mu A^\nu - \partial^\nu A^\mu - i g [A^\mu, A^\nu].
\]

The dual of \( F^\mu_\nu \), which is denoted \( \tilde{F}^\mu_\nu \), is defined as

\[
\tilde{F}^\mu_\nu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F^\rho_\sigma.
\]

In Euclidean space there is no need to distinguish between upper and lower indices. The fields transform as usual under gauge transformations:

\[
A'_\mu = SA_\mu S^{-1} - \frac{i}{g} (\partial_\mu S) S^{-1}
\]

\[
F'_{\mu\nu} = SF_{\mu\nu} S^{-1}.
\]

We now define

\[
K_\mu = \epsilon_{\mu\nu\kappa\lambda} \text{Tr} \left( \frac{1}{2} A_\nu \partial_\kappa A_\lambda - \frac{ig}{3} A_\nu A_\kappa A_\lambda \right)
\]

and some algebra shows

\[
\partial_\mu K_\mu = \frac{1}{4} \text{Tr} \tilde{F}^\mu_\nu F_{\mu\nu}.
\]

After this introduction, we start with the real construction. We consider a 4-dimensional volume \( V^4 \) in \( E^4 \), and we denote the boundary \( \partial V^4 \) by \( S^3 \). We first
suppose it is a pure vacuum $A_\mu = 0$. Then $F_{\mu \nu} = 0, K_\mu = 0$. We now perform a gauge transformation, which makes $A_\mu$ a pure gauge vacuum:

$$A_\mu \rightarrow A'_\mu = -\frac{i}{g} (\partial_\mu S) S^{-1} \quad \text{(on } S^3)$$

and we take the particular transformation

$$S = \frac{x_4 + ix_j \sigma_j}{\sqrt{\tau^2}}$$

where $\tau^2 = x_4^2 + x^2$. Some algebra yields

$$A_i = \frac{i}{g\tau^2} [x_i - \sigma_i(\sigma_j x_j + i x_4)]$$

$$A_4 = -\frac{1}{g\tau^2} \sigma_j x_j$$

$$K_\mu = \frac{2 x_\mu}{g \tau^4}$$

If we now integrate (18) over $V^4$ and use Gauss’ theorem, we find

$$\int \text{Tr} F_{\mu \nu} \tilde{F}_{\mu \nu} = 4 \int_{S^3} K_\perp d^3 \sigma = \frac{16 \pi^2}{g^2},$$

as the area of a 3-sphere of radius $\tau$ equals $2\pi^2 \tau^3$. But this shows that the field strength $F_{\mu \nu}$ cannot be zero over the entire volume $V^4$. This also means that (19) cannot be the entire solution, but simply the asymptotic form as $\tau^2 \rightarrow \infty$. The required form is

$$A_\mu = \frac{\tau^2}{\tau^2 + \lambda^2} \left( \frac{-i}{g} \right) (\partial_\mu S) S^{-1}$$

where $\lambda$ is a constant.

If we look at (19) again, we see that, given a certain gauge field $A_\mu$, this provides a mapping of $S^3$ to the gauge group SU(2). As a manifold SU(2) $\simeq S^3$, so the question is: are there mappings of $S^3$ to $S^3$ which cannot be deformed to the identity mapping? The answer is yes: in fact, there are an infinite number of mappings, labelled by an integer $q$, which are not deformable into each other. This ensures the stability of this solution $q \neq 0$: it is never possible to deform it to the true vacuum, so there are non-trivial local minima of the action.

If we interpret the time coordinate as an evolution in time, we can perform another gauge transformation, which makes the transition clear. Under this transformation, we have the limiting pure gauge forms

$$A'_i \rightarrow i (g_n)^{-1} (\partial_i g_n) \quad \text{as } x_4 \rightarrow \infty,$$

$$A'_i \rightarrow i (g_{n-1})^{-1} (\partial_i g_{n-1}) \quad \text{as } x_4 \rightarrow -\infty,$$

where $g_n := (g_1)^n$ and $g_1$ is a fixed element of SU(2), such that $g_m$ and $g_n$ are not homotopic for $m \neq n$. The instanton solution therefore describes a solution
of the gauge-field equations in which, as \( x_4 \) evolves from \(-\infty\) to \( \infty \), the vacuum of class \( n - 1 \) evolves into a vacuum of class \( n \). It can furthermore be shown that this amplitude is proportional to \( \exp(-S_E) \), where \( S_E \) is the Euclidean action. For this solution, we find

\[ S_E = \frac{8\pi^2}{g^2} q, \]

where \( q \) is the amount of change of the homotopy classes; in the example above \( q = 1 \). Notice how the action is inversely proportional to the coupling strength. So the Yang-Mills vacua contains an infinite number of instanton solutions. The true vacuum is therefore a combination of the vacua in a homotopy class:

\[ |\text{vac}\rangle_\theta = \sum_{q=-\infty}^{\infty} e^{i\pi \theta} |\text{vac}\rangle_q, \]

just as in the periodic potential. These vacua are known as \( \theta \)-vacs, and with \( \theta \neq 0 \), we have a complex vacuum state and hence time reversal. From the CPT theorem, we therefore also have violation of CP invariance.

The angles \( \theta \) are also known as theta-angles. They can also be added to the Lagrangian. If we add a the term proportional to

\[ L_\theta = i\theta \frac{g^2}{16\pi^2} \int \Tr F_{\mu\nu} \tilde{F}_{\mu\nu}, \]

then equation (18) shows that this is a total derivative, and has no physical consequence. Without the factor \( i\theta \), this term is a topological index, called the Pontryagin index. It is a term which is always an integer, being the Brouwer degree of the mapping \( S^3 \to S^3 \).

### 3.2 General string theory

This is only a very short introduction. For more details, the reader is advised to look at one of the many good books on string theory. Zwiebach [8] is a good book for those new to fundamental physics. The books by Green, Schwartz and Witten [9, 10] are good books for those more familiar with theoretical physics.

#### 3.2.1 Classical strings

String theory is, in a sense, nothing but a quantum theory of classical strings.

We describe a string as an embedding of a two dimensional manifold into a higher dimensional space. The latter is called the target space, and the image of the embedding is called the world sheet. The embedding is specified by the coordinate functions \( X^\mu \), where \( \mu \) runs from 1 to \( n \), the dimension of the target space. It turns out that the dimension of the target space is not arbitrary, as quantum computations fix the dimension. The action for this simple string is given by

\[ S = \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \sqrt{h} h^{ab} \partial_a X^\mu \partial_b X_\mu, \]
where \( h \) is the metric of the worldsheet, which is specified by the embedding. It turns out that it is more easy to work with the action where this metric is unspecified, so we can use the gauge freedom associated with it.

Variation of the action gives

\[
\delta S = -\frac{1}{2\pi\alpha'} \int d\tau \int d\sigma \sqrt{h} (\delta X_\mu) \partial^a \partial_a X_\mu + \frac{1}{2\pi\alpha'} \int d\tau [\sqrt{h} \partial_\sigma X_\mu \delta X_\mu]_{\sigma=0}.
\]

The first term gives the wave-equation-like equations of motions, if the second term vanishes. We therefore have three distinct types of boundary conditions:

- **Closed strings.** We take periodic boundary conditions on \( \sigma \):
  \[ X_\mu(\tau, \sigma) = X_\mu(\tau, \sigma + \pi). \]

- **Open string, Neumann boundary conditions**
  \[ \partial_\sigma X(\tau, \sigma = 0) = \partial_\sigma X(\tau, \sigma = \pi) = 0. \]

- **Open string, Dirichlet boundary conditions**
  \[ X_\mu(\tau, \sigma = 0) = \text{constant}, X_\mu(\tau, \sigma = \pi) = \text{constant}. \]

The Neumann boundary conditions ensure that there is no momentum flow at the end of the string. The Dirichlet boundary conditions fix the endpoints of the string to a hypersurface. As it is possible to choose Dirichlet b.c. for some values of \( \mu \) and Neumann conditions for the others, this hypersurface can have any dimension. The object the string endpoints end on is called a D-brane, and to specify the number of dimensions \( p \) it extends on, it is called a \( Dp \)-brane.

An \( D0 \)-brane is an object with zero spatial dimensions, so it is a point particle, and it would imply a string with the endpoints fixed in all dimensions (like a guitar string). At the other end, there is a \( D(n-1) \) brane, which fills the entire space, and it would allow the string to move freely (like a rope thrown into the air).

It turns out that quantization of this theory restricts us to the case \( n = 26 \). Furthermore, the spectrum of this theory contains a graviton (a massless, spin 2 particle), but it also contains a tachyon (a particle with negative mass). The most striking problem, however, is the lack of fermions.

### 3.2.2 Fermions

We can add fermions to the theory, by adding a multiplet of Majorana spinors on the world sheet, to obtain the action

\[
S = \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \sqrt{h} \left( h^{ab} \partial_a X_\mu \partial_b X_\mu + 2i \bar{\psi}^\mu \rho^a \partial_a \psi_\mu \right),
\]

where \( \rho^a, a = 1, 2 \), are a representation of the Clifford algebra. Quantization of this theory makes only sense if the dimension of the target space is equal to \( n = 10 \).
3.2.3 RR and NS sectors

We use light-cone coordinates on the world sheet to simplify the expressions. In the variation of the action with respect to the fermions we then find a boundary term

\[ \int d\tau [\psi_+ \delta \psi_+ - \psi_- \delta \psi_-]_{\sigma = 0}^{\sigma = \pi}. \]  

(21)

If we require this expression to vanish, we can establish this by requiring that it vanishes at \( \sigma = 0 \) and \( \sigma = \pi \). We therefore get the equations

\[ \psi_+ \delta \psi_+ = \psi_- \delta \psi_-, \quad \text{at } \sigma = 0, \sigma = \pi. \]

This is possible if we take \( \psi_+ = \pm \psi_- \). Although we could a priori take a different sign for \( \sigma = 0 \) and \( \sigma = \pi \), we can always remove an overall sign. The two different possibilities are usually described as

\begin{align*}
\psi_+(\tau, 0) &= \psi_-(\tau, 0) \\
\psi_+(\tau, \pi) &= e^{\pm 2\pi i \nu} \psi_-(\tau, \pi).
\end{align*}

The first possibility, where \( \nu = 0 \), gives periodic boundary conditions, and these are called Ramond (R) boundary conditions. The second possibility, where \( \nu = \frac{1}{2} \), gives anti-periodic boundary conditions, and these are called Neveu-Schwarz (NS) boundary conditions. The states with those different boundary conditions are also referred to as elements of a certain sectors, so we speak about the R sector and the NS sector. The R sector will lead to space-time fermions and states in the NS sector will describe space-time bosons.

In the case of a closed string, we can take different conditions for \( \psi_+ \) and \( \psi_- \), and therefore we find four different boundary conditions, namely (R,R), (R,NS), (NS,R) and (NS,NS). Combining those sectors adds their spins, so the sectors (R,R) and (NS,NS) will describe space-time bosons, and the sectors (NS,R) and (R,NS) will describe space-time fermions.

The lowest excitations are given in the following tables. As will become clear, only the excitations with low mass concern us, so these tables will stop at the massless level. The columns show respectively the mass of each state, the representation under SO(8), the value of \( G \) and the representation content under SO(9,1). The columns with \( G \) and \( \tilde{G} \) will be explained below.

The open string:

<table>
<thead>
<tr>
<th>( \alpha' m^2 )</th>
<th>SO(8)</th>
<th>( G )</th>
<th>Rep. content</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS ( -\frac{1}{2} )</td>
<td>{0}_\text{singlet}</td>
<td>-1</td>
<td>(1)</td>
</tr>
<tr>
<td>0 ( b_{1/2} )</td>
<td>{0}(8) _v _u</td>
<td>+1</td>
<td>(8) _v</td>
</tr>
<tr>
<td>0 ( \bar{b}_{1/2} )</td>
<td>{(8)}_v _u</td>
<td>-1</td>
<td>{(8)}_v</td>
</tr>
<tr>
<td>R ( 0 )</td>
<td>{a}8_s _u</td>
<td>+1</td>
<td>8_s</td>
</tr>
<tr>
<td>0 ( \bar{a} )</td>
<td>{(8)}_c _u</td>
<td>-1</td>
<td>8_c</td>
</tr>
</tbody>
</table>

36
3.2.4 The GSO projection

As we can see, the spectrum contains a tachyon: in the NS and the (NS,NS) sector, there is a state with \( m^2 < 0 \). Furthermore, there is no symmetry between the bosonic and the fermionic states. There is a procedure, introduces by Gliozzi, Scherk and Olive which projects out precisely the right states. This so-called GSO projection uses the quantum numbers \( G \) and \( \tilde{G} \). The projection amounts to simply keep the states with a fixed number \( G = \pm 1 \) and \( \tilde{G} = \pm 1 \).

Let us first treat the open string. If we take the NS sector and keep states with \( G = -1 \), we find as lowest state the tachyon. This sector will be called NS-. Likewise we can make the NS+ sector, which contains a massless boson as ground state. The R+ and R- both contain a massless fermion as ground level, with different chiralities.

For the closed string, we need to combine the NS± and R± sectors, so in total we can build 16 different sectors. As explained in Polchinsky \[11\], paragraph 10.5, there are some requirements in combination of these sectors, and the only consistent models are called IIA and IIB, which contain the following sectors:

<table>
<thead>
<tr>
<th></th>
<th>IIA</th>
<th>IIB</th>
</tr>
</thead>
<tbody>
<tr>
<td>(NS,NS)</td>
<td>(NS+,NS+) (R+,NS+) (NS+,R-) (R+,R-)</td>
<td></td>
</tr>
<tr>
<td>(R,R)</td>
<td>(NS+,NS+) (R+,NS+) (NS+,R+) (R+,R+)</td>
<td></td>
</tr>
</tbody>
</table>

In both theories, there are the massless vector-spinor gravitinos in the NS-R and R-NS sectors. The II states that there are two different gravitinos. In the IIA theory, both gravitinos have opposite chirality, while in IIB theory they have the same chirality.

The spectrum in IIA has the following massless states:

Bosons: \( [(1) + (28) + (35)_v] + [(8)_v + (56)_v] \)

Fermions: \( [(8)_c + (56)_c] + [(8)_s + (56)_s] \)

where we have indicated, between brackets, respectively NS-NS, R-R, NS-R and R-NS. For the bosonic content, we have the following names and properties:
The fermions can be interpreted as two gravitinos of spin 3/2 and two dilatinos of spin 1/2, one for each handedness.

In type IIB we have the following massless states:

**Bosons:** \([1] + (28) + (35)\) + \([1] + (28) + (35)\)

**Fermions:** \([8] + (56)\) + \([8] + (56)\)

The bosonic contents has the following names and properties:

| (1) | \(\phi\) | Real scalar; dilaton |
| (28) | \(B_{\mu\nu}\) | Antisymmetric rank two tensor |
| (35) | \(g_{\mu\nu}\) | Graviton |
| (8) | \(Z_{\mu}\) | Vector |
| (56) | \(A_{\mu\nu\rho}\) | Anti-symmetric rank three tensor |

The fermions can be interpreted as two gravitinos \((56)\) of spin 3/2 and two fermions \((8)\) of spin 1/2, with the same handedness.

One could try to make other theories, by interchanging every R+ by R- in type IIA and IIB. These are called IIA’ and IIB’ respectively, but they turn out to be equivalent. Finally, there are theories 0A and 0B, but these contain the (NS-,NS-) sector, which contains tachyons.

This provides us with two consistent string theories. There are three other consistent string theories. The first is the **type I string theory**, which contains open and closed strings, with a SO(32) group symmetry. The last two string theories are the so-called **heterotic string theories**, which mean that the left-moving and right-moving strings are different. They have group symmetries of SO(32) and \(E_8 \times E_8\) respectively.

### 3.2.5 Supersymmetry

Another nice result of the GSO projection of the last subsection, is that the remaining theory is a supersymmetric theory. Supersymmetry (SUSY) is a symmetry between bosons and fermions. Every boson has a fermion superpartner, and for every fermion there is a bosonic superpartner. The theory was developed in the 1970s, and after a sceptical start, it is now a vital part of many proposed theories of physics. There is no direct experimental evidence of supersymmetry in nature, but there are indications that we might find the superpartners of the presently know particles at energies just above those of modern particle accelerators. Adding supersymmetry to the Standard Model resolves some problems, but it also induces new problems.

The total number of supercharges divided by the smallest possible spinor representation is usually denoted by a capital \(N\). In \(d = 4\), which is used most in
this thesis, the smallest spinor has four components, so the number of supercharges is $4N$. The cases with $N > 1$ are called extended supersymmetry. In general, if the number $N$ is higher, more of the physical theory is constrained by supersymmetry. The Lagrangian, for instance, has to be invariant under more symmetries. For $N = 1$ supersymmetry, there are not much constraints. For $N = 2$, there is a reasonable number of constraints, as we will see later. For $N = 4$ and higher, the constraints are very severe, and do not lead to physically interesting theories.

### 3.2.6 Interactions and the dilaton

![Figure 2: The scattering of two strings to two strings.](image)

To study interactions in string theory, we need to know to compute scattering amplitudes involving various string states. There is a nice and unique way to compute those interactions. Suppose we want to study the process of figure 2. If time runs from left to right, this shows the process where two strings join into one string and separate again. It is the analogue of a tree diagram in quantum field theory. To compute the scattering amplitude, we need to sum over all possible world-sheets which have those four strings as bound, and weigh each contribution with a factor $e^{-S}$.

![Figure 3: A more complicated scattering of two strings to two strings.](image)

A slightly more complicated figure is where two strings join into one string, which separate and join and finally separate into two strings. This is the analogue of a loop diagram. There is no consistency argument which fixes the
relative amplitude of these diagrams, and hence a new free parameter appears: the string coupling constant \( g_s \). It turns out that if this relative amplitude is fixed, all other relative amplitudes are also fixed. So besides the dimensionful parameter \( \alpha' \), there is one coupling constant \( g_s \).

Now the effective action of all five string theories is invariant under a rescaling of the dilaton field \( \Phi \):

\[
\Phi \rightarrow \Phi - 2C, \quad g_s \rightarrow e^C g_s
\]

where \( C \) is a constant. This implies that the string coupling is related to the vacuum expectation value \( \langle \Phi \rangle \) of the dilaton \( \Phi \). As the perturbative action does not contain a potential for \( \Phi \), the value of \( \langle \Phi \rangle \) can take on any value. Non-perturbative effects can introduce a potential for \( \Phi \) which then fix \( \langle \Phi \rangle \).

### 3.2.7 Branes

Until so far, the D-branes have only played a static role as the endpoints for open strings. It turns out, however, that we can also view the D-brane as a dynamical object, coupling to the string [12]. Remember that a \( Dp \)-brane is an object, extending over \( p \) spatial dimensions, which forces the endpoints of the string on a \( p + 1 \) dimensional (worldvolume) hyperplane. Consistency relations with the helicity of the spinors, forces us to have only even-dimensional branes in type IIA theory, and odd-dimensional branes in type IIB theory [13]. Note the beautiful consistency with T-duality; T-duality interchanges IIA and IIB, and changes a \( Dp \)-brane into a \( D(p+1) \) or \( D(p-1) \) brane, depending on the direction of the T-duality.

We can see the branes as sources for the Ramond-Ramond gauge potentials by modifying their Bianchi identities. In type IIA, we have gauge potentials \( C_{2k-1} \), and if we include sources they satisfy

\[
\dd C_{2k-1} = \rho.
\]

As there are potentials for \( 0 < k < 6 \), we find \( D0 \)-branes, \( D2 \)-branes, \( D4 \)-branes, \( D6 \)-branes and, if we allow \( k = -1 \) also \( D8 \)-branes. Likewise, in type IIB we find \( D(-1) \)-branes, \( D1 \)-branes, \( D3 \)-branes, \( D5 \)-branes and \( D7 \)-branes.

Furthermore, the tension of the brane is inversely proportional to the string coupling constant \( g_s \) [11] (we will also see this later in more detail), these objects cannot be found by perturbation around \( g_s = 0 \).

### 3.2.8 Effective field theories

As the string scale turns out to be the Planck scale, all presently known particles should be contained in the massless string states. We therefore investigate the limit \( \alpha' \rightarrow 0 \), so all massive states are integrated out. This yields an effective field theory. Such a theory does describe the physical phenomena at this low energy scale, but it ignores the effects at higher energies. Because effective theories ignore the high-energy behavior, they might not be renormalizable.
The best example of an effective field theory is the Fermi theory of beta decay. In this theory, a four-fermi interaction is responsible for the decay of neutrons and muons. This four-fermi coupling is not renormalizable. We now see this theory as a part of the Standard Model, which uses the W gauge bosons to mediate this interaction. The great success of the Fermi theory is due to the heavy mass of the W boson, which is around 80 GeV. The experiments at that time used energy scales less than 10 MeV. Due to an energy difference of 3 orders of magnitude, the effective theory was highly successful.

3.3 N=2 Supergravity coupled to matter

Using concepts introduced before, we will now describe the content of N = 2 supergravity coupled to matter, which is a result of compactification of type IIA string theory on a Calabi-Yau.

Being a supersymmetric theory, there is gravity. The supergravitation multiplet has a bosonic content of the graviton g_μν, two gravitino’s ψ_μ, and the graviphoton A_μ. Besides gravity, there are other fields. They organize themselves under N = 2 supersymmetry into so-called vector multiplets and hypermultiplets.

A vector multiplet has a content of \{A_Iμ, Ω_iI, χ_I\}, where I is an index labeling the multiplet. The field A_Iμ is a vector field, the Ω_iI, i = 1, 2 are two Majorana spinors and the field χ_I is a complex scalar. The kinetic terms of the vector multiplet form a non-linear sigma model. N = 2 supersymmetry now requires that this space is a hyper-Kähler space. A hypermultiplet has a content of four bosonic scalars and two spinors, and again N = 2 supersymmetry requires the kinetic terms to parameterize a quaternionic-Kähler manifold.

Supersymmetry forbids the vector and hypermultiplets from talking to each other at this stage (it might change when we gauge isometries, see section 3.6). We can therefore only consider the hypermultiplets, as we will do in this thesis.

The number of vector multiplets and hypermultiplets are determined in terms of the Hodge-numbers (see subsection 2.3) of the internal Calabi-Yau space. In type IIA, we have h^{1,1} vector multiplets and h^{1,2} + 1 hypermultiplets. In type IIB, we have h^{1,2} vector multiplets and h^{1,1} + 1 hypermultiplets. Hence, there is one hypermultiplet (the ”+1”), which is present in each compactification, which is called the universal hypermultiplet. If we compactify on a Calabi-Yau space with h^{1,2} = 0, which is a called a rigid Calabi-Yau, this hypermultiplet is the only hypermultiplet. The four real scalars in the hypermultiplet are denoted \{ϕ, χ, ϕ, σ\}, and we can see that χ, ϕ descend from the R-R sector, whereas ϕ, σ come from the NS-NS sector. In a pictorial way, we can display this as follows:

This field should not be confused with the graviphoton from the supergravitation multiplet, although the name is unfortunately the same.
The last two rows describe the various fields in local coordinate systems. This will be explained later, in section 4.2. The scalars $\chi, \varphi$ are known as the RR scalars, the scalar $\sigma$ is the (four-dimensional) axion, not to be confused with the (ten-dimensional) axion $a$ in the RR sector of type IIB. The UHM thus contains the dilaton, an axion and two RR scalars.

We see that if we interchange $h^{1,1} \leftrightarrow h^{2,1}$, we interchange the number of multiplets and hypermultiplets between type IIA and type IIB. It turns out that this is a real symmetry of the theories, called mirror symmetry\(^2\). It was later shown that mirror symmetry is, in fact, a special type of T-duality. If we write our internal Calabi-Yau manifold as a fiber bundle, whose fiber is a three-dimensional torus, then the action of T-duality on all three dimensions of this torus is equivalent to mirror symmetry\(^{15}\).

The action for the fields can be obtained from $d = 11$ supersymmetry. Reducing the 11-dimensional action
\[
2\kappa_1^2 S_{11} = \int d^{11}x (-G)^{1/2} \left( R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{6} \int A_3 \wedge F_4 \wedge F_4
\]
on a circle, yields exactly the massless content of type IIA. Identifying the fields from the compactification with those already know from type IIA, we find the action
\[
cS_{10} = \int \sqrt{g} R d^{10}x
- 2 \int \left( e^{-\varphi} H \wedge * H + d\phi \wedge * d\phi + e^{\frac{2\varphi}{7}} F \wedge * F + e^{\varphi/2} \tilde{F} \wedge * \tilde{F} + B \wedge F \wedge F \right)
\]
where $\tilde{F} := dA^3 - A^1 \wedge H^3$ and $c := \alpha'^{4}(2\pi)^7$. We compactify this action on a rigid $CY_3$ by making the assumption that each 10-dimensional field is a product of a 4-dimensional field and a 6-dimensional harmonic field on the $CY_3$. Because these are severe restrictions, we can perform all the integrations on the $CY_3$ to

\(^2\) A collection of introductory articles on this subject can be found in “Essays on Mirror Manifolds” S.-T. Yau, ed., International Press, 1992.

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obtain
\[ S_4 = -16 \frac{\text{Vol} CY_3}{2c} \int d\phi \wedge *d\phi + e^{2\phi}(d\chi \wedge *d\chi + d\varphi \wedge *d\varphi) + e^{4\phi}(\chi d\varphi + d\sigma) \wedge *(\chi d\varphi + d\sigma) \] (22)

where the fields \( \phi, \sigma, \varphi \) and \( \chi \) are defined in terms of the fields \( A^3, B \) and the dilaton \( \phi \). These four scalar fields are the universal hypermultiplet (UHM).

From this compactification we can find the metric for the universal hypermultiplet, and after rescaling it reads
\[ ds^2 = d\phi^2 + e^{-\phi}(d\chi^2 + d\varphi^2) + e^{-2\phi}(d\sigma + \chi d\varphi)^2. \] (23)

### 3.3.1 Isometries of the UHM

Classically, the UHM has a SU(1, 2) group of isometries. These isometries will, in general, be broken by quantum corrections, both perturbative and non-perturbative. The three-dimensional Heisenberg subalgebra generates the following shifts on the fields
\[ \phi \rightarrow \phi, \quad \chi \rightarrow \chi + \gamma, \quad \varphi \rightarrow \varphi + \beta, \quad \sigma \rightarrow \sigma - \alpha - \gamma \phi. \]

The Heisenberg group is preserved in perturbation theory [16], as we will explicitly see in the next subsection. A detailed view of the perturbative corrections show that these will modify the metric, but leave the isometries intact. The isometries on \( \chi \) and \( \varphi \) are generated by gauge transformations in the 10-dimensional string theory, while the isometry on \( \sigma \) arises in the process of dualization leading to this description of the UHM. The 5-brane instanton will break the shift symmetry on \( \sigma \) to a discrete symmetry, while the membrane instantons will break the shifts on \( \varphi \) and \( \chi \).

### 3.3.2 Perturbative corrections to the UHM

The perturbative corrections to the UHM are completely known [16, 17, 18]. These results have later been derived from a superspace perspective [19]. The result is surprisingly simple: we have to replace the classical solution \( e^h = r \) by
\[ e^h = r + c, \] (24)

where the constant \( c \) is proportional to the Euler number \( \chi \) of the internal manifold:
\[ c := \frac{4\zeta(2)\chi}{(2\pi)^3}. \]

This accounts for the one-loop correction, but all higher corrections can be absorbed in field redefinitions, so this accounts for all corrections. Inserting this solution into the metric shows that the three isometries of the Heisenberg group do survive. Conversely, if one knows that all three isometries survive, then an analysis of the metric shows that the function \( h \) can only depend on \( r \), and the solution [24] is simply the most general solution which only depends on \( r \). The hard part is, of course, the exact determination of this constant \( c \).
3.3.3 Non-perturbative corrections to the UHM

The non-perturbative corrections to the UHM in type IIA consist of membrane instantons and fivebrane instantons. These effects were first analyzed in [20]. In type IIA, we have corrections to the universal hypermultiplet when the ten-dimensional type IIA membrane wraps a supersymmetric three-cycle and when the fivebrane wraps the six-dimensional Calabi-Yau manifold.

The fivebrane instantons were analyzed in [21]. In this thesis we only focus on membrane instantons. As explained in section 2.3.2, if we take a rigid Calabi-Yau, we have 2 supersymmetric three-cycles that the membrane can wrap around. There are then three different instantons: one fivebrane and two possible membrane. These two different instantons solutions will yield two membrane instanton charges, corresponding to the shift symmetries in $\chi$ and $\phi$.

The explicit calculations of the instanton corrected metric is a highly non-trivial task. In this thesis, we therefore take a different approach: we know that the resulting non-linear sigma model should stay a quaternionic-Kähler manifold. We can therefore look at all possible quaternionic-Kähler manifolds, and see if we can find interesting manifolds among these.

3.3.4 The work by Davidse et al.

In this section I will describe the results, obtained by Davidse et al. [22]. They constructed solutions to the Toda equation (11), using an elaborate Ansatz:

$$e^h = r + \sum_{n \geq 1} \sum_m f_{n,m}(u,v) r^{-m/2 + \alpha} e^{-\alpha n \sqrt{r}}.$$ (25)

The physical interpretation is a sum over all instantons $m$, and around each instanton a power series in $r$ to describe the perturbative effects. Using the coefficients $f_{n,m}$, this solution can satisfy the Toda equation to any order required. Furthermore, calculations in string theory confirm this solution. The leading order contribution is

$$e^h = r + c + e^{\frac{1}{2} r^{-m/2}} (A e^{iv} + A^* e^{-iv} + B e^{-iu} + B^* e^{iu}) e^{-2\alpha \sqrt{r}}.$$ (25)

The problem with this solution is the large amount of unknown integration constants. As will be shown later in this thesis, the one-instanton solution can produce meta-stable deSitter vacua, but only if the constants are fine-tuned. We therefore hope that these constants can be fixed by some other procedure.

3.3.5 The work by Cvetič

Linearizations, such as made by Cvetič [23], of the Toda equation also give some insight. If we make the Ansatz $e^h = r + \lambda$, with $\lambda$ small compared to $r$, we find that the linear terms give

$$t \lambda_{tt} + \lambda_{uu} + \lambda_{vv} = 0,$$
and further specification of $\lambda$ to $\lambda = f(r) \exp(ik_1u + k_2v)$ gives two solutions for $f$ in terms of Bessel function. The regular solution is of the form $f \sim rK_1(\lambda r)$, where $k = \sqrt{k_1^2 + k_2^2}$. Asymptotically we therefore have

$$f \sim \sqrt{r}e^{-\lambda r}.$$

### 3.3.6 SL(2, Z) duality

There is an additional symmetry in type IIB string theory, called **S-duality** (or **strong-weak duality**). In the simplest case, we transform $g_s \to 1/g_s$. This is equivalent to changing the dilaton $\phi \to -\phi$, as $g_s = e^{-\phi}/2$. This interchanges the fundamental string $F1$ and the $D1$ brane. In fact [24, 25], the complete transformation group turns out to be SL(2, Z). Under the complete transformation, more objects transform, and we have the singlets and doublets

$$\left(\begin{array}{c} \phi \\ D(-1) \end{array}\right), \left(\begin{array}{c} F1 \\ D1 \end{array}\right), \left(\begin{array}{c} NS5 \\ D5 \end{array}\right), \left(\begin{array}{c} D7 \end{array}\right).$$

At low energy and tree-level, the transformation SL(2, Z) is even SL(2, R), because the restriction to integers is only visible at the quantum level.

### 3.4 Removal of singularities

An exciting property of the inclusion of non-perturbative effects is that they might be able to remove singularities that arise at the perturbative level.

A nice example where this does happen is in the work of Ooguri and Vafa [26]. They also study type IIA supergravity, but they focus on one of the non-universal hypermultiplets. Furthermore, they look at the geometry of this hypermultiplet in the limit where gravity decouples, which implies that the moduli space is a hyper-Kähler space. Some more information about this limit can be found in the article of Ketov [27].

The hyper-Kähler hypermultiplet space is again described by a metric, which is formulated in terms of a solution $V$ to a partial differential equation. The proposed solution $V$ reads

$$V = \frac{1}{4\pi} \log \left(\frac{\mu^2}{zz}\right) + \sum_{m \neq 0} \frac{1}{2\pi} e^{2\pi imz} K_0 \left(2\pi |mz|/\lambda\right),$$

where $\lambda$ plays the role of the string coupling constant and $\mu$ is a constant. Notice how the string coupling is a constant here; in the UHM the string coupling is linked to the dilaton expectation value and therefore a dynamical variable.

The first term is the classical metric, which is singular at $z = 0$. The second term is a summation over instantons. The combination of this two terms is regular at $z = 0$: using Poisson resummation we find

$$V = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \left(\frac{1}{\sqrt{(x-n)^2 + zz/\lambda} - \frac{1}{n}}\right) + \text{constant}.$$
So the instanton terms precisely cancel the singularity, caused by the non-perturbative effects. We can then interpret the singularity: if we ignore the non-perturbative effects, we have neglected some aspects of the theory. If the singularity does appear, it shows that these aspects could not be neglected. Including then these aspects gives a sensible theory. Of course, there could also be more fundamental aspects we have ignored (such as the transition to a low-energy effective action) which then show as a singularity.

3.5 The cosmological constant

The cosmological constant is the constant $\Lambda$ in the Einstein field equations

$$G + \Lambda g = T,$$

where $G$ is the Einstein tensor, $g$ the metric and $T$ the stress-energy tensor. Originally included by Einstein to allow for a static universe, it was abandoned by observations of the Hubble telescope and the fact that the static universe was unstable. Einstein called it the “biggest blunder [he] ever made”. Ironically, there are observations in the 1990’s that can be explained very well by a small positive cosmological constant. The current standard model of cosmology, the Lambda-Cold Dark Matter model uses a cosmological constant of $10^{-47}$ GeV$^4$.

One of the major problems of today’s theoretical physics concerns this cosmological constant. Calculation of this constant from today’s theories is off by an order of magnitude 120. To obtain the observed cosmological constant there must be another contribution, opposite in sign and almost cancelling the theoretical value. Such a fine-tuning indicates that there is something we miss. A comparison of the many different solutions is presented in the work of S. Nobbenhuis [25].

One possible source for the cosmological constant is the potential function of a scalar field. Rewriting the Einstein equation (without cosmological constant) in vacuum $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$ for a constant field $\phi$ gives

$$R_{\mu\nu} \propto V(\phi) g_{\mu\nu},$$

so there is a contribution of the potential energy of this field to the cosmological constant.

Cosmological data suggest that after the big bang there was a period of rapid, accelerated expansion, which is known as inflation. In this period the scale factor grows exponentially, but at the present time the acceleration is many orders of magnitude smaller. This could be modelled with a potential with a local minimum, as shown in figure [3.5]. At the beginning of the universe, the scalar field $\phi$ did not start at the minimum, but it is now rolling towards this minimum. This gives a larger acceleration in the past, but now our universe is heading towards a lesser acceleration. If the minima of the potential would be positive, this would give a finite maximum value for the cosmological constant.
3.6 Gauging isometries

In a later stage we want to gauge one of the isometries. Gauging symmetries is known from Yang-Mills theories such as the Standard model. When gauging isometries in supergravity, we however want the theory to remain supersymmetric. Now the replacement of partial derivatives by covariant derivatives is not enough in $N=2$ supersymmetry. To keep supersymmetry, there must be additional terms added to the Lagrangian, which have the form of a potential function. Details of the calculations can be found in appendix $B$.

It is a non-trivial fact that one isometry of the quaternionic-Kähler manifold from the UHM can be gauged using the graviphoton $A_\mu$ from the gravity multiplet. This makes the entire supergravity multiplet consist of gauge fields: $g_{\mu\nu}$ is the gauge field for local diffeomorphisms, and the gravitinos are gauge fields for local supersymmetry. Gauging one isometry of the UHM when $h^{1,2} = 0$ does not require additional vector fields. If we do want to gauge more isometries of the universal hypermultiplet, we need additional vector fields, and they can only be found in the vector multiplets. We now do need to include the vector multiplets, and there will be mixing terms in the Lagrangian between vector and hypermultiplets.
4 Connection between the metrics

This section will combine the various ingredients of previous sections. We will see connections between the different descriptions of the possible geometry of the universal hypermultiplet, and how this connection might be useful to us.

4.1 Introduction

The previous two sections described the mathematical and physical foundations. Type IIA string theory compactified on a Calabi-Yau produces various multiplet, and the universal hypermultiplet is a multiplets which is always present. The Lagrangian of the bosonic part of the UHM is a non-linear sigma model, and the four scalar fields parameterize a quaternionic-Kähler space, as explained in section 3.3.

In section 3.3.1 the isometries of the universal hypermultiplet were described. If we only consider one of the membrane instantons, there will be two commuting shift symmetries, as described in section 3.3.3. So we are looking at the geometry of a four-dimensional quaternionic-Kähler space with two commuting, continuous isometries.

As explained in section 2.7, the PT and CP metrics describe, respectively, self-dual Einstein manifolds with one (an U(1)) and two commuting (U(1) × U(1)) isometries. So both these metrics describe our quantum corrected UHM. Furthermore, the CP metric is a special case of the PT metric, and it should in principle be possible to write each solution of the CP metric as a solution of the PT metric. In this section we will derive this exact procedure. Furthermore, we will study the connection between CP and PT metrics and the general Przanowski setting.

4.2 Universal Hypermultiplet

We use the relative simple metric of the UHM to find a first relation between the metrics. The metric for the UHM was given by equation (23):

\[ ds_{UHM}^2 = d\phi^2 + e^{-\phi}(d\chi^2 + d\varphi^2) + e^{-2\phi}(d\sigma + \chi d\varphi)^2. \]  

As this metric describes the quaternionic-Kähler space \( U(1, 2)/U(2) \), which has a \( U(1) \times U(1) \) symmetry, we should be able to express this metric in terms of both the PT and the CP metric.

A way to find the UHM metric in the CP framework is to require that the function \( F(\rho, \eta) \) does not depend on \( \eta \). We find one solution \( F(\eta, \rho) = \rho^{3/2} \). A short calculation gives

\[ ds_{CP}^2 = \frac{4}{\rho^2} ((d\rho)^2 + (d\eta)^2) + 4\rho^{-4}(d\phi_1 + \eta d\phi_2)^2 + \frac{1}{\rho^2} (d\phi_2)^2 \]

\[ = \rho^{-2}[(d\phi_2)^2 + (d(2\eta))^2] + \rho^{-4}[(d(2\phi_1) + 2\eta d\phi_2)^2 + (d(ln\rho^2))^2]. \]

There is another solution of the differential equation, given by \( F = \rho^{-1/2} \), but for this solution \( \det Q = 0 \), so it does not give rise to a metric.
Comparison shows that we can take the identifications
\[ e^h = \rho^2, \quad \chi = 2\eta, \quad \sigma = 2\phi_1, \quad \varphi = \phi_2. \] (27)
Furthermore, if we use the PT metric, we can take \( e^h = r \), and we find
\[
ds_{PT}^2 = \frac{1}{r^2} dr^2 + \frac{1}{r} (du^2 + dv^2) + \frac{1}{r^2} (dt + u dv)^2.
\] (28)
So we have the following mappings, explaining the last two lines of the table on page 41:

<table>
<thead>
<tr>
<th></th>
<th>PT</th>
<th>UHM</th>
<th>CP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shift coordinates</td>
<td>( t )</td>
<td>( \sigma )</td>
<td>( 2\phi_1 )</td>
</tr>
<tr>
<td>Non-shift coordinates</td>
<td>( r )</td>
<td>( e^\varphi )</td>
<td>( \rho^2 )</td>
</tr>
</tbody>
</table>

The Killing vector fields in the CP metric are simply global shifts in the \( \phi_1 \) and the \( \phi_2 \) coordinates.

### 4.3 General case

Let us assume that in the general case, the identification of \( \phi_1 \) and \( \phi_2 \) with \( t \) and \( v \) are the same as in the case of the universal hypermultiplet.

Thus, we assume that \( \rho \) and \( \eta \) do not depend on \( t \) and \( v \), and so \( r \) and \( u \) do not depend on \( \phi_1 \) and \( \phi_2 \), so the shift-coordinates and the non-shift coordinates do not mix. As the CP metric has no crossterms between \( \{d\rho, d\eta\} \) and \( \{d\phi_1, d\phi_2\} \), the PT should have none corresponding either between \( \{dr, du\} \) and \( \{dt, dv\} \), so expanding the \( (dt + \Theta)^2 \) term, we find \( \Theta_u = \Theta_v = 0 \).

We first compare the metric parts of the shift coordinates. We write the PT metric in the coordinates of the CP metric. The relevant part of equation (10) is
\[
ds_{PT, \text{shift}}^2 = \frac{1}{r^2} \left[ e^h d\phi_2^2 + f^{-1}(4d\phi_1^2 + 4\Theta_v d\phi_1 d\phi_2 + \Theta_v^2 d\phi_2^2) \right],
\]
which is in matrix form, against a vector \( \phi = (d\phi_1, d\phi_2) \)
\[
\frac{1}{r^2} f^{-1} \begin{pmatrix} 4 & 2\Theta_v \\ 2\Theta_v & \Theta_v^2 \end{pmatrix} + \frac{1}{r^2} \begin{pmatrix} 0 & 0 \\ 0 & f e^h \end{pmatrix}.
\]
This matrix should be equal to the matrix \( M^{IJ} \) of equation (15). Solving the three independent equations, we find the important result
\[ e^h = \frac{r^4}{F^4} = \frac{r(\rho, \eta)^4}{F(\rho, \eta)^4}, \] (29)
and the results
\[ f = \frac{-8\rho F^2 \det Q}{\omega r^2}, \] (30)
\[ \Theta_v = \frac{2\omega \eta - 8\rho^2 F F_{\eta}}{\omega}. \] (31)
where we have defined
\[ \omega := (F + 2\rho F_\rho)^2 + 4\rho^2 F_\eta^2. \]

The equation for \( f \) provides us immediately with a consistency check. Remember that the sign of \( \det Q \) is the same as the sign of the scalar curvature (see section 2.7.3). From equation (30) we see that \( f \) and \( \det Q \) have opposite signs, because all other factors are positive: \( \rho > 0 \) as it is the coordinate of the upper half plane, and all other quantities are (sums of) squares of real numbers. With equation (12) we now conclude that \( \Lambda \) and the scalar curvature have the same sign, as they should. Furthermore, a solution where \( f = 0 \) is singular in the Przanowski-Tod case, and \( \det Q = 0 \) is singular in the Calderbank-Pedersen case, again as it should be.

The only remaining problem now is how to write \( \rho \) and \( \eta \) in terms of \( r \) and \( u \). The part of the metric containing \( \{dr, du\} \) in (10) is given by
\[ ds^2_{\text{PT, non-shift}} = f r \rho^2 (dr)^2 + f e^h u \rho^2 (du)^2, \]
which should be compared to the non-shift part of the CP metric (15):
\[ ds^2_{\text{CP, non-shift}} = -2 \frac{\det Q}{F^2 \rho^2} (d\rho^2 + d\eta^2). \]

Substitution of the result found for \( f \) and rearranging gives the equation
\[ e^h \omega \frac{\omega}{4\rho^3} (d\rho^2 + d\eta^2) = dr^2 + e^h du^2. \] (32)

There are now two different routes; we can view \( r, u \) as functions of \( \rho, \eta \) or vice versa. The first approach is most useful in our current situation, so we present it first.

4.3.1 First approach

Substitution of \( r = r(\rho, \eta) \) and \( u = u(\rho, \eta) \) gives the set of equations
\[ r^2_\rho + e^h u^2_\rho = r^2_\eta + e^h u^2_\eta = e^h \frac{\omega}{4\rho^3}, \]
\[ r_\rho r_\eta + e^h u_\rho u_\eta = 0. \] (33)

Finally, a solution to these equations must be a solution to the Toda-equation, which now reads
\[ \partial^2_r \log \left( \frac{r}{F} \right)^4 + \partial^2_r \left( \frac{r}{F} \right)^4 = 0. \] (34)

Remember that the function \( r, F \) are both functions of \( \rho \) and \( \eta \) so this is a complicated expression. This system is underdetermined, hence we make the important Ansatz\(^5\)
\[ r(\rho, \eta) = F(\rho, \eta) \sqrt{\rho}. \] (35)

\(^4\)With the minor exception for the case where also \( \omega = 0 \).

\(^5\)This Ansatz is very natural; in fact this equation, seen as an Ansatz for \( F(\rho, \eta) \) is used in the original article of Calderbank and Pedersen \cite{CalderbankPedersen}.
which gives equation \(29\) the simple form \(e^h = \rho^2\). The system of equations \(33\) now reduces to

\[
\begin{align*}
\rho u_\rho^2 &= F_\eta^2, \\
2\rho^2 u_\eta u_\rho &= -F_\eta \left( F + 2\rho F_\rho \right), \\
4\rho^3 u_\eta^2 &= (F + 2\rho F_\rho)^2,
\end{align*}
\]

and if we take the solutions

\[
\begin{align*}
u_\rho &= -\frac{F_\eta}{\sqrt{\rho}}, \\
u_\eta &= \frac{F + 2\rho F_\rho}{2\rho^{3/2}},
\end{align*}
\]

we find that both the integrability conditions and the Toda-equation \(34\) are implied by the equation \(14\) for the function \(F\). The latter claim was verified using Mathematica.

The system \(36\) is solved by

\[
\begin{align*}
u(r, \eta) &= \int_{c_1}^{\rho} \frac{\partial_x F(x, 0)}{\sqrt{x}} \, dx + \int_{c_2}^{\eta} \frac{F(\rho, y) + 2\rho \partial_y F(\rho, y)}{2\rho^{3/2}} \, dy + c_3,
\end{align*}
\]

where the \(c_i\) are constants, with \(c_1 \geq 0\). The function \(u\) only appears in the metric \(10\) as a derivative or as a one-form, so any constant added to \(u\) has no significance at all. In the following, we set \(c_3 = 0\).

We have completed the mapping; given a solution \(F(\rho, \eta)\), we can determine \(r\) and \(u\), and solve those in terms of \(\rho\) and \(\eta\) to express \(e^h = \rho^2\) in terms of \(r\) and \(u\). This determines the function \(e^h\) in terms of \(r, u\) which defines the metric.

However, as we shall see, it is usual not possible to explicitly solve \(r\) and \(u\) in terms of \(\rho\) and \(\eta\).

### 4.3.2 Second approach

Substitution of \(\rho = \rho(r, u)\) and \(\eta = \eta(r, u)\) in \(32\) gives the equations

\[
\begin{align*}
\omega e^h (\rho_r^2 + \eta_u^2) &= 4\rho^3 \\
(\rho_r \rho_u + \eta_u \eta_r) &= 0 \\
\omega (\rho_u^2 + \eta_u^2) &= 4\rho^3.
\end{align*}
\]

Again this is underdetermined, so we again substitute equation \(35\), which now takes the form \(F = r\rho(r, u)^{-1/2}\), and \(\rho(r, u) = e^{h/2}\). The presence of various derivatives of \(F\) in the expression for \(\omega\) gives complicated expressions. Rewriting all the derivatives using the chain rule, we however find a very simple solution, which is given by

\[
\eta_r = -\rho^{-1} \rho_u, \quad \eta_u = \rho \rho_r,
\]

and leads to

\[
\eta(r, u) = -\int_0^r e^{-\frac{1}{2} h(x, 0)} \partial_x e^{\frac{1}{2} h(x, 0)} \, dx + \int_0^u e^{\frac{1}{2} h(r, y)} \partial_r e^{\frac{1}{2} h(r, y)} \, dy
\]

\[
= -\int_0^r \frac{1}{2} \partial_x h(x, 0) \, dx + \int_0^u \frac{1}{2} e^{h(r, y)} \partial_r h(r, y) \, dy.
\]
So we find a similar solution using this approach. Again we have the integration of the specified function, and we should solve this integration to obtain the new coordinates.

### 4.3.3 Examples

We now present some important examples.

The situation of the universal hypermultiplet is recovered by using either $F = \rho^{3/2}$ or $e^h = r$. We then find

\[
\begin{align*}
  r &= \rho^2, \\
  u &= 2\eta, \\
  \rho &= \sqrt{r} \\
  \eta &= u/2 \\
  F &= \rho^{3/2} = r^{3/4} \\
  e^h &= \rho^2 = r \\
  f &= 1 \\
  \Theta_v &= 2\eta = u,
\end{align*}
\]

as expected.

If we correct the universal hypermultiplet with the one-loop correction, we find

\[
F = \rho^{3/2} - c\rho^{-1/2},
\]

where $c$ is a constant parameter, see section 3.3.2. This gives the result

\[
\begin{align*}
  r &= \rho^2 - c, \\
  u &= 2\eta, \\
  \rho &= \sqrt{r + c} \\
  \eta &= u/2 \\
  e^h &= \rho^2 = r + c \\
  f &= \frac{r + 2c}{r + c} \\
  \Theta_v &= 2\eta = u.
\end{align*}
\]

The function $F$ defined by $F = \sqrt{\rho + \frac{\eta^2}{\rho}}$ is a solution to the differential equation for $F$, but it gives singular metrics in both the PT and CP settings, as we find

\[
\begin{align*}
  r &= \sqrt{\eta^2 + \rho^2}, \\
  u &= \text{arcsinh} \left( \frac{\eta}{\rho} \right), \\
  \eta &= r\tanh u \\
  \rho &= r\sech u \\
  e^h &= \rho^2 = r^2 \sech^2 u \\
  f &= 0 \\
  \Theta_v &= 0.
\end{align*}
\]

### 4.4 The derivation by Przanowski

To make the connection between the Przanowski metric and the CP and PT metrics, we first study the universal hypermultiplet in those cases.
We can follow this procedure in section 2.7.1 backwards to obtain the function \( u \) for the classical UHM potential \( F = \ln v \). From \( F = \tilde{g}_{vv} \) we find that \( \tilde{g}_{vv} = v \ln v - v + f(z^2, \bar{z}^2) \), where \( f = f(z^2, \bar{z}^2) \) is a (unknown) function of \( z^2 \) and \( \bar{z}^2 \). This implies that \( h = -v \ln v + v + f(z^2, \bar{z}^2) \). From the differential equation \( h_{2\bar{2}} + (v^2 e^h v) = 0 \) we see \( f_{2\bar{2}} = -1 \) and hence we can take \( f = -\frac{1}{2}(z^2 + \bar{z}^2)^2 \), which gives

\[
h = -v(\ln v - 1) - \frac{1}{2}(z^2 + \bar{z}^2)^2.
\]

From the contact transformation we find \( v = p^{-1}, u = \ln v = \ln p^{-1} \), and hence

\[
h = w - up^{-1} = -v(\ln v - 1) - \frac{1}{2}(z^2 + \bar{z}^2)^2,
\]

and solving \( u \) in terms of \( w = z^1 + z^1, z^2, \bar{z}^2 \) gives

\[
u = \ln(w + z^2 \bar{z}^2) = \ln (z^1 + z^1 + \frac{1}{2}(z^2 + \bar{z}^2)^2).
\]

This function satisfies the differential equation (41). The transformation

\[
z^1 = e^\phi - \frac{1}{2} \chi^2 + i\sigma, \quad z^2 = \chi - i\varphi,
\]

transforms the metric (41) into the metric (23) of the universal hypermultiplet.

We will try to repeat this, using the one-loop corrected potential \( F = \ln(v + c) \). This gives, performing the same steps as above

\[
h = (v + c) \ln(v + c) - 2v \ln v + v - \frac{1}{2}(z^2 + \bar{z}^2)^2.
\]

The contact transformations now give

\[
u = -\ln \left( \frac{v + c}{v^2} \right), \quad w = c \ln(v + c) + v - \frac{1}{2}(z^2 + \bar{z}^2)^2,
\]

and we should solve this for \( u \) in terms of \( w, z^2, \bar{z}^2 \), which are transcendental equations.

### 4.5 General mapping of the Przanowski metric

For the most general comparison between the Przanowski metrics and the CP metric, we use the same procedure as in section 4.3. We keep the identification of the shift coordinates we found for the universal hypermultiplet, so from equation (41) we use

\[
\sigma = \frac{1}{2}(z^1 - \bar{z}^1), \quad \varphi = \frac{1}{2}(z^2 - \bar{z}^2),
\]

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and we define coordinates $\alpha, \beta$ for the real part of the complex $z^1, z^2$ coordinates as

$$
\alpha = \frac{1}{2}(z^1 + z^1), \quad \beta = \frac{1}{2}(z^2 + z^2).
$$

(43)

Substitution and comparison finally leads to the equation

$$
\omega d\alpha^2 + (2\omega \eta - 9\rho^2 F F_\eta)(d\alpha d\beta + d\beta d\alpha) + 4(\omega \eta^2 + \rho^2(\overline{\omega} - 8\eta F F_\eta))d\beta^2 = \frac{16 \det^2 Q}{\rho}(d\rho^2 + d\eta^2),
$$

(44)

where we have defined

$$
\det Q := \frac{1}{4} F^2 - \rho^2 F^r_\rho - \rho^2 F^\eta_\eta
$$

$$
\omega := (F + 2\rho F_\rho)^2 + 4\rho^2 F^2_\eta
$$

$$
\bar{\omega} := (F - 2\rho F_\rho)^2 + 4\rho^2 F^2_\eta.
$$

I have not been able to solve equation (44) in general. For the classical UHM metric, it can be easily checked this equation is true if one substitutes $F = \rho^{3/2}$ and the coordinates of equations (41) and (43).

For the one-loop corrected metric, $F = \rho^{3/2} - c\rho^{-1/2}$, we find from equation (44) that equation (43) is modified to

$$
\alpha = \rho^2 - 2\eta^2 + 2c \log \rho, \quad \beta = 2\eta.
$$

Inverting those relations (solving for $\rho, \eta$ in terms of $\alpha, \beta$) gives transcendental equations. This is the same problem as encountered in (42).

Finally, using the mapping derived in section 4.3, we can transform equation (44) from the CP to the PT setting, but this does not offer any simplifications.

### 4.6 Conclusion

In this section we investigated the connections between the three different classes of metrics. It turns out that the transformation between the CP and PT metrics is relatively simple, although it might not yield explicit solutions. The classical and one-loop metrics can be transformed explicitly.

The transformation between the Przanowski metric and the CP/PT metrics is more difficult, and poses already problems at the one-loop level.
The Eisenstein series

In this section we will analyze a special solution \( F(\rho, \eta) \) to the differential equation (14). We will first motivate the usage of this solution and show its properties. Using the tools developed in previous chapters, we try to transform this solution from the CP setting to the PT setting. If this transformation turns out to be the solution found by Davidse, we can determine the unknown constants. We then demonstrate some intrinsic problems with the Eisenstein series. We then show how a modification of the Eisenstein series gives rise to meta-stable deSitter and anti-deSitter vacua.

5.1 The series

In section 2.7 we studied the CP metric, and found that metrics of self-dual Einstein manifolds with an \( U(1) \times U(1) \) isometry are described by solutions \( F = F(\rho, \eta) \) of the differential equation

\[
F_{\rho\rho} + F_{\eta\eta} = \frac{4F}{3\rho^2}.
\]

This states that \( F \) is an eigenfunction with eigenvalue \( 3/4 \) for the Laplace equation of the complex upper-half plane \( H \) with the Poincaré metric. In section 3.3.6 we studied \( SL(2, \mathbb{Z}) \) duality. There is a natural action of \( SL(2, \mathbb{Z}) \) on the upper half plane, which is an isometry for the Poincaré metric. This differential equation is therefore also invariant. It is therefore natural to find a solution to the differential equation which is also invariant under \( SL(2, \mathbb{Z}) \). This solution is a modular form, called the Eisenstein series for \( s = 3/2 \).

The Eisenstein series \( E_s(z) \) for \( s \in \mathbb{C} \) with \( \text{Re } s > 1 \) and \( z \in H \) is defined as

\[
E_s(z) = \sum_{\gamma \in \Gamma \setminus \Gamma_{\infty}} (\gamma z)^s,
\]

where

\[
\Gamma_{\infty} = \left\{ \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix} \mid a \in \mathbb{Z} \right\},
\]

which is the stabilizer subgroup of each point. The construction is very natural: the functions \( p_s(z) = (\text{Im } z)^s = y^s \) are clearly eigenfunctions of the Laplace operator on the upper half plane \( H \), and we sum over the group \( SL(2, \mathbb{Z}) \) to make this solution invariant under \( SL(2, \mathbb{Z}) \).

In terms of \( z = \eta + i\rho \) we can write this as

\[
F(\rho, \eta) = E_{3/2}(z) = \rho^{3/2} + \rho^{3/2} \sum_{(p,n)=1, n \geq 1} \frac{1}{|p + n\tau|^3}
\]

\[
= \rho^{3/2} + \sum_{(p,n)=1, n \geq 1} \left( \frac{\rho}{n^2\rho^2 + (n\eta + \rho)^2} \right)^{3/2},
\]

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where summations run over the integers, and \((p, n)\) is the greatest common divisor of \(p\) and \(n\). This form already shows the classical solution \(\rho^{3/2}\), with a summation of corrections.

Being invariant under \(\eta \to \eta + 1\), the Eisenstein series has an expansion in terms of Fourier series. For the general Eisenstein series we have

\[
\Lambda(s)\mathcal{E}_s(p, \eta) = \rho^s \Lambda(s) + \rho^{1-s} \Lambda(1-s) + 2\rho^{1/2} \sum_{m \neq 0} |m|^{s-1/2} \sigma_1(m) K_{s-1/2} (2\pi|m| \rho) e^{2\pi i m \eta},
\]

where the so-called divisor function is defined by \(\sigma_s(m) = \sum_{0 < d | m} d^s\) and

\[
\Lambda(s) = \pi^{-s} \Gamma(s) \zeta(2s).
\]

In the case of \(s = 3/2\), it can be shown that this series can be written as

\[
\mathcal{E}_{3/2}(p, \eta) = \rho^{3/2} + \frac{\pi^2}{3\zeta(3)} \rho^{-1/2} + \frac{4\pi}{\zeta(3)} \rho^{1/2} \sum_{m \neq 0} \frac{|m|}{n} e^{2\pi i m \eta} K_1(2\pi|m| \rho)
\]

\[
= \rho^{3/2} + \frac{\pi^2}{3\zeta(3)} \rho^{-1/2} + \frac{8\pi}{\zeta(3)} \rho^{1/2} \sum_{m, n \geq 1} \frac{m}{n} \cos(2\pi m n \eta) K_1(2\pi m n \rho).
\]

Notice that each of these three parts are a solution to the differential equation for \(F(p, \eta)\): the first is the classical solution, then we have the one-loop correction, and the remainder is a summation over instantons.

This form can also be obtained by directly using separation of variables on the Laplace equation on \(H\). Each term inside the double summation is already a solution to this equation, and the Eisenstein series fixes the relative factors of each term.

It is of course possible to rescale the entire Eisenstein series with a fixed overall constant, but this does not change the line elements as this factor drops out. We have chosen the normalization such that the classical term, proportional to \(\rho^{3/2}\), has no additional factor.

### 5.2 Calculations with the Eisenstein series

Given the Eisenstein serie, we can now try to transform this solution to the \(CP\) equation to the setting of the PT metric, as explained in section 4.3.1. We therefore first investigate the basic solution

\[
F(p, \eta) = \left( \frac{\rho}{n^2 \rho^2 + (n \eta + p)^2} \right)^{3/2},
\]
which is, however, not a solution of the differential equation (14). Inserting this
function gives the relations

\[
\begin{align*}
    r &= \frac{\rho^2}{(n^2\rho^2 - (n + p/\eta))^{3/2}} \\
    u &= \frac{1}{n\rho^2} - \frac{(\eta + p/n)F(\rho, \eta)}{\rho^{3/2}} \\
        &= \frac{1}{n} \left( \frac{1}{\rho^2} - \frac{p + n\eta}{((p + n\eta)^2 + n^2\rho^2)^{3/2}} \right),
\end{align*}
\]

which we should solve for \(\rho, \eta\) as functions of \(r, u\). These equations do not appear
to be analytically solvable.

### 5.3 Fourier expansion

Another way to proceed is using the Fourier expansion. Inserting this expansion gives

\[
\begin{align*}
    u &= 2\eta + \frac{8\pi}{\zeta(3)} \sum_{m,n \geq 1} \frac{m}{n} K_0(2\pi mn\rho) \sin(2\pi mn\eta),
\end{align*}
\]

and we should again solve for \(\rho, \eta\) as functions of \(r, u\). Using asymptotic behavior,

\[
K_s(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x},
\]

still does not give algebraic equations. We use the lowest order terms, so we use
only \(n = 1, m = 1\). This gives

\[
\begin{align*}
    r &= \rho^2 + \frac{\pi^2}{3\zeta(3)} + \frac{4\pi}{\zeta(3)} \sqrt{\rho} \cos(2\pi \eta) e^{-2\pi \rho} \quad (45) \\
    u &= 2\eta - \frac{4\pi}{\zeta(3)} \frac{1}{\sqrt{\rho}} \sin(2\pi \eta) e^{-2\pi \rho}. \quad (46)
\end{align*}
\]

It appears that the transformation of the Eisenstein series from the CP setting
to the TP setting is not easy. Using the entire series is far too difficult, but even
using a small part leads to transcendental equations.

We can, however, already extract some leading-order behavior. From section 4.3.1 we know that classically \(e^h = \rho^2\), so using (45) and substitution of the classical solution (37) gives

\[
e^h = r - \frac{\pi^2}{3\zeta(3)} - \frac{4\pi}{\zeta(3)} r^{1/4} \cos(2\pi u) e^{-2\pi \sqrt{\tau}}.
\]

This solution looks like the solution found by Davidse et al., as described in
section 3.3.3. There is however a striking difference: this solution has as exponential argument of \(-2\pi \sqrt{\tau}\) whereas the solution (25) has \(-2\sqrt{\tau}\).
There is a way to solve this problem. We change the periodicity of the Eisenstein series: the original series were invariant under $\eta \rightarrow \eta + 1$, we now use $\eta \rightarrow \eta + \pi$.

We then use
\[
F(\rho, \eta) = \rho^{3/2} + \frac{\pi^2}{3\zeta(3)} \rho^{-1/2} + \frac{4\pi^{3/2}}{\zeta(3)} \cos(2\eta)e^{-2\rho},
\]
which shows
\[
eh = \rho^2 = r - \frac{\pi^2}{3\zeta(3)} - r^{1/4} \frac{4\pi^{3/2}}{\zeta(3)} \cos(u)e^{-r}.
\]

This has the same exponential behavior and the same periodicity as the leading-order solution in section 3.3.4. By taking $k_1 = 1, k_2 = 0$, it also has the same form as the asymptotical solutions in section 3.3.5. It furthermore provides values for the unknown constants in these solutions, but as we will see below, they do not seem to be correct.

In the next section, we will find another physical property of the Eisenstein series which corresponds with the solution found by Davidse.

5.4 Gauging isometries

We have seen that the transformation of the Eisenstein series is difficult; we have furthermore seen that imposing the SL(2, Z) symmetry also leads to problems. In this section we will show a property of the Eisenstein series that does agree with its interpretation as an instanton corrected solution.

To study this property, we first slightly modify the Eisenstein series. The first change is that we treat the constants before each of the three main terms as parameters. The second change is the change in periodicity, as we have made before. Dropping the factor $\pi$ from the cosine requires also the removal of this factor from the Bessel function, to keep this function a solution. Finally, we use only the lowest order term. Making these changes gives the exact solution to the Calderbank-Pedersen equation
\[
F_{\text{pert}} = \rho^{3/2} - c\rho^{-1/2} \\
F_{\text{inst}} = dp^{1/2} \cos(2\eta)K_1(2\rho) \\
F = F_{\text{pert}} + F_{\text{inst}}.
\]

Finally, we want to gauge the isometries of the non-linear sigma model, as explained in section 3.6. The details of the calculation can be found in section B. If we gauge the isometry corresponding to the unit length Killing vector in the $t = \phi_1$ direction, we find
\[
V = 4G_{\phi_1}\phi_1 - \frac{1}{\sqrt{\rho F}}. \\
\text{(47)}
\]

For the CP metric we find
\[
V = \frac{-1}{\rho F^2} \left( \frac{1}{\det Q} \left(F^2 + 2\rho F F_\rho \right) + 1 \right)
\]
If the function $F$ furthermore has the property that $F_{\eta} = 0$ when $\eta = 0$ (such as the Eisenstein series), this further simplifies to

$$V = \frac{-1}{\rho F} \left( \frac{4}{F - 2\rho F_\rho} + \frac{1}{F} \right) \text{ for } \eta = 0.$$  \hspace{1cm} (48)$$

If we substitute this function $F$ and the CP line element (15) into (47), we get a complicated expression. The terms corresponding to $F_{\text{pert}}$ give

$$V_{\text{pert}} = \frac{\rho^2 - 3c}{(\rho^2 + c)(\rho^2 - c)^2},$$

and if we only keep the term in $F_{\text{inst}}$ linear in the parameter $d$ we have

$$V_{\text{inst}} = d \frac{2\rho (2\rho^2 - c)^2 K_0(2\rho) + (\rho^4 + 3c^2) K_1(2\rho)}{(\rho^2 - c)^3 (\rho^2 + c)^2} \cos(2\eta).$$

We now finally approximate the Bessel functions as $K_s(x) \approx \sqrt{\pi x} \exp(-x)$, and set $c = 0$ in $V_{\text{inst}}$ to arrive at

$$V = \frac{\rho^2 - 3c}{(\rho^2 + c)(\rho^2 - c)^2} + d\sqrt{\pi} \rho^{-11/2} e^{-2\rho(1 + 2\rho)} \cos(2\eta),$$  \hspace{1cm} (49)$$

which is the leading term instanton correction to the potential. We now even drop the 1 in the factor $(1 + 2\rho)$.

We can substitute the relations $\rho^2 = r + c$ and $2\eta = u$, valid at the perturbative level, to obtain

$$V = \frac{r - 2c}{r^2(r + 2c)} + 2\sqrt{\pi} d r^{-9/4} e^{-2\sqrt{r}} \cos(u),$$

which can be compared to the conclusion, found by Davidse e.a. (see section 3.3.4)

$$V = \frac{r - 2c}{r^2(r + 2c)} - 4r^{-(m_1 + 5)/2} e^{-2\sqrt{r}} (\hat{A} \cos(u) - \hat{\tilde{A}} \sin(u)),$$

where $\hat{A}, \tilde{A}$ and $m_1$ are parameters. Again we find that the parameter $m_1$ should equal $-\frac{1}{2}$.

### 5.5 Problems with the Eisenstein series

In the previous subsection, we did see agreement of the Eisenstein series with the solution of Davidse, if we adjust the periodicity a bit. There are, however, some problems with the Eisenstein series or with its interpretation.

The Eisenstein series has been used in type IIB string theory. In this theory, there are tree-level and one-loop terms in the action which are proportional to $R^4$, where $R$ is the Riemann curvature. Computations of instanton effects produces terms which also contributed to the $R^4$ term.
However, this does not imply that it should be useful in our current setting, as we are studying type IIA theory. If we take the first part of the Eisenstein series

\[ F(\rho, \eta) = \rho^{3/2} + \frac{\pi^2}{3\zeta(3)} \rho^{-1/2}, \]

and compare this with the results from perturbative theory, as described in section 3.3.2, we then find that the constant parameter \( \hat{\chi} \) equals

\[ \hat{\chi} = -\frac{4\zeta(2)}{(2\pi)^3} \chi = -\frac{\pi^2}{3\zeta(3)} \approx -2.74, \]

and hence the Euler number of the internal manifold equals \( \chi = \frac{4\pi^3}{\zeta(3)} \approx 103.2, \) which is not an integer, as it should be.

Another desirable feature of the Eisenstein series would be the removal of perturbative singularities, as described in section 3.4. There clearly is a singularity in (49) at \( \rho^2 = -c \) (remember that \( c \) is a negative constant). The second term does not remove this singularity, but this could be because we only took the lowest order term; maybe the inclusion of all terms will remove the singularity. From equation (48) we see that, because \( F > 0, \) there is a singularity at \( \eta = 0 \) if and only if

\[ F - 2\rho F_\rho = 0. \quad (50) \]

In appendix A, we rigourously prove that there is a \( \rho \) such that \( F - 2\rho F_\rho = 0 \) for the full Eisenstein series. There is therefore no removal of singularities with the Eisenstein series.

These two facts imply that we need to think again about the Eisenstein series. The series were interpreted as a classical, perturbative corrected and an instanton summation. The first argument shows that at least this interpretation is flawed, as the perturbative effects do not match. Maybe the second term is only partially a perturbative correction, and the other part is a non-perturbative correction. The second argument could have many explanations. We might need to include the other membrane instanton and the five-brane instanton to remove the singularity. If that does not remove the singularity, there might be a more fundamental reason for this singularity, such as the transition to a low-energy effective action.
5.6 Plot of the Eisenstein solution

We can now make some plots of this potential. We plot the function of equation 49

\[ V = \frac{\rho^2 - 3c}{(\rho^2 + c)(\rho^2 - c)^2} + d\rho^{-9/2}e^{-2\rho}\cos(2\eta), \]

which is shown along $\eta = 0$ for values of $c = -10, d = -2.7 \cdot 10^3$ in figure 5.

If we choose $d$ to be negative, the dependence of $V$ on $\eta$ shows that we have a minimum in the $\eta$ direction for every value of $\rho$, so the minimum in figure 5 is really a stable minimum.

Figure 5: A plot of the potential function, with $c = -10, d = -3.2 \cdot 10^3$ and $\eta = 0$.

5.7 Problems with the Eisenstein series

The Eisenstein series have been used in type IIB string theory. In this theory, there are tree-level and one-loop terms in the action which are proportional to $R^4$, where $R$ is the Riemann curvature. Computations of instanton effects produces terms which also contributed to the $R^4$ term.

However, this does not imply that it should be useful in our current setting, as we are studying type IIA theory. If we take the first part of the Eisenstein series

\[ F(\rho, \eta) = \rho^{3/2} + \frac{\pi^2}{3\zeta(3)}\rho^{-1/2}, \]

and compare this with the results from perturbative theory, as described in
section 3.3.2 we then find that the constant parameter $\hat{\chi}$ equals

$$\hat{\chi} = -\frac{4\zeta(2)}{(2\pi)^3} \chi = -\frac{\pi^2}{3\zeta(3)} \simeq -2.74,$$

and hence the Euler number of the internal manifold equals $\chi = \frac{4\pi^3}{\zeta(3)} \simeq 103.2$, which is not an integer, as it should be.

Another desirable feature of the Eisenstein series would be the removal of perturbative singularities, as described in section 3.4. Judging from the plot in the previous section, the singularity still remains, but this plot used only the lowest order term. We will show that inclusion of all terms still does not resolve the singularity. As we have seen in equation (48), for a solution with $F_\eta = 0$ where $\eta = 0$, such as the Eisenstein series, the potential simplifies to

$$V = -\frac{1}{\rho F} \left( \frac{4}{F} - 2\rho F_\rho + \frac{1}{F} \right) \text{ for } \eta = 0.$$ 

Because the function $F$ is always positive (the Eisenstein series is a summation of positive terms), the only singularity can appear when $\frac{1}{2} F = \rho F_\rho$. For the Eisenstein series, this is equivalent to

$$-\rho^2 + \frac{\pi^2}{3\zeta(3)} + \frac{8\pi^2}{\zeta(3)} \rho^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^2 \left( K_0(2mn\rho) + K_2(2\pi mn\rho) \right) = 0. \quad (51)$$

But there is always a $\rho$ such that this expression becomes zero, as rigourously proven in appendix A. Intuitively one can see that for small values of $\rho$ this expression is positive, as the second terms is greater than the first, and the last term is always positive. For large values of $\rho$, the first term will dominate the third, because this one is exponentially suppressed. We therefore have a continuous function which attains both positive and negative values, and should therefore also attain zero, which is a singularity in the potential function.
6 Conclusion

We have studied non-perturbative corrections to the universal hypermultiplet. The stringent demands of supersymmetry provides us with information on the geometry of this multiplet. Using this information, we have been able to determine some properties of the instanton corrections.

In this thesis, we have learned about the different mathematical descriptions of the geometry and the interplay between them. Various solutions, such as those from Davidse and the Eisenstein series, have been investigated. There are some physical properties which indicated that they are related, but an explicit transformation between these solutions has not been demonstrated.

There are numerous suggestions for further research. It would be nice if the solution from Davidse could be further analyzed. What does happen with the potential function if one includes more instantons? It is possible to give proofs of convergence? Can one include other complex structure moduli, by taking $h^{1,2}$ nonzero, and do the conclusions still persist?

The other solution, the Eisenstein series, also has open questions. How should the Eisenstein series be interpreted in type IIA string theory, where there is no direct $SL(2, \mathbb{Z})$ duality? We have seen that the Euler number, implied by the Eisenstein series, was not an integer. Is this really a problem of the Eisenstein series, or can there be other effects that cause this discrepancy? One question I personally would like to know, is to what solution of the Toda equation does the Eisenstein series transform?

The best solution is of course a full calculation in string theory of instanton effects. While this may not be possible at the present time, maybe general considerations in string theory can be used again as input in procedures as we have used. Such calculations might, for instance, give additional geometrical data to work with. Maybe looking at the dual theories might also provide us with more characterizations of the geometry of these spaces.
In this appendix we rigorously prove that there is a $\rho > 0$ such that equation (51) is satisfied.

Define, for $x > 0$

$$f(x) = -x^2 + \frac{\pi^2}{3\zeta(3)} + \frac{8\pi^2}{\zeta(3)} x^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^2 (K_0(2\pi mn x) + K_2(2\pi mn x)),$$

where $K_\nu(x)$ is the modified Bessel function of the second kind and $\zeta(3)$ is the Riemann zeta function.

I will prove that $f(x)$ has a zero for $x > 0$. Notice that this function is a continuous function, as it is the result of differentiating the continuous differentiable Eisenstein series and the addition of and multiplication with continuous functions.

**Lemma 28.** $f(1) > 0$

*Proof.* Plugging in $x = 1$ shows that the summation of the first two parts is strictly positive, and the remainder is also strictly positive. □

**Lemma 29.**

$$K_0(z) \leq \sqrt{\frac{\pi}{2z}} e^{-z},$$

$$K_2(z) \leq 3 \sqrt{\frac{\pi}{2z}} e^{-z}, \text{ for } x > 2.$$

*Proof.* From Abramowitz and Stegun, equation 9.7.2 and the discussion below prove that

$$K_0(z) \leq \sqrt{\frac{\pi}{2z}} e^{-z},$$

$$K_2(z) \leq \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{15}{8z} + \frac{16 \cdot 6}{4 \cdot 8z^2}\right),$$

and for $z > 2$ the second estimate can be replaced by

$$K_2(z) \leq 3 \sqrt{\frac{\pi}{2z}} e^{-z}.$$

□

**Lemma 30.**

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{3/2} m^{-1/2} e^{-20\pi mn} \leq 10^{-3}.$$
Proof. We compare the summation with an integration. This is possible, as the function \( n \mapsto n^{3/2}e^{-20\pi nm} \) is monotonically decreasing for \( n \geq 1 \) for all \( m \geq 1 \). We then perform the integration and estimate the result.

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{3/2}m^{-1/2}e^{-20\pi mn} = \sum_{m=1}^{\infty} m^{-1/2} \left( \sum_{n=1}^{\infty} n^{3/2}e^{-20\pi mn} \right)
\leq \sum_{m=1}^{\infty} m^{-1/2} \left( \int_{n=0}^{\infty} \, dn \, n^{3/2}e^{-20\pi mn} \right)
= \sum_{m=1}^{\infty} m^{-1/2} \left( \frac{3}{3200 \sqrt{5} m^{5/2} \pi^2} \right)
\leq 10^{-4} \sum_{m=1}^{\infty} m^{-3} = 10^{-4} \zeta(3) < 10^{-3}.
\]

□

In fact, using Mathematica’s \texttt{NSum} method gives that this sum is approximately \( 5.16 \cdot 10^{-28} \).

**Lemma 31.** \( f(10) < 0 \)

Proof. For all \( x > 2 \) and by lemma 29

\[
f(x) = -x^2 + \frac{\pi^2}{3\zeta(3)} + \frac{8\pi^2}{\zeta(3)}x^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^2 (K_0(2\pi mnx) + K_2(2\pi mnx))
\leq -x^2 + \frac{\pi^2}{3\zeta(3)} + \frac{8\pi^2}{\zeta(3)}x^2 \sum_{m,n}^{\infty} n^2 \left( \sqrt{\frac{\pi}{2 \cdot 2\pi mnx}} + \sqrt{\frac{3\pi}{2 \cdot 2\pi mnx}} \right) e^{-2\pi mnx}
= -x^2 + \frac{\pi^2}{3\zeta(3)} + \frac{12\pi^2}{3\zeta(3)}x^{3/2} \sum_{m,n} n^{3/2}m^{-1/2}e^{-2\pi mnx}.
\]

Now using the value \( x = 10 \) and using lemma 30 gives

\[
f(10) \leq -(10)^2 + \frac{\pi^2}{3\zeta(3)} + 10^{3/2} \frac{12\pi^2}{3\zeta(3)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{3/2}m^{-1/2}e^{-20\pi mn}
\leq -100 + \frac{\pi^2}{3\zeta(3)} + \frac{12\pi^2}{3\zeta(3)} 10^{-3} 10^{3/2} < 0.
\]

□

**Theorem 32.** The function \( f(x) \) has a zero \( x > 0 \).

Proof. This follows from the combination of the observation that the function \( f(x) \) is continuous and lemmata 31 and 28. □
B Moment maps and the potential function

In this section we will describe properties of the quaternionic-Kähler manifolds, and we will describe how one can calculate the potential function. This section will also describe the conventions we use; the final potential function is highly sensitive to the different conventions, and a wrong choice will drastically change the behavior.

Using local coordinates $q^A, A = 1, \ldots, 4n$, where $4n$ is the real dimension of the manifold. We then have the metric

$$ds^2 = G_{AB} dq^A \otimes dq^B,$$

and three almost complex structures $J^r$ which satisfy the quaternionic algebra.

$$J^s J^t = - \delta^{st} 1 - \epsilon^{str} J^r.$$ 

Associated with each complex structure is a Kähler form, defined by

$$K^r_{AB} = G_{AC} J^r CB.$$ 

We introduce a set of quaternionic one-form vielbeins $V^a_i = V^a_i A dq^A$, with $i = 1, 2$ and $a = 1, \ldots, 2n$, and we can write the metric as

$$G_{AB} = V^a_i A V^b_j B \epsilon^{ij} \epsilon_{ab},$$

Using the complex conjugate $V^i a = (V^i a)^*$ of the vielbein, we can write

$$ds^2 = G_{ab} V^i a \otimes V^j b,$$

$$G_{ab} = \frac{1}{2} G_{AB} V^i A \overline{V}^B,$$

and we can express the hyperkähler 2-forms as

$$K^r = \frac{i}{2} G_{ab} V^i a \wedge \overline{V}^j b (\tau^r)^i j.$$ 

The $SU(2)$ bundle $S$ which is defined by the complex structures has a connection, defined by the one-forms $\omega^r$, and the covariant derivative reads

$$DK^r = dk^r - \epsilon^{rst} \omega^s \wedge K^t = 0.$$ 

For a quaternionic-Kähler manifold we introduce the $SU(2)$ curvature

$$\Omega^r = d\omega^r - \frac{1}{2} \epsilon^{rst} \omega^s \wedge \omega^t,$$

and we find that the $SU(2)$ curvature is proportional to the Kähler form:

$$\Omega^r = \nu K^r,$$

where $\nu$ is a real, non-vanishing number. We can express $\nu$ in terms of the Ricci scalar $R$. Using the Einstein property of quaternionic-Kähler manifolds, we can write

$$R_{AB} = \frac{R}{4n} G_{AB},$$

$$\nu = \frac{1}{4n(n+2)} R.$$
To compute the potential we need the moment maps $P^I_I$, which are implicitly defined from

$$K^I_{AB} k^B_I = D_A P^I_I = \partial_A P^I_I - \epsilon^{st} \omega_A^s \wedge P^t_I,$$

where $k^B$ are the coordinates of the Killing vector, and $I$ is a label for the different isometries. In this thesis we only have one isometry, so we omit the label $I$ from now on. This definition agrees with [29], and the moment maps differ a factor minus two from the ones in [30 31]. This relation can be solved, to obtain

$$P^I_I = -\frac{1}{2n\nu} K^I_{AB} D^A k^B_I.$$

The kinetic terms of the scalars and the graviton in the supergravity action read [32]

$$e^{-1} \mathcal{L} = -\frac{1}{2\kappa^2} R - \frac{1}{2} G_{AB} \partial_\mu \phi^A \partial^\mu \phi^B - V,$$

$$V = 2\kappa^{-2} G_{AB} k^A k^B - 3P^I P_I.$$

Imposing local supersymmetry fixes the parameter $\kappa$ as $\nu = -\kappa^2$.

Solving the Toda-equations provides us with metrics with a Ricci scalar of $-6$, so we should use $\kappa^2 = 1/2$. The metric of Calderbank-Pedersen have a Ricci scalar of $-12$, which gives $\kappa^2 = 1$, as in [33]. However, we multiplied the Calderbank-Pedersen metrics with a factor 2 in equation (15) to give easier comparison between the metrics. This implies that we should take $\kappa^2 = 1/2$, which shows a geometrical reason why this form of the metric is easier to work with.

We can now gauge the isometry to obtain

$$e^{-1} \mathcal{L} = -\frac{1}{2\kappa^2} R - \frac{1}{2} G_{AB} \partial_\mu \phi^A \partial^\mu \phi^B - V,$$

$$V = 2\kappa^{-2} G_{AB} k^A k^B - 3P^I P_I.$$

We now turn to the isometry we want to gauge. In the language of the Toda equation, we can take the Killing vector to have only a non-vanishing length in the $t$ direction, which we will set to unity. In terms of coordinates $r, v, u, t$ we therefore have $k^\mu = (0, 0, 0, 1)$.

For the Calderbank-Pedersen metric, the SU(2) connection has been computed in [34 4] and reads

$$K^1 = \frac{F_\eta}{F} d\rho + \frac{1}{\rho F} \left( \frac{1}{2} F + \rho F_\rho \right) d\eta,$$

$$K^2 = -\frac{\rho}{F} d\phi_2,$$

$$K^3 = \frac{1}{\sqrt{\rho F}} (d\phi_1 + \eta d\phi_2),$$

and yields the moment maps

$$P^1 = 0, \quad P^2 = 0, \quad P^3 = \frac{1}{\sqrt{\rho F}}.$$
The potential function therefore becomes

\[ V = 4G_{\phi_1\phi_1} - \frac{1}{\sqrt{\rho F}}. \]
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