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Master Thesis

The Quantum Hall Effects
Casimir Operators and Anyonic Quasi-Particle Excitations

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Abstract

After an extensive mathematical treatment of the system comprised of a single charged particle in a uniform perpendicular magnetic field, we provide an introduction to the integer and the fractional quantum Hall effect. The quantization of kinetic energy in terms of Landau levels is discussed, followed by an explanation of the quantization of the Hall resistance at integer filling factor, Laughlin’s variational approach to the fractional quantum Hall effect, and a short discussion on composite fermion theory. We then focus on the fractional quantum Hall effect, where all low energy excitations lie within the fractionally filled Landau level. Because the kinetic energy is constant within each Landau level, the effective dynamics are described by the density operators projected to the fractionally filled Landau level. Gaining insight in the projected density operators is therefore of prime importance for understanding the fractional quantum Hall effect.

We derive the algebra that the projected density operators satisfy, and, in order to gain a better understanding of this algebra of projected density operators, we compute several Casimir operators of the algebra: first on the infinite plane, which results in a Casimir that is rife with infinities due to the infinite degeneracy of the Landau levels, and then on the torus, where we find explicit expressions for the Casimir operators of order two, three and four, generalizing Haldane’s expression [10] for the second order Casimir. Additionally, we recognize a pattern in the expressions found for the Casimir operators that can be generalized to order $N_B$, which is equal to the degeneracy of each Landau level.

Having computed the Casimir operators, we proceed to a related topic of interest: that of the low energy quasi-particle excitations of the fractional quantum Hall fluid. We show that the quasi-particles are anyonic in nature. Because the quasi-particles carry (fractional) charge, we expect that they feel the external magnetic field that causes the quantum Hall effect. Motivated by this realization, we construct an effective model that takes anyons as fundamental particles, rather than quasi-particle excitations, and we produce a method for generating exact $N$-particle anyonic energy eigenfunctions for non-interacting charged anyons in a uniform perpendicular magnetic field. Comparing our method to other known methods for constructing exact anyonic wavefunctions, we conclude that, despite being less general, our method is more thorough and finds additional exact anyonic eigenfunctions that have not been constructed before.
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Chapter 1

Introduction

Consider a gas of spinless electrons confined to the $x$-$y$ plane, subject to a uniform perpendicular magnetic field $\mathbf{B} = B\mathbf{e}_z$. Contrary to ordinary electrical systems, a steady current $I$ corresponds to a transverse Hall voltage difference $V_H$. The ratio of $V_H$ and $I$ is, by definition, the Hall resistance,

$$R_H = \frac{V_H}{I} = \frac{B}{-en_{el}}, \quad (1.1)$$

where $n_{el}$ is the two-dimensional electron density, and $-e$ is the electron charge. In particular, the Hall resistance $R_H$ is perpendicular to the magnetic field strength $B$.

If there is no electric field present, the electrons move in circles, called cyclotron orbits. The frequency of such an orbit is given by the cyclotron frequency, $\omega_C := eB/m$, where $m$ is the mass of the electron. Because the frequency is constant, the velocity and the radius are proportional to each other, and the latter is of the order of the magnetic length $l_B := \sqrt{\hbar/eB}$, which is the typical length scale of the system.

Due to the magnetic field, the kinetic energy of the electrons becomes quantized, forming Landau levels: equidistant energy levels in steps of $\hbar\omega_C$. This quantization becomes apparent when the system is cooled to temperatures $T$ for which the thermal energy $k_B T$ is much smaller than the kinetic energy $\hbar\omega_C$. Moreover, the Hall resistance, instead of being proportional to the magnetic field $B$ as shown in equation (1.1), shows plateaus around values of the magnetic field for which the Hall resistance equals

$$R_H = \frac{\hbar}{e^2 \nu},$$

where $\nu$ is the filling factor, which is defined as the number of electrons divided by the degeneracy of a single Landau level (which depends on $B$). There are actually two underlying phenomena that cause the Hall resistance to form plateaus: the plateaus formed at integer filling factor are a consequence of the integer quantum Hall effect (IQHE), and the plateaus occurring at fractional filling factor are explained by the fractional quantum Hall effect (FQHE). The IQHE can be understood in terms of non-interacting particles, whereas the FQHE requires that electron-electron interactions be taken into account.

There are several descriptions of the FQHE. The oldest is due to Laughlin [14], which explains the FQHE at so called Laughlin filling factors $\nu = \frac{1}{p}$ with $p$ an odd integer. Laughlin provided a variational wavefunction for the ground state based on symmetry properties, his knowledge of the non-interacting ground state, and physical intuition; as well as an explanation of the fractionally charged quasi-particle excitations that obey fractional statistics. Continuing in the same line of reasoning, it is possible to view the FQHE states at filling factor $\frac{p}{2ps+1}$, with $p, s \in \mathbb{N}$, as integer quantum states of composite particles. These composite particles go by the name composite fermions (due to Jain [11]).
In addition to the construction of multi-particle ground state wavefunctions a la Laughlin, there are also field theoretical approaches to the quantum Hall effects, such as the extended Hamiltonian formalism by Murthy and Shankar [19], or the Chern-Simons theory approach to composite fermions by Lopez and Fradkin [18], which puts Jain’s composite fermion theory on a firm field-theoretical footing by describing the attachment of flux to electrons as a coupling between the electrons and a Chern-Simons gauge field.

The discovery of the quantum Hall effects had a great impact on condensed matter physics. For example, the quantization of the Hall resistance is so precise, that nowadays it provides the most accurate way to measure the fine structure constant. Unrelated to the previous statement, but no less influential, is the fact that quantum Hall effects provided the first example of a state of matter that cannot be understood in terms of spontaneous symmetry breaking, because, contrary to ordinary matter which condenses into crystal-like solids when the temperature is lowered, no symmetry is lost when the temperature of the FQHE system is decreased. Instead of positional order that occurs in crystals, the order in the FQHE fluid, known as topological order (see Wen [24]), refers to the systematic way in which the electrons move around: the electrons are localized in the bulk, but net charge transport does occur at the edges of the sample, where, irrespective of the geometry of the sample, each edge states contributes one quantum of conductance $e^2/h$ to the total conductance. The reason that the quantum Hall fluid does not condense into a crystal when the temperature is lowered is that the mass of the constituents, the electrons, is too small, and quantum fluctuations prevent the FQHE fluid from becoming a solid. Additionally, the quantum Hall effects are home to a number of fascinating collective phenomena, such as fractionally charged particles, fractional statistics and even non-abelian statistics.

This thesis consists of two distinct but related parts:

- the first part focuses on Casimir operators of the algebra of projected density operators in the FQHE,
- the second part discusses the anyonic quasi-particle excitations of the quantum Hall fluid at fractional (Laughlin) filling factor.

In the low temperature limit $k_B T \ll \hbar \omega_C$, the Landau levels are filled from bottom to top. This means that, at fractional filling, the highest non-empty Landau level is fractionally filled. Because $k_B T \ll \hbar \omega_C$, all effective dynamics occur within this fractionally filled Landau level, where the kinetic energy is constant. Hence, the low energy excitations are governed by the electron-electron interactions, which can be completely described in terms of the density operators projected to the fractionally filled Landau level. The so obtained projected density operators form an algebra, first discovered by Girvin, MacDonald and Platzman [5], which is therefore also known as the GMP algebra. Understanding the algebra of projected density operators is of prime importance for understanding the FQHE fluid at low temperatures.

The aim of the first part of this thesis is to obtain a better understanding of the algebra of projected density operators, by computing several Casimir operators of the algebra. These are specific operators\(^1\) that commute with all the elements of the algebra. Common examples of Casimir operators are the total spin and total angular momentum operator, which are physically significant operators, and our initial conjecture was that so too are the Casimir operators of the projected density operator algebra.

Haldane [10] computed the second order Casimir of the projected density operator algebra. We will show how to generalize Haldane’s result up to fourth order and provide explicit expressions for the Casimir operators in terms of sums of products of permutation operators. The computed Casimir operators turn out to be constant on fixed-particle-number subspaces of Fock space, on

\(^1\)In a general Lie algebra $g$ with a countable basis $\{X_i\}, i \in I$, the second order Casimir $\zeta$ is the element of the universal enveloping algebra $U(g)$ of $g$ given by $\zeta = \sum_{i \in I} X_i X^i$, where $X^i$ is the dual vector of $X_i$ with respect to a fixed invariant bilinear form. There are additional Casimir elements for each order.
which each Casimir operator becomes a polynomial in the particle number. In these polynomial
expressions, a clear pattern appears that can be generalized up the the \( N_B \) order Casimir, where
\( N_B \) is the degeneracy of each Landau level.

After obtaining the expressions for the Casimirs, we will proceed with the second part of this
thesis, in which we attempt to give an exact description of the quasi-particle excitations of the
FQHE fluid at the Laughlin fractions \( \nu = \frac{1}{p} \), where \( p \) is an odd integer. It has been experimentally
verified (see Goldman and Su [9], and M. Heiblum et. al. [21]) that the quasi-particle excitations
carry fractional charge, and it is theoretically predicted that they obey anyonic statistics. Because
the quasi-particles are charged, it is expected (see for example Khare [13], Stern [22], Johnson and
Canright [12] or Lerda [17]) that the quasi-particles themselves feel the external magnetic field that
causes the FQHE.

We will construct exact energy eigenfunctions for non-interacting charged anyons in a uniform
perpendicular magnetic field. Our approach differs from Laughlin’s in that we take the anyon to
be a fundamental particle, instead of a collective excitation. This means that we are providing an
effective description of the system of quasi-particle excitation of the FQHE fluid.

Wilczek ([25] and [26]) introduced anyons as flux tube-charged particle composites. Wilczek’s
description allows for two equivalent ways of describing anyons. One sees the anyon as a bosonic
charge to which a flux tube is attached by coupling the charge to a Chern-Simons gauge field (see
e.g. Dunne [2]). The other involves a singular gauge transformation, which gauges away the vector
potential due to the flux tube and transforms the flux tube-charged particle composites into non-
interacting charged particles that obey anyonic exchange statistics. It is this second description that
is closest to the formalism implemented in this thesis: we will construct wavefunctions that obey
anyonic exchange statistics and are defined on the universal cover of the configuration space of \( N \)
identical particles as introduced by Leinaas and Myrheim [16].

Motivated by the single-particle energy eigenstates for a charged particle in a uniform perpen-
dicular magnetic field, we come up with a general ansatz, and develop a theory with which we
can construct exact \( N \)-particle energy eigenfunctions for non-interacting charged anyons in a uniform
perpendicular magnetic field. The energy spectrum of the found eigenstates is the same as
that of \( N \) non-interacting charged fermions in a uniform perpendicular magnetic field, given by
\[ E_{N,K} = \frac{1}{2} \hbar \omega_C (N + 2K), \]
where \( K \in \mathbb{Z}_{\geq 0} \) labels the excitation energy that the system has on top of its ground state energy \( \frac{1}{2} N \hbar \omega_C \). Observing that the energy eigenvalues do not explicitly depend
on the statistical parameter \( \alpha \) that determines the phase \( e^{i\alpha \pi} \) picked up by the wavefunction under
the exchange of two identical anyons, we further diagonalize the energy eigenstates with respect to
the (\( z \)-component of the) total angular momentum operator, the spectrum of which \( \alpha \) explicitly
depend on \( \alpha \). We conclude that the total angular momentum operator could in principle lead to a
method for directly measuring the anyonic statistical parameter \( \alpha \).

The main advantage of our approach is that it provides exact anyonic energy eigenfunctions.
The disadvantage, as shown by comparing to Khare [13], Johnson and Canright [12] or Lerda [17],
is that we are not able to generate a complete set of eigenstates\(^2\). The latter reference of the three
provides the most extensive list of anyonic energy eigenstates, which were obtained by a method
that is similar to ours, but with the use of a more general ansatz: Lerda considers what he calls
\textit{type I} and \textit{type II} wavefunctions, the former of which coincide with the wavefunctions that we find.
However, as discussed in the conclusion, we find additional wavefunctions of type I that Lerda does
not produce. We are therefore led to conclude that, despite the fact that it is less general, our
approach is more thorough than Lerda’s. This brings us to the further conclusion that it might be
viable to repeat our approach for Lerda’s type II wavefunctions.

\(^2\)As of today, no complete exact solution to the \( N \) non-interacting charged anyon problem in a uniform perpendicular
magnetic field is known, see Khare [13].
The outline of this thesis is as follows. Chapter 2 provides an introduction to the Hall effect, followed by a mathematically rigorous description of a single charged particle confined to two dimensions in a uniform perpendicular magnetic field. Chapter 3 gives an introduction to both the IQHE and the FQHE. In Chapters 4 and 5 the Casimir operators in the FQHE are discussed: first on the infinite plane, and then on the torus. Chapter 6 is about anyons in the FQHE, and the construction of exact anyonic energy eigenfunctions. Finally, in Chapter 7 we present our conclusions and provide an outlook for possible further research.
Chapter 2

Free Electrons in a Magnetic Field

In order to get acquainted with the Quantum Hall Effects, it is necessary to obtain a firm understanding of a simple toy model — that of a single electron moving freely in a constant perpendicular magnetic field — which, despite its simplicity, exhibits most of the key features underlying the quantum Hall effects.

In the first part of this chapter, we will give a classical description of a free electron in a magnetic field. In the second part, we will set the stage for the remainder of this thesis by describing the same system from a quantum mechanical perspective, and in doing so, we will keep a high level of mathematical rigour.

2.1 The Hall Effect

Consider a gas of non-interacting electrons confined to an impurity free two-dimensional plane, which we take to be the $x$-$y$ plane. Suppose we add a uniform magnetic field $B = Be_z$ that points in the $+z$ direction. It turns out that this setup leads to very peculiar behavior of the electrons: especially when a current is applied to the 2D surface. Even though the electrons do not interact with anything, they still experience some kind of resistance; i.e. a tendency of the material (in this case meaning the conduction plate together with the magnetic field) to stop the current from flowing. This resistance is called the Hall resistance. It is different from normal electrical resistance in the sense that the corresponding voltage difference is perpendicular to the direction of the current.

2.1.1 Resistance and Resistivity

In ordinary conducting materials, a given voltage difference $V$ generates a current $I$, and both are related by Ohm’s law:

$$V = IR.$$  \hfill (2.1)

The resistance $R$ signifies to which extend the conducting material prevents the current from flowing.

Ohm’s law is only applicable to conducting materials of a particular geometry wherein the current is restricted to flow in one specific direction (e.g. wires), in which case $I$ measures the amount of charge passing through a fixed area per unit of time. More generally, we can speak of a current density $J$, which, in case of a conducting wire is the current per unit area $I/A$. The current density also makes sense in materials where the charge can flow in more than one direction.

When there are many electrons that contribute to the total charge flow, and we are looking at distance scales on which the electron distribution can be approximated by a three dimensional density function $n_{el}(r, t)$, the current density $J$ due to electrons moving around is given by

$$J = -e n_{el}(r, t) v(r, t),$$  \hfill (2.2)
2.1. THE HALL EFFECT

i.e. the product of the electron charge \(-e\), the electron density \(n_{el}(r, t)\), and the electron velocity vector field \(v(r, t)\), which are functions of the position \(r\) and the time \(t\). In this more general setup, Ohm’s law takes the form:

\[ E = \rho J. \quad (2.3) \]

\(\rho\) is called the resistivity of the material. Contrary to resistance, resistivity is, generally speaking, a local property of the material. In the case of a conducting wire of length \(l\), cross sectional area \(A\) and uniform resistivity \(\rho\) subject to a constant longitudinal electric field with field strength \(E\), we have \(V = El\). Combining (2.1) and (2.3) yields,

\[ R = \rho \frac{l}{A}. \quad (2.4) \]

The above formula intuitively makes sense: decreasing the cross-sectional area or increasing the length of the wire should indeed increase the resistance.

2.1.2 Cyclotron Motion

Let us focus on the situation of a single electron confined to move in the \(x\)-\(y\) plane under the influence of a perpendicular magnetic field \(B = B_0 e_z\), without any external electric field being present. The electron experiences a magnetic Lorentz force \(F_m\) given by

\[ F_m = -e v \times B. \quad (2.5) \]

The force \(F_m\) also lies in the \(x\)-\(y\) plane, and it is perpendicular to \(v\). It has a constant magnitude of \(F_m = e v B_0\). As a consequence, any electron with a non-zero velocity will display a circular motion, called a cyclotron orbit. We will adopt the following coordinates for such an orbit: \(r\) denotes the position of the electron, \(R\) the centre of the cyclotron orbit, and \(\eta\) is the vector that points from \(R\) to \(r\) – see Figure 2.1.

![Figure 2.1: Electron at position \(r\) making a cyclotron orbit. The magnetic field \(B\) points out of the page.](image)

In any circular motion of radius \(\eta\) with constant speed \(v\), the acceleration towards the centre is given by \(a = \frac{\omega^2 \eta}{\eta}\). By equating \(m a = F_m\), we get

\[ v = \frac{e B}{m} \eta = \omega_C \eta, \quad (2.6) \]

where

\[ \omega_C := \frac{e B}{m} \quad (2.7) \]
2.1. THE HALL EFFECT

is called the cyclotron frequency. It is the angular frequency at which the electron preforms cyclotron orbits. Contrary to for example planetary orbits, the frequency of a cyclotron orbit is independent of the radius and the velocity of the orbiting particle: it only depends on the strength of the magnetic field.

From equation (2.6), we see that the radius and the speed are proportional to each other. Hence, there are two dynamical variables that describe the cyclotron orbit completely: the speed \( v \) at which the orbit is performed — or equivalently, the radius \( \eta \) — and the position \( R \) of the centre of the cyclotron orbit.

### 2.1.3 The Classical Hall Effect

In addition to the magnetic field \( B = Be_x \), let us turn on an external electric field \( E = E_x e_x + E_y e_y \) pointing in the \( x-y \) plane. The equation of motion for a single electron is

\[
\dot{p} = F = -eE - e v \times B.
\] (2.8)

That is,

\[
\begin{pmatrix}
\dot{p}_x \\
\dot{p}_y
\end{pmatrix}
= -e
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix}
+ \omega_c
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
p_x \\
p_y
\end{pmatrix},
\] (2.9)

where we have left out the \( z \)-component because the electron is confined to the \( x-y \) plane. Taking the time derivative of the above equation yields

\[
\begin{pmatrix}
\dot{p}_x \\
\dot{p}_y
\end{pmatrix}
= \omega_c
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
p_x \\
p_y
\end{pmatrix},
\] (2.10)

which is solved by

\[
\begin{pmatrix}
\dot{p}_x \\
\dot{p}_y
\end{pmatrix}
= R
\begin{pmatrix}
\cos(\omega_c t) \\
\sin(\omega_c t)
\end{pmatrix},
\] (2.10)

where \( R \) is an integration constant. Plugging equation (2.10) into equation (2.9) and solving for \( p \) gives the general solution to equation (2.9):

\[
\begin{pmatrix}
p_x \\
p_y
\end{pmatrix}
= R
\begin{pmatrix}
\sin(\omega_c t) \\
-\cos(\omega_c t)
\end{pmatrix}
+ e
\begin{pmatrix}
E_y \\
-E_x
\end{pmatrix}
= p_c + p_H.
\]

From the solution above, we see that the motion of the electron is a superposition of a circular cyclotron motion \( p_c \) and a linear motion \( p_H = mv_H \), where \( v_H \) is the Hall drift velocity,

\[
v_H := \frac{e}{m \omega_c} \begin{pmatrix}
E_y \\
-E_x
\end{pmatrix} = \frac{E \times B}{B^2}.
\] (2.11)

The linear term \( p_H \) is responsible for any net electrical current in the material: it specifies the direction in which the charge carriers — in this case electrons — move.

**Remark 2.1.** There are two important things to note. First, contrary to ordinary electrical systems, the current flows in, or opposite to, the direction of \( v_H \), which is perpendicular to the direction of \( E \). This phenomenon is called the Hall drift. It is the above mechanism that relates an electric current to a perpendicular voltage difference, which in turn gives rise to the Hall resistance.

Second, the Hall drift velocity only depends on the electric and magnetic fields, and not on the mass or charge of the charge carriers. In particular, this means that if the charge carriers would have been positively charged, the current \( J = +en_d v_H \) would have flowed in the opposite direction.
Another way of thinking about the Hall resistance is the following. Suppose we have a current $I$ in the $+x$ direction (meaning that the electrons, which have a negative charge, move in the $-x$ direction). The magnetic Lorentz force acting on the electrons points in the $-y$ direction: it tries to stop the electrons from moving in a straight line. This is the tendency of the system to resist the current. To ensure that the current is maintained, we need to apply an electric field in the $-y$ direction (perpendicular to $I$) that counters the magnetic Lorentz force.

In more mathematical language; we are looking for steady currents, i.e. currents for which the electrons move at a constant velocity. That is to say that we are looking for stationary ($\dot{p} = 0$) solutions to equation (2.8), i.e. $0 = -eE - e\mathbf{v} \times \mathbf{B}$. These are given by $\mathbf{v} = \mathbf{v}_H$, or, in terms of the current density (2.2),

$$
\begin{pmatrix}
J_x \\
J_y
\end{pmatrix} = -en_{el} \mathbf{v}_H = \begin{pmatrix}
e n_{el} & -\frac{en_{el}}{B} \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix},
$$

(2.12)

Comparing to equation (2.3), we see that the matrix in equation (2.12) is the inverse of the resistivity, also called the conductivity. Its inverse $\rho$ is given by

$$
\rho = \rho_H \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
$$

(2.13)

where

$$
\rho_H := \frac{B}{e n_{el}}
$$

(2.14)

is the Hall resistivity. The fact that the resistivity matrix is off-diagonal reflects the fact that the currents are induced by transverse electric fields. The ordinary resistivity terms will appear on the diagonal of $\rho$.

For example, if we alter equation (2.9) such that it incorporates interactions of the electrons with impurities in the sample – e.g. by adding a term $-p/\tau$ which causes $p$ to decrease over time,

$$
\begin{pmatrix}
\dot{p}_x \\
\dot{p}_y
\end{pmatrix} = -e \begin{pmatrix}
E_x \\
E_y
\end{pmatrix} + \omega_c \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
p_x \\
p_y
\end{pmatrix} - \frac{1}{\tau} \begin{pmatrix}
p_x \\
p_y
\end{pmatrix},
$$

(2.15)

then the steady currents will be given by

$$
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix} = \begin{pmatrix}
\rho_L & \rho_H \\
-\rho_H & \rho_L
\end{pmatrix} \begin{pmatrix}
J_x \\
J_y
\end{pmatrix},
$$

where $\rho_L = \frac{m}{n_{el} \tau \sigma}$ is the longitudinal resistivity. This is known as the Drude model.

The Hall effect was first measured by Edwin Herbert Hall in 1879. When passing a current $I$ through a thin metallic plate of thickness $l_z$, he measured a transverse voltage difference given by

$$
V_H = -I \frac{\rho_H}{l_z},
$$

(2.16)

corresponding to a Hall resistance of

$$
R_H = -\frac{\rho_H}{l_z} = \frac{B}{-en_{el} l_z},
$$

(2.17)

where $n_{el}$ now refers to the average electron density. The minus sign is due to the fact that that the charge carriers are negatively charged. Indeed, if the charge carriers would have been positively charged, then, due to remark 2.1, $\mathbf{v}_H$ would have been the same, but the sign in equation (2.12) would have changed. Hence, we conclude that for charge carriers of charge $Q$ inside a conductor of thickness $l_z$, the measured Hall resistance is

$$
R_H = \frac{B}{Q n_{el} l_z}.
$$

(2.18)

As a matter of fact, one of the applications of the Hall effect is to determine the sign of the charge of the charge carriers by determining the sign of the Hall resistance.
Remark 2.2. If \( n_{el} \) is independent of the \( z \) direction, and the length \( l_z \) is constant and small enough for us to be able to treat the system as being effectively two-dimensional, then \( n_{el} l_z \) is equal to the two-dimensional electron density \( n_{el}^{2D} \), and the Hall resistance (2.17) becomes

\[
R_H = \frac{B}{-en_{el}^{2D}},
\]

which is in agreement with formula (1.1) in the Introduction.

The measured value of the Hall resistance found in (2.17) can be understood in terms of the Hall resistivity as follows. Let us assume that the conducting plate has dimensions length \( l_x \) times width \( l_y \) times thickness \( l_z \). Inject a uniform current \( I = I_x e_x = J_x l_y l_z e_x \) through the plate. According to equation (2.12), this current comes with a transverse electric field \( E = E_y e_y \), where

\[
E_y = -\rho_H J_x.
\]

The Hall voltage difference \( V_H \) is related to the electric field \( E = E_y e_y \) by \( V_H = E_y l_y \). Plugging this into the equation above yields

\[
V_H = -I_x \frac{\rho_H}{l_z},
\]

which is in agreement with equation (2.16).

Remark 2.3. The Hall resistance is independent of both the length \( l_x \) and the width \( l_y \) of the sample: only the thickness \( l_z \) plays a role. This is a feature that holds for any kind of sample, not only for rectangular ones. As a matter of fact, it has been experimentally verified that the measured Hall resistance is independent of the shape of the sample as seen from the \( +z \) direction.

2.1.4 The Quantum Hall Effects

In case of the Classical Hall Effect, the Hall resistance (2.17) is proportional to the magnetic field,

\[
R_H \propto \frac{B}{e n_{el}}.
\] (2.19)

However, in 1980, Klaus von Klitzing measured a Hall resistance curve in a certain MOSFET (metal-oxide-semiconductor field-effect transistor) that is not linear in the magnetic field, but instead has a stair-case like shape as shown in Figure 2.2. This phenomenon, where the Hall resistance curve remains constant for certain values of \( B \), is commonly referred to as quantization of the Hall resistance.

It turns out that there are actually two different processes that cause the formation of plateaus in the Hall resistance curve. For values of \( B \) for which the Hall resistivity equals

\[
\rho_H = \frac{h}{e^2 \nu},
\] (2.20)

where \( \nu \) is a positive integer, the underlying phenomenon goes by the name integer quantum Hall effect, or IQHE for short. The discovery of the IQHE is attributed to Klaus von Klitzing, for which he was awarded the Nobel Prize in 1985. We will discuss the IQHE in Section 3.1.

There is also the fractional quantum Hall effect (FQHE), corresponding to plateaus in the Hall resistance at resistivities for which \( \nu \) in (2.20) is fractional. Understanding the FQHE requires an understanding of the IQHE, so we will postpone any further discussion on the FQHE until we have treated the IQHE. See Section 3.2 for the details.

However, before we can start discussing either the IQHE or the FQHE, we need to understand how 2D electrons in a perpendicular magnetic field can be described within the framework of quantum mechanics. This requires the setup of the appropriate mathematical formalism, which is the topic of the next section.
2.2 A QUANTUM MECHANICAL APPROACH

Consider a free electron with charge \(-e\) confined to move in the \(x-y\) plane, subject to a perpendicular magnetic field \(B = Be_z\). In order to describe this system quantum mechanically, we need a Hamiltonian. We will take a small detour and set up the appropriate Lagrangian and Hamiltonian formalism for free electrons — in 3D — subject to a generic electric and a magnetic field, before turning to the 2D case.

2.2.1 The Lagrangian and Hamiltonian Formalisms

Suppose we have a charged particle with charge \(-e\) and mass \(m\). The particle moves in configuration space \(Q\), which can be any submanifold of \(R^3\). Furthermore, we define the phase space \(TQ\) to be the tangent bundle of \(Q\). We will adopt local coordinates \((r,v)\) on \(TQ\), where \(v_i\) is the \(i\)-component of \(\frac{\partial}{\partial r_i}\). Physical observables are functions on the phase space; for example, we have the mechanical momentum of the particle, which is defined by

\[
\Pi := mv. \tag{2.21}
\]

Let us denote the path the particle follows through \(TQ\) by \((r(t), \dot{r}(t))\). The movement of the particle in question is dictated by Newton’s law:

\[
F = m\ddot{r}, \tag{2.22}
\]

where, in the system that we are considering, the force \(F\) is the Lorentz force, given by,

\[
F_L = -e(E + \dot{r} \times B). \tag{2.23}
\]

The above equation of motion can be derived from a action,

\[
S[r(t)] = \int dt L(r(t), \dot{r}(t)),
\]
where \( L = L(r, v) \) is the Lagrangian, which we will specify below. Because (2.22) is a second order differential equation, any point \((r_0, v_0)\) in phase space together with a choice of a moment in time \(t_0\) determines a unique curve \( r(t) \) such that \( (r(t_0), \dot{r}(t_0)) = (r_0, v_0) \), fixing the complete future and past of the particle in question. Such a point \((r_0, v_0)\) is called a classical state.

The action is a functional on the space of paths \( r(t) \) in \( Q \). Fixing two endpoints \( r_0 \) and \( r_1 \), the action principle says that the particle will go from \( r_0 \) to \( r_1 \) in such a way that its path extremizes the action \( S \). This is equivalent to saying that \( r(t) \) satisfies the Euler-Lagrange equations,

\[
\frac{\partial L}{\partial r} \bigg|_{(r(t), \dot{r}(t))} = \frac{d}{dt} \frac{\partial L}{\partial v} \bigg|_{(r(t), \dot{r}(t))}.
\]

Next, we deliberately choose a Lagrangian \( L \) such that equation (2.24) reduces to equation (2.22) with \( F = F_L \). A Lagrangian that fulfills this condition is

\[
L(r, v) = \frac{1}{2} m v^2 - eA(r) \cdot v + eV(r),
\]

where \( V \) and \( A \) are the electric and magnetic vector potential — which we will elaborate on in the next section — respectively, and \( v^2 = v \cdot v \). The dot refers to the standard Euclidean metric on \( T^*Q \). Because the system that we want to describe is of a quantum mechanical nature, we want to work in the Hamiltonian formalism. The Hamiltonian can be obtained by preforming a fiberwise Legendre transformation to the Lagrangian in the variables \( v_1, \ldots, v_3 \), making it a function on the cotangent bundle \( T^*Q \). Note that, for fixed \( r \), \( L \) is quadratic (hence convex) in the velocity, so we can indeed perform a fiberwise Legendre transformation.

Having done the transformation, \( H \) is now a function of \( p \) instead of \( v \), where \( p \) is given by:

\[
p := \frac{\partial L}{\partial v} = mv - eA(r).
\]

\( p \) is called the canonical momentum; it is not a physical quantity, because it depends on the gauge dependent vector potential \( A \). The Hamiltonian is given by

\[
H(r, p) = \frac{1}{2m} \left[ p + eA(r) \right]^2 - eV(r).
\]

Before we proceed to canonical quantization, let us take some time to review the concept of gauge fields within the context of electromagnetism.

### 2.2.2 Gauge Fields in Electromagnetism

The electric and magnetic field are vector fields on \( \mathbb{R}^3 \). In the presence of a charge density \( \rho \) and current density \( J \), they satisfy Maxwell’s equations:

\[
\nabla \cdot E = \frac{\rho}{\varepsilon_0} \quad (2.27)
\]

\[
\nabla \cdot B = 0 \quad (2.28)
\]

\[
\nabla \times E = -\frac{\partial B}{\partial t} \quad (2.29)
\]

\[
\nabla \times B = \mu_0 J + \rho_0 \varepsilon_0 \frac{\partial E}{\partial t} \quad (2.30)
\]

The first two equations are Gauss’s law for the electric and the magnetic field. The third equation is Faraday’s law of induction, and the last equation is Ampere’s circuitual law with Maxwell’s correction. Loosely speaking, equations (2.27) and (2.30) relate the electric and magnetic field to their respective
When going from classical to quantum mechanics, the degrees of freedom make sense in any dimension other than three.

We now turn to the co-chain complex of differential forms on \( \mathbb{R}^3 \). Under the identification of vector fields with 1 and 2-forms on \( \mathbb{R}^3 \), given by: \( e_i \mapsto dx^i \) and \( e_i \mapsto \frac{1}{2} \epsilon_{ijk} dx^j \wedge dx^k \), we can identify the exterior derivative \( d \) with the gradient, rotation and divergence operators, as follows:

\[
0 \to \Omega^0(\mathbb{R}^3) \xrightarrow{d_0 = \text{grad}} \Omega^1(\mathbb{R}^3) \xrightarrow{d_1 = \text{rot}} \Omega^2(\mathbb{R}^3) \xrightarrow{d_2 = \text{div}} \Omega^3(\mathbb{R}^3) \to 0
\]

By de-Rham’s theorem, the \( k \)-th de-Rham cohomology group \( H^k_{dR}(\mathbb{R}^3) \) of \( \mathbb{R}^3 \) is isomorphic to the \( k \)-th singular cohomology group \( H^k_\Delta(\mathbb{R}^3) \) of \( \mathbb{R}^3 \):

\[
H^k_{dR}(\mathbb{R}^3) \cong H^k_\Delta(\mathbb{R}^3).
\]

Also, by contractibility of \( \mathbb{R}^3 \), \( H^k_\Delta(\mathbb{R}^3) \) can easily be computed to be equal to \( \mathbb{R} \) when \( k = 0 \), and 0 otherwise. Thus, we conclude that

\[
H^k_{dR}(\mathbb{R}^3) := \frac{\text{Ker}(d_k)}{\text{Im}(d_{k-1})} = \begin{cases} \mathbb{R} & \text{when } k = 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.31)
\]

Now, equation (2.28) precisely states that \( B \in \text{Ker}(d_2) \), which, by equation (2.31) equals \( \text{Im}(d_1) \). Thus, there exists another vector field \( A \in \Omega^1(\mathbb{R}^3) \), such that

\[
B = \nabla \times A. \tag{2.32}
\]

This vector field is called the magnetic vector potential, or vector potential for short. Additionally, from equation (2.29), we have \( E + \partial A / \partial t \in \text{Ker}(d_1) \), which, again by equation (2.31), is equal to \( \text{Im}(d_0) \). Hence, there also exists a function \( V \) such that

\[
E = -\nabla V - \frac{\partial A}{\partial t}.
\]

\( V \) is called the electric potential.

Because the exterior derivatives \( d_0 \) and \( d_1 \) have a nonzero kernel, there is a choice in which \( A \) and \( V \) to pick. Given two vector potentials \( A \) and \( A' \) that satisfy \( \nabla \times A = \nabla \times A' = B \), the difference \( A - A' \in \text{Ker}(d_1) = \text{Im}(d_0) \), i.e. \( A' = A + \nabla \chi \) for some function \( \chi \). Likewise, two different \( V \)'s satisfying \( \nabla V = E + \partial A / \partial t \) differ by a constant (since \( \text{Ker}(d_0) = H^0_{dR}(\mathbb{R}^3) = \mathbb{R} \)).

A particular choice of the electric and the magnetic vector potentials is called a gauge. The fields \( A \) and \( V \) are called gauge fields. They are not physical fields in the sense that there is an arbitrariness to them, contrary to the electric and magnetic fields, which are completely determined by their physical sources \( \rho \) and \( J \). Thus, working with gauge fields is always a little bit tricky, because there are many gauge fields that correspond to the same physical situation, and we require that every measurable quantity is independent of the choice of the gauge.

**Remark 2.4.** When we restrict ourselves to two-dimensional systems, keep in mind that the electric and magnetic field should always be thought of as three-dimensional fields existing on all of \( \mathbb{R}^3 \) wherein the two-dimensional configuration space \( \mathcal{Q} \) is embedded, because Maxwell’s equations do not make sense in any dimension other than three.

### 2.2.3 Canonical Quantization and Gauge Invariance

When going from classical to quantum mechanics, the degrees of freedom \( r \) and \( p \) are promoted to self-adjoint operators on a — soon to be specified — Hilbert space \( \mathcal{H} \),

\[
r \leadsto r \text{ seen as a multiplication operator}
\]

\[
p \leadsto -i\hbar \nabla
\]
satisfying the commutation relations

\[ [r_i, p_j] = i\hbar \delta_{ij} \]

for \( i = 1, 2, 3 \), with all other commutators vanishing.

Pure states are represented by vectors \( \psi \in \mathcal{H} \) with unit norm. Their time dependence is governed by the *Schrodinger equation*,

\[
  i\hbar \frac{\partial \psi}{\partial t} = H\psi,
\]

where \( H \) is the Hamiltonian, which in our case is given by equation (2.26).

For any operator \( O \) on \( \mathcal{H} \), the *expectation value* of \( O \) with respect to \( \psi \) is given by

\[
  \langle \psi, O\psi \rangle,
\]

where \( \langle, \rangle \) denotes the inner product on \( \mathcal{H} \). The expectation values are the measurable quantities that can be obtained from a quantum mechanical system.

Back to the quantum Hall effect. Consider an electron with charge \(-e\) confined to a two-dimensional surface under the influence of a constant perpendicular magnetic field \( B = B\hat{e}_z \). For now, we will set \( E = 0 \). Because of the gauge freedom in electrodynamics, a fixed physical state can be described mathematically in many different ways, and we require that all these mathematical descriptions are equivalent, in the sense that the extractable physical information should be the same for each description.

More concretely, *after choosing a gauge*, we obtain an isomorphism between the Hilbert space of states \( \mathcal{H} \) and the space of square-integrable complex valued functions on \( \mathbb{R}^2 \);

\[
  \mathcal{H} \cong L^2(\mathbb{R}^2, \mathbb{C}),
\]

the mechanical momentum \( \Pi \) (2.21) becomes

\[
  \Pi = p + eA(r),
\]

and the Hamiltonian takes the form

\[
  H = \frac{\Pi^2}{2m} = \frac{1}{2m} \left[ p + eA(r) \right]^2.
\]

If we would have chosen a different gauge, then both the isomorphism in (2.34) and the explicit formulas for the mechanical momentum (2.35) and the Hamiltonian (2.36) would have been different.

Because we want different gauges to describe the same physics, we require that to each gauge transformation

\[
  A \rightarrow A' = A + \nabla \chi
\]

there corresponds a unitary operator \( U : \mathcal{H} \rightarrow \mathcal{H}, \) such that

\[
  \Pi' = p + eA(r) + e\nabla \chi(r) = U\Pi U^\dagger,
\]

and, consequently

\[
  H' = \frac{1}{2m} \left[ p + eA(r) + e\nabla \chi(r) \right]^2 = UHU^\dagger.
\]

For, given such a unitary transformation \( U \), we can go from the \( A \) gauge to the \( A' \) gauge by applying \( U \) to all the functions in \( L^2(\mathbb{R}^2) \). Indeed, if \( \psi \in L^2(\mathbb{R}^2) \) describes some fixed physical system in the \( A \) gauge, and \( \psi' := U\psi \), then, by unitarity of \( U \),

\[
  \langle \psi, \Pi \psi \rangle = \langle \psi', \Pi' \psi' \rangle.
\]
As a matter of fact, for any physical operator \(O\), which is always an analytic function of \(r\) and \(\Pi\), we have \(O' = \mathbf{U}O\mathbf{U}^\dagger\), and therefore
\[
\langle \psi, O\psi \rangle = \langle \psi', O'\psi' \rangle.
\]
Moreover, the time evolution of \(\psi\) and \(\psi'\) is such that, even at later times, \(\psi(t) = \exp(-iHt/\hbar)\psi\) and \(\psi'(t) = \exp(-iH't/\hbar)\psi'\) are related by \(\psi'(t) = \mathbf{U}\psi(t)\). We conclude that, for all times \(t\),
\[
\langle \psi(t), O\psi(t) \rangle = \langle \psi'(t), O'\psi'(t) \rangle,
\]
which is another way of saying that \(\{O, \psi\} \) and \(\{O', \psi'\} \) describe the same physical system.

**Lemma 2.1.** The unitary transformation \(\mathbf{U}\) that satisfies equations (2.37) and (2.38) is the multiplication operator:
\[
\mathbf{U}(r) := \exp\left(\frac{-ie\chi(r)}{\hbar}\right)
\]

**Proof.** To see that (2.39) indeed solves equations (2.37) and (2.38), observe that we have the following operator equation, meaning that it holds whenever applied to an arbitrary wavefunction.
\[
p_i\mathbf{U}^\dagger(r) = -i\hbar\partial_i e^{ie\chi(r)/\hbar} = \mathbf{U}^\dagger(r)\left[ e(\partial_i\chi)(r) - i\hbar\partial_i \right] = \mathbf{U}^\dagger(r)\left[ e(\partial_i\chi)(r) + p_i \right].
\]
Therefore,
\[
\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}\left[ \mathbf{p} + e\mathbf{A}(r) \right] \mathbf{U}^\dagger = e\nabla\chi(r) + \mathbf{p} + e\mathbf{A}(r) = \mathbf{I}',
\]
and
\[
\mathbf{U}H\mathbf{U}^\dagger = \frac{1}{2m}\left[ U(\mathbf{p} + e\mathbf{A}(r))\mathbf{U}^\dagger \right]^2 = \frac{1}{2m}\left[ \mathbf{p} + e\mathbf{A}(r) + e\nabla\chi(r) \right]^2 = H'.
\]

**Example 2.1.** Consider the two most frequently used gauges: the symmetric gauge
\[
\mathbf{A}_S = \frac{B}{2}(-y\hat{e}_x + x\hat{e}_y),
\]
and the Landau gauge
\[
\mathbf{A}_L = -By\hat{e}_x.
\]
The two gauges are related by the following equation,
\[
\mathbf{A}_S = \mathbf{A}_L + \nabla\chi_{SL},
\]
where
\[
\chi_{SL}(x, y) = \frac{B}{2}xy.
\]
According to equation (2.39), the unitary operator \(\mathbf{U} : L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C})\) that transforms wavefunctions from the Landau gauge to the symmetric gauge is given by
\[
\mathbf{U}_{SL}(x, y) = \exp\left(\frac{-ieBxy}{2\hbar}\right).
\]

To summarize, we have found a unitary operator \(\mathbf{U} : L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C})\) that allows us to go from one gauge to the other. Because there are infinitely many vector potentials \(\mathbf{A}\) that generate the same magnetic field \(\mathbf{B} = B\hat{e}_z = \nabla \times \mathbf{A}\), to a fixed physical state there corresponds an infinite set of wavefunctions, each of which describes this particular physical state in a different gauge. It seems more sensible to assign a single mathematical entity to this fixed physical state, instead of an infinite set of wavefunctions. As it turns out, a physical state is naturally described by a section of a complex line bundle \(\mathcal{L}\) over \(\mathbb{R}^2\), where a choice of the gauge corresponds to a global trivialization of the line bundle. The construction of this line bundle is the topic of the next section.
2.2.4 Construction of the One-Particle Hilbert Space

We will now construct the aforementioned line bundle $\mathcal{L}$ over $\mathbb{R}^2$. A physical one-particle state will then be described by a square-integrable section of this line bundle. The line bundle will be equipped with a connection such that the connection 1-form associated to a particular trivialization corresponds to the magnetic vector potential $A = A_x \, dx + A_y \, dy$ in a specific gauge. Consequently, the curvature 2-form associated to the connection will be equal to the magnetic field 2-form $B = dA$.

The construction of $\mathcal{L}$ seems fairly abstract at first, but when most of the definitions are done the object is actually quite manageable.

**Definition 2.1.** Let $PM$ denote the space of piecewise smooth paths $\gamma : [0, 1] \to \mathbb{R}^2$ that start at $\gamma(0) = 0$. We define the line bundle $\mathcal{L}$ as the set $PM \times \mathbb{C}$ modulo the following equivalence relation,

$$
(\gamma_1, z_1) \sim (\gamma_2, z_2) \iff \gamma_1(1) = \gamma_2(1) \text{ and } z_2 = \exp \left[ 2\pi i \kappa \int_{\gamma_2, \gamma_1} B \right] z_1,
$$

(2.44)

where $\kappa$ is a constant that will be fixed later on. The equivalence class of the element $(\gamma, z)$ will be denoted by $[\gamma, z]$.

**Notation:** in the above equivalence relation, the integral is over the domain $D$ enclosed by the curves $\gamma_1$ and $\gamma_2$, and oriented in such a way that the orientation on the boundary $\partial D$ corresponds to first going along $\gamma_1$, and then going back along $\gamma_2$. Hence, by Stokes theorem, we could have equally well written

$$
(\gamma_1, z_1) \sim (\gamma_2, z_2) \iff \gamma_1(1) = \gamma_2(1) \text{ and } z_2 = \exp \left[ 2\pi i \kappa \left( \int_{\gamma_1} A - \int_{\gamma_2} A \right) \right] z_1,
$$

for any $A$ with $B = dA$.

Despite the use of the word line bundle, we have not yet shown that $\mathcal{L}$ actually is a line bundle over $\mathbb{R}^2$.

**Definition 2.2.** Define the projection

$$
\pi : \mathcal{L} \to \mathbb{R}^2 \quad \text{by} \quad \pi([\gamma, z]) = \gamma(1).
$$

This makes the fibre $\mathcal{L}_r$ at the point $r \in \mathbb{R}^2$ the set of equivalence classes of curves ending at $\gamma(1) = r$.

Next, we will construct global trivializations of $\mathcal{L}$, and in the process we will also show that each $\mathcal{L}_r$ is actually a one-dimensional complex vector space. A trivialization of $\mathcal{L}$ will be given by a set of paths $\beta_r : [0, 1] \to \mathbb{R}^2$ for each point $r \in \mathbb{R}^2$, satisfying $\beta_r(0) = 0$ and $\beta_r(1) = r$, such that the paths $\beta_r$ vary nicely with the endpoint $r$. Given such a set $\{\beta_r : r \in \mathbb{R}^2\}$, for fixed $r \in \mathbb{R}^2$, we can then construct a map $\lambda_r : \mathcal{L}_r \to \mathbb{C}$, defined by

$$
\lambda_r([\gamma, z]) := \exp \left[ 2\pi i \kappa \int_{\beta_r, \gamma} B \right] z.
$$

(2.45)

**Lemma 2.2.** For each $r \in \mathbb{R}^2$, the map $\lambda_r : \mathcal{L}_r \to \mathbb{C}$ given by equation (2.45) is well-defined. It is also a bijection.

**Proof.** Suppose that $(\gamma_1, z_1) \sim (\gamma_2, z_2)$. Then,

$$
\lambda_r((\gamma_2, z_2)) = \exp \left[ 2\pi i \kappa \int_{\beta_r, \gamma_2} B \right] z_2 = \exp \left[ 2\pi i \kappa \int_{\beta_r, \gamma_2} B \right] \exp \left[ 2\pi i \kappa \int_{\gamma_2, \gamma_1} B \right] z_1
$$

$$
= \exp \left[ 2\pi i \kappa \int_{\beta_r, \gamma_1} B \right] z_1 = \lambda_r((\gamma_1, z_1)).
$$

Also, for any fixed path $\gamma$, $\lambda_r([\gamma, z] : z \in \mathbb{C}) = \mathbb{C}$, which implies surjectivity. For injectivity, observe that $\exp(2\pi i \kappa \int_{\beta_r, \gamma_1} B) z_1 = \exp(2\pi i \kappa \int_{\beta_r, \gamma_2} B) z_2$ precisely when $(\gamma_1, z_1) \sim (\gamma_2, z_2)$. 

\[\square\]
We now turn $\mathcal{L}_r$ into a $\mathbb{C}$-vector space such that $\lambda_r$ becomes a $\mathbb{C}$-linear isomorphism.

**Definition 2.3.** Define addition on $\mathcal{L}_r$ by

$$\gamma_1, z_1 + \gamma_2, z_2 := \gamma_1, z_1 + z_2 \exp \left( 2\pi i \kappa \int_{\gamma_1, \gamma_2} B \right),$$

and multiplication by a complex number $c$ as

$$c[\gamma, z] := [\gamma, cz].$$

The last definition should not come as a surprise. The first one simply ensures additivity of $\lambda_r$. At first sight, it might look like a non-abelian form of addition, but because $\lambda_r$ maps bijectively into $\mathbb{C}$, which is abelian, so is the sum defined above.

**Corollary 2.1.** For each choice of paths $\{\beta_r : r \in \mathbb{R}^2\}$, using the $\lambda_r$'s constructed above, we obtain a global trivialization of $\mathcal{L}$ over $\mathbb{R}^2$ through $(r, [\gamma, z]) \mapsto (r, \lambda_r([\gamma, z]))$, making $\mathcal{L}$ a line bundle over $\mathbb{R}^2$.

**Definition 2.4.** The space of sections of the line bundle $\mathcal{L}$ over $\mathbb{R}^2$ is denoted by $\Gamma(\mathbb{R}^2, \mathcal{L})$.

Having constructed the line bundle, we can define the Hilbert space that we will be working with.

**Definition 2.5.** Define an inner product on $\Gamma(\mathbb{R}^2, \mathcal{L})$ by

$$\langle \psi, \phi \rangle := \int_{\mathbb{R}^2} \overline{\psi}^\beta(r) \phi^\beta(r) dr,$$

where $\psi^\beta, \phi^\beta : \mathbb{R}^2 \to \mathbb{C}$ are the trivialized expressions for $\psi$ and $\phi$ through some trivialization $\{\beta_r\}$. From (2.46), we obtain a norm in the usual way,

$$||\psi||^2 := \langle \psi, \psi \rangle.$$  

Note that the inner product defined above does not depend on the choice of trivialization, because two trivialized expression $\psi^\beta$ and $\psi'^\beta$ differ by a locally varying complex phase, which cancels out in (2.46) due to the bar on the first argument. However, the inner product might be infinite for some sections of $\mathcal{L}$. Hence, to define the actual Hilbert space, we have to be a little bit more restrictive.

**Definition 2.6.** The one-particle Hilbert space $\mathcal{H}$ is defined as the space of measureable square-integrable sections of $\mathcal{L}$ modulo the equivalence relation that identifies two sections if the set on which they disagree has zero measure, i.e. $\mathcal{H} = L^2(\mathbb{R}^2, \mathcal{L})$. For all practical purposes however, we can simply think of $\mathcal{H}$ as the space of differentiable sections of $\mathcal{L}$ with finite norm.

**Remark 2.5.** The inner product (2.46) is well-defined on $\mathcal{H}$ due to Hölders inequality. As a consequence, $\mathcal{H}$ is closed under addition: for $\psi, \phi \in \mathcal{H}$, $||\psi + \phi||^2 = ||\psi||^2 + \langle \psi, \phi \rangle + \langle \phi, \psi \rangle + ||\phi||^2 < \infty$. Also, $\mathcal{H}$ is complete because $L^2(\mathbb{R}^2, \mathbb{C})$ is, making $\mathcal{H}$ a true Hilbert space.

As noted at the end of Section 2.2.3, we want to describe physical states as sections of the line bundle $\mathcal{L}$. In order get a better idea of what the relation is between a section of $\mathcal{L}$ and a wavefunction, consider the following example.

**Example 2.2.** The symmetric gauge corresponds to the case where each of the paths $\beta_r = \beta^S_r$ is a straight line from the origin to the point $r = (x, y)$, and the Landau gauge corresponds to paths $\beta_r = \beta^L_r$ that go from 0 in a straight horizontal line to the point $(x, 0)$, and then in a straight vertical
2.2. A QUANTUM MECHANICAL APPROACH

In order to distinguish the connection from the partial derivative operator, we use Remark 2.7. A the particular gauge expression \( \psi \) provided that we take \( \kappa \) for the connection, and reserve \( \nabla \) out to be very closely related (see Remark 2.8) to the gauge field \( A \) we have to construct a specific connection on \( L \). \( \beta \) transformations. However, we still have to show that the trivializations \( \lambda^S \) and \( \lambda^L \) chosen above actually agree with the symmetric and the Landau gauges in the first place. In order to do so, we have to construct a specific connection on \( L \), which we will do in the next section. It is the connection that distinguishes the line bundle \( L \) from the trivial bundle \( \mathbb{R}^2 \times \mathbb{C} \).

2.2.5 The Vector Potential as a Connection 1-Form

There exists a natural connection \( \tilde{\nabla} \) on the space of section of \( L \): the one that is associated to the notion of parallel transport coming from glueing paths together. After choosing a trivialization \( \{ \beta_r \} \), the connection takes the trivialized form \( \tilde{\nabla} = d + A \), where \( A \) is the connection 1-form that will turn out to be very closely related (see Remark 2.8) to the gauge field \( A \), and the resulting trivialized expression \( \psi^\beta : \mathbb{R}^2 \to \mathbb{C} \) for \( \psi \in \mathcal{H} \) will then agree with the original wavefunction corresponding to the particular gauge \( A \).

Remark 2.7. In order to distinguish the connection from the partial derivative operator, we use \( \tilde{\nabla} \) for the connection, and reserve \( \nabla \) for the vector operator (\( \partial_x, \partial_y \)).

Remark 2.8. There is a slight ambiguity about the magnetic vector potential. The physicists convention is that of equation (2.32), where \( A \) has units volt-second per meter. The mathematicians convention uses \( A \) for the connection 1-form – see equation (2.50) – where we require \( A \) to have the same units as the derivative operator, i.e. 1/meter. To distinguish between the two, we keep \( A \) for the ordinary physicists vector potential, and use \( \tilde{A} \) for the mathematicians vector potential. The difference between the two is a mere constant,

\[
\tilde{A} = \frac{ie}{\hbar} A.
\]

The factor \( ie/\hbar \) comes from comparing \( i\Pi/\hbar = (\nabla + ieA/\hbar) \) to the connection \( d + \tilde{A} \).
In this section, want to construct the aforementioned connection \( \tilde{\nabla} \). In order to do so, we will first define the corresponding notion of parallel transport. But first, a preliminary definition.

**Definition 2.7.** For two paths \( \gamma, \gamma' : [0, 1] \to \mathbb{R}^2 \) such that \( \gamma(1) = \gamma'(0) \), we define the composition \( \gamma' \circ \gamma \) by gluing both paths together. We will parametrize \( \gamma' \circ \gamma \) such that it does \( \gamma \) on \([0, \frac{1}{2}]\) and \( \gamma' \) on \([\frac{1}{2}, 1]\).

Note that, for our purposes, it doesn’t really matter how the new curve is parametrized as the equivalence relation in equation (2.44) considers all curves that only differ by a reparametrization to be equivalent.

Having defined the composition of paths, we can make sense of the notion of parallel transport.

**Definition 2.8.** Given two points \( r, s \in \mathbb{R}^2 \) and a path \( \eta : [0, 1] \to \mathbb{R}^2 \) such that \( \eta(0) = r \) and \( \eta(1) = s \), we construct a linear isomorphism

\[
P^\eta_{r,s} : L_r \to L_s,
\]

defined by

\[
P^\eta_{r,s}(\{\gamma, z\}) := [\eta \circ \gamma, z].
\]

This map is trivially well-defined, and it is straightforward to check that it is an isomorphism.

**Notation:** For a vector \( w \in \mathbb{R}^2 \) and a point \( r \in \mathbb{R}^2 \), denote the path that connects \( r \) and \( r + w \) in a straight line by \( \eta_{r,w} : [0, 1] \to \mathbb{R}^2 \). That is, for \( s \in [0, 1] \), \( \eta_{r,w}(s) = r + sw \).

**Definition 2.9.** The connection \( \tilde{\nabla} \) is defined as follows. Given a vector field \( v \in \text{Vec}(\mathbb{R}^2) \), \( \tilde{\nabla} \) acts on an arbitrary section \( \psi \in \Gamma(\mathbb{R}^2, L) \) by

\[
\tilde{\nabla}_v(\psi)|_r = \lim_{t \to 0} \frac{(P^\eta_{r,r+vt})^{-1}(\psi_{r+vt}) - \psi_r}{t}.
\]

Now, suppose we have a trivialization \( \{\beta_r : r \in \mathbb{R}^2\} \) of \( L \). Each \( \beta_r \) induces an isomorphism \( \lambda_r : L_r \to \mathbb{C} \). Let \( e_r := (\lambda_r)^{-1}(1) \) be the corresponding global frame. More concretely, \( e_r = [\beta_r, 1] \).

Using the definition of \( \nabla_v \) above, we will construct an explicit formula for the connection 1-form \( \tilde{A} \) corresponding to the trivialization \( \{\beta_r\} \) of \( L \).

When acting on the frame \( e \), we have,

\[
\tilde{\nabla}_v(e)|_r = \tilde{A}(v)e|_r
\]

for any point \( r \in \mathbb{R}^2 \) and any tangent vector \( v \in T_r \mathbb{R}^2 \). Using (2.49), the left hand side of (2.50) can also be obtained by taking a limit of parallel transported vectors,

\[
\tilde{\nabla}_v(e)|_r = \lim_{t \to 0} \frac{(P^\eta_{r,r+vt})^{-1}(e_{r+vt}) - e_r}{t}.
\]

We can rewrite the RHS of (2.51) using the definition of parallel transport (2.48),

\[
(P^\eta_{r,r+vt})^{-1}(e_{r+vt}) = [\eta_{r,vt}^{-1} \circ \beta_{r+vt}, 1] = [\beta_r, \exp \left( -\frac{ie}{\hbar} \int_{D(\eta_{r,vt})} B \right)] = \exp \left( -\frac{ie}{\hbar} \int_{D(\eta_{r,vt})} B \right) e_r.
\]

In the above expression, \( D(\eta_{r,vt}) \) refers to the domain enclosed by and oriented according to the curve \( \beta_{r+vt}^{-1} \circ \eta_{r,vt}^{-1} \circ \beta_r \) — see Figure 2.3. Also, for any path \( \gamma \), we have adopted the notation \( \gamma^{-1} \) for the path \( \gamma \) parametrized backwards. We conclude that parallel transporting the frame \( e \) results in an extra phase factor determined by the path over which we do the parallel transportation.
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\[ \beta_r \]

\[ \beta_{r+vt} \]

\[ r = \eta_{r,vt}(0) \]

\[ v \]

\[ r + vt = \eta_{r,vt}(1) \]

\[ O \]

Figure 2.3: Parallel transporting the vector \( e_{r+vt} \) over a straight line to the point \( r \) yields an additional phase factor \( \exp \left( \frac{-ie}{\hbar} \int_{D(\eta_{r,vt})} B \right) \), where \( D(\eta_{r,vt}) \) is the region enclosed by the curves as shown above.

Plugging the above expression into equation (2.51) yields

\[ \nabla_v(e)_{|r} = \lim_{t \to 0} \frac{\exp \left( \frac{-ie}{\hbar} \int_{D(\eta_{r,vt})} B \right) - 1}{t} e_r. \] (2.52)

Comparing equations (2.50) and (2.52) gives

\[ \tilde{\mathbf{A}}(v)_{|r} = \lim_{t \to 0} \frac{\exp \left( \frac{-ie}{\hbar} \int_{D(\eta_{r,vt})} B \right) - 1}{t} \frac{d}{dt} \exp \left( \frac{-ie}{\hbar} \int_{D(\eta_{r,vt})} B \right) \bigg|_{t=0}. \] (2.53)

It might not be entirely obvious that formula (2.53) defines a 1-form, so let us verify this.

Lemma 2.3. Through formula (2.53), a trivialization \( \{ \beta_r : r \in \mathbb{R}^2 \} \) defines a 1-form \( \tilde{\mathbf{A}} \). (Recall that \( D(\eta_{r,vt}) \) depends on \( \beta_r \).)

Proof. The smoothness of \( \tilde{\mathbf{A}} \) comes from the fact that the paths \( \beta_r \) vary smoothly with \( r \). Hence, in order to prove that (2.53) defines a 1-form, we have to show that it is linear in \( v \). To see that this is indeed the case, we introduce some more notation. For each \( r \in \mathbb{R}^2 \), define \( E_r : \mathbb{R}^2 \to \mathbb{R} \) by

\[ E_r(w) = \exp \left( \frac{-ie}{\hbar} \int_{D(\eta_{r,w})} B \right). \]

Writing \( w = vt \), equation (2.53) becomes

\[ \tilde{\mathbf{A}}(v)_{|r} = \left. \frac{d}{dt} E_r(vt) \right|_{t=0}. \] (2.54)

Using the chain rule on the composition \( t \mapsto w(t) \mapsto E_r(w(t)) \),

\[ \frac{d}{dt} E_r(w(t)) = \left. w'(t) \cdot E'(w(t)) \right|_{w=vt}, \]

equation (2.54) reads

\[ \tilde{\mathbf{A}}(v)_{|r} = \left. \left[ (v_1 \partial_{w_1} + v_2 \partial_{w_2}) \right] E_r(w) \right|_{w=vt}, \]

which is clearly linear in \( v \).
Remark 2.9. We see that the components and the Leibniz rule, $\psi$ can express \[ \tilde{\psi} \] as required.

$\text{Lemma 2.4.}$ Suppose we have two 1-forms, $A$ and $A'$, corresponding to trivializations $\{\beta_r\}$ and $\{\beta'_r\}$ respectively. Denote the frames that correspond to $\{\beta_r\}$ and $\{\beta'_r\}$ by $e$ and $e'$, and let $u : \mathbb{R}^2 \to \mathbb{C}$. Then, the 1-forms are related by

\[ \tilde{A} = (u^{-1}du + \tilde{A}')e. \]

$\text{Proof.}$ We have

\[ \begin{align*}
\tilde{A}(v)e \big|_r &= \lim_{t\to0} \frac{\exp\left(-ie \int_{D(\beta_{r+vt})} B\right) - 1}{t} e_r \\
&= \lim_{t\to0} \frac{(P^{\beta_{r+vt}}_{\beta_{r+vt}})^{-1}(e_{r+vt}) - e_r}{t} \\
&= \lim_{t\to0} u(r + vt) \frac{(P^{\beta_{r+vt}}_{\beta_{r+vt}})^{-1}(e_{r+vt}) - e_r}{t} \\
&= \lim_{t\to0} u(r + vt) \frac{\exp\left(-ie \int_{D'(\beta_r)} B\right) e'_r - e_r}{t} \\
&= \lim_{t\to0} u(r + vt) \frac{\exp\left(-ie \int_{D'(\beta_r)} B\right) - 1}{t} u(r)^{-1}e_r + \lim_{t\to0} \frac{u(r + vt) - u(r)}{t} u(r)^{-1}e_r \\
&= [\tilde{A}'(v)e + u^{-1}du(v)e]_r,
\end{align*} \]

as required. \qed

We know how the connection $\tilde{\nabla}$ acts on a global frame $e$ corresponding to a trivialization $\{\beta_r\}$, where, recall that the frame $e$ is defined by $e_r = [\beta_r, 1]$. What about arbitrary sections $\psi \in \mathcal{H}$? We can express $\psi$ in terms of $e$ as follows: $\psi = \psi^\beta e$, where $\psi^\beta : \mathbb{R}^2 \to \mathbb{C}$. Now, using equation (2.50) and the Leibniz rule,

\[ \nabla(\psi) = \tilde{\nabla}(\psi^\beta e) = d\psi^\beta \otimes e + \psi^\beta \tilde{\nabla}(e) = (d + \tilde{A})\psi^\beta \otimes e. \]

That is, $\tilde{\nabla}$ acts on the trivialized wavefunction $\psi^\beta$ as $d + \tilde{A}$. In terms of $x$-$y$ coordinates, we have

\[ (d + \tilde{A})\psi^\beta = (\partial_x + \tilde{A}_x)\psi_x^\beta dx + (\partial_y + \tilde{A}_y)\psi_y^\beta dy = \nabla_x \psi_x^\beta dx + \nabla_y \psi_y^\beta dy. \]

$\text{Remark 2.9.}$ We see that the components $\Pi_x$ and $\Pi_y$ of the mechanical momentum operator are, up to the numerical factor $\frac{i}{\hbar}$ mentioned in Remark 2.8, equal to the covariant derivative operators $\nabla_x = \partial_x + \tilde{A}_x$ and $\nabla_y = \partial_y + \tilde{A}_y$ respectively.

We can also express the Hamiltonian in terms of the connection $\tilde{\nabla}$. From equation (2.36), we have

\[ H = \frac{1}{2m} \left( [p_x + eA_x]^2 + [p_y + eA_y]^2 \right) = -\frac{\hbar^2}{2m} \left( \left[ \partial_x + \frac{ie}{\hbar} A_x \right]^2 + \left[ \partial_y + \frac{ie}{\hbar} A_y \right]^2 \right) = \frac{\hbar^2}{2m} \left( \nabla_x^2 + \nabla_y^2 \right), \]

which, in coordinate-free notation reads

\[ H = \frac{\hbar^2}{2m} \tilde{\Delta}, \quad (2.55) \]
where $\hat{\Delta}$ is the Bochner Laplacian, defined by

$$\hat{\Delta} := -\text{tr} \hat{\nabla}^2.$$ 

Finally, we will revisit Example 2.2 and show that equation (2.53) indeed gives the correct expression for the magnetic vector potential in the Landau and symmetric gauge when using the aforementioned trivializations $\{\beta^L_r\}$ and $\{\beta^S_r\}$ respectively.

**Example 2.3.** Let $r = (x, y) \in \mathbb{R}^2$ be arbitrary, and let $v = (a, b) \in T_r \mathbb{R}^2$ be any tangent vector at $r$. For the Landau gauge, we have $\int_{D_{(y^a, z^b)}} B = B(yat + \frac{1}{2} abt^2)$. Therefore,

$$\hat{A}^L(v)|_r = \frac{d}{dt} \bigg|_{t=0} \exp \left[ -\frac{ie}{\hbar} B \left( yat + \frac{1}{2} abt^2 \right) \right] = -\frac{ieB}{\hbar} ya.$$ 

Because $\hat{A}^L = \hat{A}^L_x dx + \hat{A}^L_y dy$, we have $\hat{A}^L(v) = \hat{A}^L_x a + \hat{A}^L_y b$. Comparing with the above equation yields

$$A^L = \frac{\hbar}{ie} \hat{A}^L = -Bydx,$$

which agrees with the vector potential in the Landau gauge (2.42). Likewise, for the symmetric gauge we have $\int_{D_{(y^a, z^b)}} B = \frac{1}{2} B(ya - xb) t$. Thus,

$$\hat{A}^S(v)|_r = \frac{d}{dt} \bigg|_{t=0} \exp \left[ -\frac{ieB}{\hbar} \left( ya - xb \right) t \right] = -\frac{ieB}{2\hbar} (ya - xb).$$

Comparing to $\hat{A}^S(v) = \hat{A}^S_x a + \hat{A}^S_y b$ gives

$$A^S = \frac{\hbar}{ie} \hat{A}^S = \frac{B}{2} (-ydx + xdy),$$

which agrees with the vector potential in the symmetric gauge (2.41).

In conclusion, we have constructed a line bundle $\mathcal{L}$ over $\mathbb{R}^2$, such that its square-integrable sections correspond to the one-particle states available to a free electron in a constant perpendicular magnetic field $B$. We have equipped $\mathcal{L}$ with a connection $\hat{\nabla}$, to which the Hamiltonian is related by (2.55). Globally trivializing $\mathcal{L}$ yields a connection 1-form $\hat{A}$ that is proportional to the magnetic vector potential $A$. Moreover, the curvature 2-form $\Omega$ of $\hat{\nabla}$ is proportional to the magnetic field 2-form: $\Omega = d\hat{A} + \hat{A} \wedge \hat{A} = d\hat{A} \propto dA = B$; that is, we can view the magnetic field as curving the line bundle $\mathcal{L}$, making it a topologically trivial line bundle over $\mathbb{R}^2$, with no canonical trivialization.

In practice, we will always choose a gauge — i.e. a global trivialization of $\mathcal{L}$ — and work in that particular gauge, allowing us to regard a section $\psi \in \mathcal{H}$ as a wavefunction $\psi \in L^2(\mathbb{R}^2, \mathbb{C})$. For the remainder of this thesis, we will use the words “section” and “wavefunction” interchangeably. However, it is good to keep in mind that a wavefunction $\psi \in L^2(\mathbb{R}^2, \mathbb{C})$ is only a particular realization of a section of $\mathcal{L}$, the latter of which represents the actual physical state.
Chapter 3

The Quantum Hall Effects

Now that we have set up the formalism with which we will be working, we can start to do some actual physics. In this chapter, we will discuss both the integer and the fractional quantum Hall effect. Section 3.1 deals with the quantization of kinetic energy in the form of Landau levels and how this leads to the integer quantum Hall effect. In order to describe the fractional quantum Hall effect, we have to turn on the electron-electron interactions, which will be discussed in Section 3.2.

Throughout the rest of this thesis, we will assume that the system is in the low temperature limit. Concretely put, this meant that we require the temperature $T$ to be such that thermal energy scale of the system $k_B T$ is much smaller than the kinetic energy scale $\hbar \omega_C$. This ensures that the Landau level structure is maintained in the presence of thermal fluctuations.

### 3.1 The Integer Quantum Hall Effect

As in the previous chapter, consider a 2D spinless free electron gas confined to the $x$-$y$ plane in the presence of a perpendicular magnetic field $B = Be_z$. The magnetic field causes the kinetic energy of the electrons to become quantized. This is known as Landau quantization, which forms the basis of our understanding of the QHE’s. For now, we leave the the electric potential out of the equation; it will be turned on in Section 3.1.4.

#### 3.1.1 Landau Quantization

Recall that the Hamiltonian for a single electron in a magnetic field is given by equation (2.36)

$$H = \frac{1}{2m} |p + eA(r)|^2.$$  

where $p$ is the canonical momentum, given by equation (2.25)

$$p = m \nu - eA(r) = \Pi - eA(r),$$

and $\Pi = m \nu$ is the mechanical momentum (2.21). Since we are working in the Hamiltonian picture, $\Pi$ should be viewed as a function of $r$ and $p$:

$$\Pi(r, p) = p + eA(r).$$

When performing canonical quantization, the canonical momentum $p$ gets replaced by the derivative operator $p = -i\hbar \nabla$, and $\Pi$ (up to a multiplicative constant, see Remark 2.8) becomes the covariant derivative operator $\nabla$ that acts on sections of the one particle Hilbert space $\mathcal{H}$. In particular, we have the commutation relations

$$[r_i, p_j] = i\hbar \delta_{ij}.$$
In terms of the mechanical momentum (3.2), the Hamiltonian reads

$$H = \frac{\Pi^2}{2m} = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2).$$

We compute,

$$[\Pi_x, \Pi_y] = e[p_x, A_y] + e[A_x, p_y] = -i\hbar \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) = -i\hbar(\nabla \times A)_z = -i\hbar eB. \quad (3.4)$$

Or, in terms of the magnetic length

$$l_B = \sqrt{\frac{\hbar}{eB}},$$

which is the typical length scale in our system, equation (3.4) becomes

$$[\Pi_x, \Pi_y] = -\frac{i\hbar^2}{2l_B^2}. \quad (3.5)$$

This is a remarkable result. We see that, because of the magnetic field, the $x$ and $y$ components of the mechanical momentum (which correspond to physical properties of the moving electron) no longer commute. This means that we cannot know both $\Pi_x$ and $\Pi_y$ at the same time! Moreover, the Hamiltonian in equation (3.3) mathematically resembles that of an ordinary quantum mechanical harmonic oscillator.

Define the ladder operators $a$ and $a^\dagger$ by

$$a := \frac{l_B}{\sqrt{2\hbar}}(\Pi_x - i\Pi_y) \quad \text{and} \quad a^\dagger := \frac{l_B}{\sqrt{2\hbar}}(\Pi_x + i\Pi_y), \quad (3.6)$$

where the normalization is chosen precisely such that

$$[a, a^\dagger] = 1.$$ 

Following exactly the same procedure as for the quantum mechanical harmonic oscillator, $H$ can be rewritten in terms of $a$ and $a^\dagger$,

$$H = \frac{\hbar^2}{2ml_B^2}(aa^\dagger + a^\dagger a) = \hbar\omega_C \left( a^\dagger a + \frac{1}{2} \right), \quad (3.7)$$

where $\omega_C = \frac{eB}{m}$ is the cyclotron frequency that we encountered in the classical Hall effect. The eigenvalues of the Hamiltonian are

$$E_n = \hbar\omega_C \left( n + \frac{1}{2} \right), \quad (3.8)$$

for $n$ a non-negative integer.

In conclusion, we see that our Hamiltonian is of the same form as the quantum mechanical harmonic oscillator, only in this case, the energy levels correspond to kinetic energy of the electron. In the semi-classical picture, an electron moves in a cyclotron orbit, where the velocity, and therefore also the radius, is quantized. Applying $a^\dagger$ to an electron in a cyclotron orbit increases both its radius and its velocity, hence its kinetic energy.

It seems that are done now, having diagonalized the Hamiltonian and found its spectrum. However, we have missed a crucial feature of the system that we are describing. Recall that a cyclotron orbit is characterized by two degrees of freedom: the radius of the orbit, and the position of the centre. This last degree of freedom we have completely neglected so far.
3.1.2 Energy Level Degeneracy

We want to introduce operators $R = (X, Y)$ for the coordinates of the centre – also called the guiding centre – of the cyclotron motion. In order to express $R$ in terms of the known operators $x, y, p_x$ and $p_y$, we refer to Figure 2.1. Clearly, $R = r - \eta$. Also, from equation (2.6), in magnitude $\eta = \frac{v \omega_C}{eB}$. Because $\eta$ is rotated clockwise over an angle of $\frac{\pi}{2}$ compared to $v = \frac{\Pi_m}{m}$, we can express $\eta$ in terms of $\Pi$ as follows,

$$\eta = \frac{\Pi}{eB} \times e_z.$$  \hspace{1cm} (3.9)

Hence, the guiding centre operators $X$ and $Y$ are given by

$$R = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x - \frac{\Pi_y}{eB} y - \frac{\Pi_x}{eB} y \\ y + \frac{\Pi_x}{eB} \end{pmatrix} = \begin{pmatrix} x - \frac{\Pi_y}{eB} y + \frac{\Pi_x}{eB} y \\ y - \frac{\Pi_x}{eB} \end{pmatrix}$$  \hspace{1cm} (3.10)

where $\Pi$ is defined in terms of $x, y, p_x$ and $p_y$ by equation (3.2).

Now that we have a proper expression for the operators corresponding to the guiding centre, let us compute the commutator $[X, Y]$.

$$[X, Y] = \left[ x - \frac{\Pi_y}{eB} y + \frac{\Pi_x}{eB}, x - \frac{\Pi_y}{eB} y - \frac{\Pi_x}{eB} y \right]$$

$$= \frac{1}{eB} [x, p_x] - \frac{1}{eB} [p_y, y] - \frac{i}{(eB)^2} \Pi_x \Pi_y$$

$$= 2i\hbar \frac{i}{eB} - \frac{1}{(eB)^2} \frac{i}{l_B^2} = i\hbar \frac{i}{eB}$$

$$= l_B^2.$$  \hspace{1cm} (3.11)

As with the mechanical momentum, also the $x$ and $y$ components of the guiding centre operator do not commute. This means that $X$ and $Y$ too, cannot be measured simultaneously.

We introduce a second set of ladder operators $b$ and $b^\dagger$, defined by

$$b = \frac{1}{\sqrt{2l_B}} (X + iY) \quad \text{and} \quad b^\dagger = \frac{1}{\sqrt{2l_B}} (X - iY),$$  \hspace{1cm} (3.12)

where, as before, the prefactors are chosen such that $[b, b^\dagger] = 1$.

Because $X$ and $Y$ commute with $\Pi_x$ and $\Pi_y$ – see appendix (A.3) and (A.4), the one-particle Hilbert space can be written as a tensor product of two harmonic oscillators. The Hamiltonian is given by

$$H = \hbar \omega_C \left( a^\dagger a + \frac{1}{2} \right) + \hbar \omega_0 \left( b^\dagger b + \frac{1}{2} \right)$$  \hspace{1cm} (3.13)

where $\omega_0 = 0$, and the one-particle eigenfunctions of $H$ are completely described by two parameters: $n$ for the energy level, and $m$ for the guiding centre,

$$|n, m\rangle := \frac{(a^\dagger)^n (b^\dagger)^m}{\sqrt{n!} \sqrt{m!}} |0\rangle.$$  \hspace{1cm} (3.14)

Physically, we can think of the system comprised of a single electron in a magnetic field as a sum of independent harmonic oscillators of frequency $\omega_C$ located at different positions labelled by the guiding center coordinate $m$. By saturating Heisenberg’s uncertainty relation, we can assign an average area to one such harmonic oscillator,

$$\Delta X \Delta Y = 2\pi l_B^2.$$  \hspace{1cm} (3.15)
Given that the (macroscopic) sample whereon the electron moves has total area $\mathcal{A}$, we can fit

$$N_B := \frac{\mathcal{A}}{2\pi l_B^2} = n_B \mathcal{A},$$  \hspace{1cm} (3.16)

where $n_B$ is the flux density

$$n_B := \frac{1}{2\pi l_B^2} = \frac{B}{\hbar/e} = \frac{B}{\phi_0},$$  \hspace{1cm} (3.17)

harmonic oscillators onto the sample. In the equation above, we have introduced the magnetic flux quantum

$$\phi_0 := \frac{\hbar}{e}.$$

In short, the situation is the following: we have $N_B$ positions to place a guiding centre, and at each position an electron can be in any of the discrete harmonic oscillator energy levels $E_n$ given by equation (3.8). Thus, we see that, in the case that we have more than one non-interacting spinless electron, a fixed energy eigenstate can be occupied by $N_B$ electrons. That is, each energy level is $N_B$-fold degenerate.

Now, suppose that we have $N_{el} = n_{el} \times \mathcal{A}$ non-interacting spinless electrons in our system, which we take to be in the ground state. Because the electrons are fermions, the first $N_B$ electrons can all go into the harmonic oscillator ground states, the second $N_B$ electrons must go into the first excited states, and so on. Hence, the multi-particle ground states are completely determined by the filling factor

$$\nu := \frac{N_{el}}{N_B} = \frac{n_{el}}{n_B}.$$  \hspace{1cm} (3.18)

A final note: the degeneracy $N_B$ depends on the magnetic field. By increasing it we can decrease the area of uncertainty $2\pi l_B^2$ in equation (3.15). Hence, we can change the filling factor either by changing the electron density $n_{el}$, or by changing the magnetic field strength $B$.

We now possess all the necessary machinery to understand the IQHE, which we will discuss in the next section. But, before we do so, we will try to make the theory explained above a little bit more precise from a mathematical viewpoint.

### 3.1.3 Mathematical Intermezzo

Let us try to put the theory on a more rigorous mathematical footing. As described in Section 2.2.4, the one-particle Hilbert space $\mathcal{H}$ is the space $L^2(\mathbb{R}^2, \mathcal{L})$ of square integrable sections of $\mathcal{L}$. The Hamiltonian is given by equation (2.55),

$$H = \frac{\hbar^2}{2m} \Delta,$$

Picking a gauge corresponds to globally trivializing the line bundle. This yields an isomorphism $\mathcal{H} \cong L^2(\mathbb{R}^2, \mathbb{C})$, and the Hamiltonian then takes the form

$$H = \frac{1}{2m} [p + eA(q)]^2 = \frac{1}{2m} \left( \Pi_x^2 + \Pi_y^2 \right) = \hbar \omega_C \left( a^\dagger a + \frac{1}{2} \right),$$  \hspace{1cm} (3.19)

which corresponds to equation (3.3). The ladder operators $a$ and $a^\dagger$ satisfy

$$[a, a^\dagger] = 1.$$  \hspace{1cm} (3.20)

**Definition 3.1.** Let $\mathcal{A}$ be the associative algebra over $\mathbb{C}$ with involution spanned by two elements $\alpha$ and its conjugate $\alpha^*$. Let $\mathcal{I} \subset \mathcal{A}$ be the ideal $\mathcal{I} = \mathbb{C}(\alpha \alpha^* - \alpha^* \alpha - 1)$. Define

$$\mathcal{A} := \mathcal{A}/\mathcal{I},$$

that is, $\mathcal{A}$ is the associative algebra spanned by $\alpha$ and $\alpha^*$ such that $[\alpha, \alpha^*] = \alpha \alpha^* - \alpha^* \alpha = 1$.

---

1The total flux through a superconducting loop is always an integer multiple of the magnetic flux quantum.

2We consider spinless electrons because we do not want the Zeeman effect to split each Landau level into two.
A choice of the gauge gives explicit expressions for the operators $a$ and $a^\dagger$ acting on $L^2(\mathbb{R}^2, \mathbb{C})$ satisfying (3.20). That is, it defines a representation
$$\alpha \mapsto a$$
of the algebra $\mathcal{A}$ on $L^2(\mathbb{R}^2, \mathbb{C})$. Different gauges correspond to unitarily equivalent representations of $\mathcal{A}$, because the explicit expressions for $\Pi$, and therefore also for $a$ and $a^\dagger$, corresponding to different gauges are related by a unitary transformation as shown in equation (2.40). Therefore, the operators $a$ and $a^\dagger$ are actually defined on sections of $L^2(\mathbb{R}^2, \mathbb{L})$.

In order to gain a better understanding in the system that we are investigating, we will now identify all relevant irreducible representations of $\mathcal{A}$.

**Definition 3.2.** The standard representation of $\mathcal{A}$ is defined as follows. Let $\mathcal{H}_0$ be the Hilbert space $l^2(\mathbb{N}_0)$, where $\mathbb{N}_0$ is the set of non-negative integers. That is,
$$\mathcal{H}_0 = \bigoplus_{n=0}^{\infty} V_n,$$
where each of the $V_n$’s is isomorphic to $\mathbb{C}$. The inner product on $\mathcal{H}_0$ is defined as
$$\langle x, y \rangle := \sum_{n=0}^{\infty} x_n \overline{y}_n$$
for $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ arbitrary elements of $\mathcal{H}_0$. We have an orthonormal basis $\{e_n\}$ of $\mathcal{H}_0$ defined by $\langle e_n, e_m \rangle = \delta_{nm}$, i.e. $e_n = 1 \in V_n$.

The standard representation of $\mathcal{A}$ is defined through the action of $\alpha$ and $\alpha^*$ on the ONB $\{e_n\}$,
$$\alpha(e_n) := \begin{cases} \sqrt{n} e_{n-1} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$
$$\alpha^*(e_n) := \sqrt{n+1} e_{n+1}.$$The most common example in physics of the standard representation of $\mathcal{A}$ is that of the quantum harmonic oscillator.

**Theorem 3.1.** Any irreducible representation of $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}$ with non-empty $\text{Ker}(\alpha) \subset \mathcal{H}$ is unitarily equivalent to the standard representation.

**Remark 3.1.** For the case of a single electron in a perpendicular magnetic field, $\text{Ker}(a)$ is actually non-empty, so the above assumption holds. We will explicitly construct the $\text{Ker}(a)$ in the symmetric gauge in Lemma 3.4.

**Proof.** Suppose we have such a representation $\mathcal{A} \to \text{End}(\mathcal{H})$. Define $V_0 := \text{Ker}(\alpha)$. Because $\text{Ker}(a) \neq \emptyset$, there exists a $0 \neq \psi \in V_0$. For each $n \in \mathbb{N}$, define
$$V_n := \mathbb{C}(\alpha^*)^n \psi.$$

**Claim 1:** $(\alpha)^n \psi \neq 0$ for all $n \in \mathbb{N}$.

The basic idea is the following: suppose $\alpha^* \psi = 0$, then $0 = \alpha \alpha^* \psi = (\alpha^* \alpha + 1) \psi = \psi$, which contradicts the fact that $\psi \neq 0$. Hence, we conclude that $\alpha^* \psi \neq 0$. More formally,

**Base case:** $\psi \neq 0$.

**Induction step:** suppose that $(\alpha^*)^n \psi \neq 0$, and assume that $(\alpha^*)^{n+1} \psi = 0$. Then, by exploiting the commutation relation in (3.20) and the fact that $\alpha \psi = 0$, $0 = \alpha(\alpha^*)^{n+1} \psi = (n+1)(\alpha^*)^n \psi \neq 0,$
proving that the assumption \((\alpha^*)^{n+1}\psi = 0\) is false, which means that \((\alpha^*)^{n+1}\psi \neq 0\).

**Claim 2:** for \(m \neq n, V_m \perp V_n\).

*Proof:* without loss of generality, assume that \(n > m\). Now,

\[
\langle (\alpha^*)^n \psi, (\alpha^*)^m \psi \rangle = \langle (\alpha^*)^m (\alpha^*)^n \psi, \psi \rangle \propto \langle (\alpha^*)^n-m \psi, \psi \rangle = \langle (\alpha^*)^{n-m-1} \psi, \alpha \psi \rangle = 0.
\]

Hence, by claims 1 and 2 above, after properly normalizing each \((\alpha^*)^n \psi\) we see that \(\mathcal{H}\) contains a copy of \(\mathcal{H}_0\) as an \(\mathcal{A}\)-invariant subspace. By irreducibility of \(\mathcal{H}\), we conclude that \(\mathcal{H} \cong \mathcal{H}_0\). \(\square\)

**Remark 3.2.** We have swept a couple of non-trivial details under the rug. Theorem 3.1 has a mathematically rigorously formulated counterpart, known as the Stone-von Neumann theorem. This theorem is more general than Theorem 3.1 because the extra assumption that \(\text{Ker}(\alpha) \neq \emptyset\) is not needed. We will now briefly discuss the Stone-von Neumann theorem.

Instead of dealing with the ladder operators \(a \) and \(a^\dagger\), the Stone-von Neumann theorem is formulated in terms of the self-adjoint operators \(Q = \frac{1}{\sqrt{2}}(a + a^\dagger)\) and \(P = \frac{1}{\sqrt{2}}(a^\dagger - a)\), which satisfy \([P, Q] = -i\).

Given a representation of \(\mathcal{A}\) on a Hilbert space \(\mathcal{H}\), it follows (by taking the trace on both sides of \([P, Q] = -i\)) that \(\mathcal{H}\) has to be infinite dimensional, and that at least one of the operators \(P\) and \(Q\) is represented by an unbounded operator. As a consequence, the relation \(PQ - QP = -i\) cannot hold on all of \(\mathcal{H}\). However, if \(P\) and \(Q\) are self-adjoint, then both generate one-parameter subgroups \(\{U(t) := e^{-itP}, t \in \mathbb{R}\}\) and \(\{V(t) := e^{-itQ}, t \in \mathbb{R}\}\) of the space of bounded linear operators \(\mathcal{B}(\mathcal{H})\) on \(\mathcal{H}\). By \(PQ - QP = -i\), the operators \(U(t)\) and \(V(s)\) satisfy

\[
U(t)V(s) = e^{i ts}V(s)U(t). \tag{3.22}
\]

Because \(U(t)\) and \(V(s)\) are bounded, equation (3.22) could possibly hold on all of \(\mathcal{H}\).

The most common representation of two operators \(P\) and \(Q\) acting on some Hilbert space \(\mathcal{H}\) and satisfying \(PQ - QP = -i\) is the example due to Schrödinger of the position and momentum operators \(Q = x\) and \(P = -i\partial_x\) acting on \(L^2(\mathbb{R})\). Indeed, both operators are unbounded, but their exponentiated versions are not. For \(\psi \in L^2(\mathbb{R}^2)\), we have

\[
U(s)\psi(x) = \psi(x-s) \quad \text{and} \quad V(s)\psi(x) = e^{-isx}\psi(x).
\]

Now, the Stone-von Neumann Theorem says that, if we have another pair of weakly continuous families \(\{\hat{U}(t), t \in \mathbb{R}\}\) and \(\{\hat{V}(t), t \in \mathbb{R}\}\) of unitary operators acting irreducibly on a separable Hilbert space \(\mathcal{H}\) such that

\[
\hat{U}(t)\hat{U}(s) = \hat{U}(t+s), \quad \hat{V}(t)\hat{V}(s) = \hat{V}(t+s), \tag{3.23}
\]

\[
\hat{U}(t)\hat{V}(s) = e^{its}\hat{V}(s)\hat{U}(t) \tag{3.24}
\]

holds, then there exists a (unitary) isomorphism \(W: \mathcal{H} \to L^2(\mathbb{R}^2)\) such that

\[
W\hat{U}(t)W^{-1} = U(t) \quad \text{and} \quad W\hat{V}(t)W^{-1} = V(t)
\]

for all \(t \in \mathbb{R}\). In other words, any \((\hat{U}, \hat{V}, \mathcal{H})\) satisfying (3.23) – (3.24) is unitarily equivalent to \((U, V, L^2(\mathbb{R}^2))\). A similar statement also holds for more than one set of conjugate operators.

We will not go any further into the details of this beautiful but complicated theorem. The interested reader is referred to [23].

Suppose we are given a general (not necessarily irreducible) representation of \(\mathcal{A}\) on a separable Hilbert space \(\mathcal{H}\) with non-empty \(\text{Ker}(\alpha)\), then \(\mathcal{H}\) can be decomposed into a direct sum of irreducible representations of \(\mathcal{A}\). The following corollary shows that this is indeed the case.
3.1. THE INTEGER QUANTUM HALL EFFECT

Corollary 3.1. Given a representation of $A$ on a separable Hilbert space $H$ with $\text{Ker}(\alpha) \neq \emptyset$. Then, $H \cong H_0 \otimes \text{Ker}(\alpha)$ as a representation of $A$, meaning that $H$ is isomorphic to $\text{dim}(\text{Ker}(\alpha))$ $A$-invariant copies of $H_0$.

Proof. Pick a basis $\{\psi_1, \psi_2, \ldots\}$ of $\text{Ker}(\alpha)$. Following the proof of Theorem 3.1, each of the $\psi_i$ generates an $A$-invariant subspace isomorphic to $H_0$, and the so obtained $A$-invariant subspaces do not intersect. For, suppose that we have linearly independent $\psi_i, \psi_j \in \text{Ker}(\alpha)$ and $n > m \in \mathbb{N}$ such that $(a^\dagger)^n \psi_i = \lambda (a^\dagger)^m \psi_j$ for some $0 \neq \lambda \in \mathbb{C}$. Applying $a^{m+1}$ to both sides yields $(a^\dagger)^{n-m-1} \psi_i = 0$, which contradicts Claim 1 of Theorem 3.1. Hence, $H \cong H_0 \otimes \text{Ker}(\alpha)$. □

Remark 3.3. Any representation of $A$ on a separable Hilbert space $H$ with $\text{Ker}(\alpha) \neq \emptyset$ is completely characterized by the dimension of $\text{Ker}(\alpha)$.

We now return to the one-particle Hilbert space $H = L^2(\mathbb{R}^2, \mathcal{L})$ for a single charged particle in a uniform perpendicular magnetic field. Recall that $A$ acts on $H$ via $\alpha \mapsto a$ with $\text{Ker}(\alpha) \neq \emptyset$, and therefore $H \cong H_0 \otimes \text{Ker}(\alpha)$ according to Corollary 3.1.

Definition 3.3. In the decomposition $H \cong H_0 \otimes \text{Ker}(\alpha)$, for each non-negative integer $n$ denote the $n$-th subspace $V_n \otimes \text{Ker}(\alpha)$ of $H$ by $L_n$. That is, $H \cong \bigoplus_{n=0}^{\infty} L_n$, where $L_n \cong \text{Ker}(\alpha)$.

Remark 3.4. Because the combination $a^\dagger a$ acts by multiplication with $n$ on $L_n$, by (3.19) each $L_n$ is an energy eigenspace with energy $\hbar \omega_C (n + \frac{1}{2})$. We call $L_n$ the $n$th Landau level.

Definition 3.4. The subspace $L_0 = \text{Ker}(\alpha) \subset H$ is called the degeneracy space, because it determines the degeneracy of each energy level. In physics, it is known as the lowest Landau level.

In order to further understand the representation of $A$ on the one-particle Hilbert space $H$, we will now investigate the degeneracy space $\text{Ker}(\alpha) \subset H$. Recall the guiding centre operators $X$ and $Y$ defined in equation (3.10).

Lemma 3.1. For each $n \geq 0$, the operators $X$ and $Y$ are endomorphisms of $L_n$.

Proof. According to (A.3) and (A.4), both $X$ and $Y$ commute with $\Pi_x$ and $\Pi_y$, and therefore with $a, a^\dagger$ and $H$. This means that $X$ and $Y$ leave each of the $L_n$’s invariant. □

As explained in the previous section, because $[X, Y] = i l^2_B$, we can define another set of ladder operators $b$ and $b^\dagger$ – see equation (3.12) – satisfying $[b, b^\dagger] = 1$, which are also endomorphisms of each Landau level $L_n$.

Corollary 3.2. The map $\alpha \mapsto b$ defines a representation of $A$ on each of the $L_n$’s. As a consequence, $\text{dim}(L_n)$ is infinite for each $n$.

Using the guiding centre operators $X$ and $Y$, we can construct an operator that plays a mathematically similar role to the Hamiltonian (3.19).

Definition 3.5. The guiding centre operator $G$ is defined by

$$G := X^2 + Y^2 = 2l^2_B \left( b^\dagger b + \frac{1}{2} \right).$$

By working in the symmetric gauge, in Corollary 3.6 we will show explicitly that $\text{Ker}(b) \cap L_0$ is one-dimensional. This statement is actually gauge independent because the ladder operators $a, a^\dagger, b$ and $b^\dagger$ are well-defined on sections of $L^2(\mathbb{R}^2, \mathcal{L})$.

Corollary 3.3. For each $n \geq 0$, $\text{Ker}(b) \cap L_n$ is one-dimensional.
Proof. For \( n = 0 \), the above statement is exactly Corollary 3.6. For \( n > 0 \), observe that \( L_n \) is the image of \( L_0 \) under \( (a^\dagger)^n \). Thus, an arbitrary vector in \( L_n \) is of the form \( (a^\dagger)^n \psi \), with \( \psi \in L_0 \). Since \( a^\dagger \) and \( b \) commute, \( \text{Ker}(b) \cap L_n \) is the space of vectors \( a^\dagger \psi \) satisfying

\[
0 = b(a^\dagger)^n \psi = (a^\dagger)^n b \psi.
\]

By injectivity of \( a^\dagger \), the above equation is equivalent to \( \psi \in \text{Ker}(b) \cap L_0 \), so \( \text{Ker}(b) \cap L_n \) is one-dimensional.

Corollary 3.4. \( A \) acts irreducibly on each of the spaces \( L_n \) through \( \alpha \mapsto b \). Moreover, for each non-negative integer \( n \), \( L_n \) together with the ladder operators \( b \) and \( b^\dagger \) forms a quantum harmonic oscillator, where the guiding centre operator \( G \) assumes the role of the Hamiltonian.

The structure of the one-particle Hilbert space \( L(\mathbb{R}^2, \mathcal{L}) \) is becoming more apparent. Let us summarize the results of this section in the following Corollary.

Corollary 3.5. The one-particle Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^2, \mathcal{L}) \) is isomorphic to a tensor product of two harmonic oscillators:

\[
\mathcal{H} \cong \mathcal{H}_0 \otimes \text{Ker}(a).
\]

Since \( \mathcal{H}_0 \) is itself a direct sum of one-dimensional vector spaces, we can also decompose \( \mathcal{H} \) into a direct sum

\[
\mathcal{H} = \bigoplus_{n=0}^{\infty} L_n
\]

of energy eigenspaces \( L_n \), where each subspace \( L_n \cong \text{Ker}(a) \) forms an harmonic oscillator with corresponding ladder operators \( b \) and \( b^\dagger \). As a consequence, we have a basis of \( \mathcal{H} \) where each basis vector \( \psi_{n,m} \) is labeled by two non-negative integers \( n \) and \( m \), where \( n \) labels the energy with respect to the Hamiltonian \( H \), and \( m \) labels the 'energy' with respect to the guiding centre operator \( G \).

Remark 3.5. The above Corollary justifies the notation \( |n,m\rangle = \psi_{n,m} \) in equation (3.14).

We conclude with a lemma on the nature of the guiding centre operators \( X \) and \( Y \). The position operators \( x \) and \( y \) do not commute with the Hamiltonian \( H \), However, we can make them commute with \( H \) if we replace them by the operators \( x \mapsto \sum_n p_n x p_n \) and \( y \mapsto \sum_n p_n y p_n \), where

\[
p_n := \mathcal{H} \to L_n
\]

is the orthogonal projection onto the \( n \)-th Landau level. It turns out that this procedure exactly gives us the guiding centre operators \( X \) and \( Y \).

Lemma 3.2. The operators \( X \) and \( Y \) are given by

\[
X = \sum_{n=0}^{\infty} p_n x p_n \quad \text{and} \quad Y = \sum_{n=0}^{\infty} p_n y p_n.
\]

Proof. We will only prove the lemma for \( X \). The proof for \( Y \) is analogous. The following equalities are equivalent:

\[
X = \sum_{n=0}^{\infty} p_n x p_n \quad \iff \quad \langle X \xi, \eta \rangle = \sum_n \langle p_n x p_n \xi, \eta \rangle \quad \text{for all} \quad \xi \in L_{n_1} \quad \text{and} \quad \eta \in L_{n_2},
\]

where \( n_1 \) and \( n_2 \) are arbitrary. We will prove the second statement. For \( \xi \in L_{n_1} \) and \( \eta \in L_{n_2} \) we have

\[
\sum_n \langle p_n x p_n \xi, \eta \rangle = \langle p_{n_1} x \xi, \eta \rangle = \langle x \xi, p_{n_1} \eta \rangle = \delta_{n_1,n_2} \langle x \xi, \eta \rangle.
\]
Thus, we have to prove that
\[ \langle X_\xi, \eta \rangle = \delta_{n_1, n_2} \langle x_\xi, \eta \rangle \] (3.26)
for all \( n_1, n_2 \in \mathbb{N}_0 \), with \( \xi \in L_{n_1} \) and \( \eta \in L_{n_2} \).

If \( n_1 \neq n_2 \), then the left hand side is zero because \( X \) maps \( L_{n_1} \) into itself, so we have an equality. If \( n_1 = n_2 \), then, from the definition of \( X \) in (3.10), \( \langle X_\xi, \eta \rangle = \langle x_\xi, \eta \rangle - \frac{1}{eB} \langle \Pi y_\xi, \eta \rangle \). However, recall that \( \Pi y \propto a - a^\dagger \), and both \( a \) and \( a^\dagger \) change the energy level of \( \xi \), so \( \langle \Pi y_\xi, \eta \rangle = 0 \) by orthogonality of the energy eigenspaces. Hence, we conclude that (3.26) is true, which proves the lemma.

We can also (trivially) express the Hamiltonian in terms of the projection operators (3.25).

**Lemma 3.3.** The Hamiltonian \( H \) is given by
\[
H = \hbar \omega C \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) p_n
\]

*Proof.* Follows immediately from equation (3.19) and Remark 3.4.

Having our feet firmly planted on strong mathematical soil, we are now confident enough to tackle the multi-particle case and discuss the integer quantum Hall effect.

### 3.1.4 The Hall Resistivity at Integer Filling Factor

The goal of this section is to explain the integer quantum Hall effect; that is, the peculiar shape of the Hall resistance curve shown in Figure 2.2. We keep the number of electrons in the system \( N_{el} \) constant, while varying the magnetic field strength \( B \), or equivalently, the filling factor \( \nu \) (3.18), i.e.
\[
\nu = \frac{N_{el}}{N_B},
\]
where \( N_B = B \mathcal{A}/\phi_0 \) is the degeneracy of each Landau level, and \( \mathcal{A} \) is the sample area. Recall that the system is assumed to be in the ground state, and hence the Landau levels are filled up from bottom to top. For the remainder of this chapter, we will follow the reasoning of Goerbig [7].

Before explaining the actual shape of the Hall resistivity curve — which we will do in Section 3.1.5 — we will first focus on the specific case where the magnetic field \( B \) is such that \( \nu \) is exactly integer. For these values of \( B \), it will turn out that the value of the Hall resistivity agrees with the classical value, given by equation (2.14):
\[
(\rho_H)_{\text{classical}} = \frac{B}{en_{el}},
\]
where \( n_{el} \) is the electron density.

Instead of computing the resistivity, we will compute the conductivity of the system. Because \( \nu \) is assumed to be integer, say \( \nu = N \in \mathbb{N} \), each Landau level is either completely filled, or completely empty. Only the completely filled Landau levels, of which we have \( N \), contribute to the conductivity. Computing the total conductivity boils down to computing the conductivity of one completely filled Landau level (which will turn out to be the same for each Landau level), and then multiplying by \( N \).

In order to compute the conductivity of a fully filled Landau level, we take a rectangular system of length \( L \) and width \( W \), such that \( L \gg W \). We choose our \( x-y \) coordinate system such that the corners of the sample are at \((0,0), (0,W), (L,W) \) and \((L,0)\). Next, we apply a current \( I \) in the \( x \)-direction. We assume that the system is translationally invariant in the \( x \)-direction, meaning that the potential \( U = -eV_H \) associated to the Hall voltage difference is independent of \( x \): \( U = U(y) \), and we impose periodic boundary conditions in the \( x \)-direction. Also, we assume that the kinetic
energy $\hbar\omega_C$ scale is much larger than the potential energy difference $\nabla U$ associated to the magnetic length $l_B$, as to not spoil the Landau level structure. More precisely, promoting an electron to the next Landau level means that its cyclotron radius increases by roughly $l_B$. Thus, the potential energy gained is of the order of $l_B|\nabla U|$. We want this increase in potential energy to be negligible compared to the gain in kinetic energy. That is, we require

$$\hbar\omega_C \gg l_B|\nabla U| = l_B eE.$$  \hfill (3.27)

The most convenient gauge for this system is the Landau gauge (2.42), which respects the translational symmetry and assures that $H$ is independent of the $x$-coordinate. When looking for eigenfunctions of $H$, given by

$$H = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2),$$  \hfill (3.28)

by separating the variables $x$ and $y$, we see that functions of the form

$$\psi_k(x,y) = e^{ikx}\zeta_k(y)$$  \hfill (3.29)

will do the trick. Using the ansatz from (3.29), we can effectively set $p_x = \hbar k$, where $k = \frac{2\pi m}{L}$ for $m \in \mathbb{Z}$. This means that the Hamiltonian (3.28) of our system takes the following shape

$$H = \frac{(\hbar k - eBy)^2}{2m} + \frac{p_y^2}{2m} + U(y) = \frac{1}{2}m\omega_C^2(y - kl_B)^2 + \frac{p_y^2}{2m} + U(y).$$  \hfill (3.30)

Consider $k$ to be fixed. We can expand the potential $U$ around the point $y = kl_B$. Because the kinetic energy dominates the potential energy, we can view the additional term $U(y)$ as a correction to the first. Neglecting $U(y)$ completely yields an harmonic oscillator centered at $y = kl_B$, which has energy eigenfunction that decay exponentially with decay constant $l_B$. Because $U$ varies very little on length scales of the order of $l_B$, we anticipate that the true energy eigenfunctions will resemble the harmonic oscillator energy eigenfunctions, which are localized around $kl_B$. Therefore, in a first order approximation, we can linearize $U(y)$ near $y = kl_B$:

$$U(y) \approx U(kl_B^2) + \frac{\partial U}{\partial y}|_{kl_B^2} (y - kl_B) = U(kl_B^2) + eE(kl_B^2)(y - kl_B).$$  \hfill (3.31)

Plugging equation (3.31) into equation (3.30) and completing the square yields

$$H \approx \frac{p_y^2}{2m} + \frac{1}{2}m\omega_C^2 \left( y - kl_B^2 + \frac{eE(kl_B^2)}{m\omega_C^2} \right)^2 + U(kl_B^2) - \frac{e^2E^2(kl_B^2)^2}{2m\omega_C^2}.$$  \hfill (3.32)

Next, define

$$y_k := kl_B^2 + \frac{eE(kl_B^2)}{m\omega_C^2}.$$  \hfill (3.33)

Note that the second term in (3.33) is much smaller than the first due to equation (3.27). Hence, we can replace $U(kl_B^2)$ in (3.32) by $U(y_k)$ at the cost of an extra term $-\frac{e^2E^2(kl_B^2)^2}{m\omega_C^2}$. Both this term as well as the last term in (3.32) are proportional to $mv_D$ — where $v_D = E^2/B^2$ is the Hall drift velocity — which is much smaller than the sum of the kinetic and the potential energy and can therefore be neglected. This results in

$$H \approx \frac{p_y^2}{2m} + \frac{1}{2}m\omega_C^2(y - y_k)^2 + U(y_k),$$  \hfill (3.34)

which, for each value of $k$, is an harmonic oscillator centered at $y_k$ shifted by a constant $U(y_k)$.
We conclude that the approximate eigenfunctions of the Hamiltonian (3.28) are of the form 
\[ \psi_{n,k}(x,y) = e^{ikx} \xi_{n,k}(y), \]
where \( \xi_{n,k} \) is an eigenfunction of (3.34). The wavefunctions \( \psi_{n,k} \) have energy eigenvalues
\[ \epsilon_{n,k} = \hbar \omega_C \left( n + \frac{1}{2} \right) + U(y_k). \] (3.35)

Here, \( n \in \mathbb{N} \) denotes the Landau level we are in, and \( y_k \) signifies where the wavefunction is localized. Hence, \( m = \frac{\hbar k}{2e} \) is interpreted as the guiding centre eigenvalue. Because we want the wavefunction to be localized somewhere on the sample, \( k \) must take values between certain \( k_{\text{min}} \) and \( k_{\text{max}} \). When \( E = 0 \), \( y_k = k_l \frac{e}{2e} \) and \( k \) runs from \( k_{\text{min}} = 0 \) to \( k_{\text{max}} = \frac{W}{L} \) since the \( y \)-coordinate runs from \( y = 0 \) to \( y = W \). Hence, the guiding centre eigenvalue \( m \) takes values between \( m_{\text{min}} = 0 \), and \( m_{\text{max}} = \frac{LW}{2\pi \ell_B} = \frac{\text{Area}}{2\pi \ell_B} = N_B \), which agrees with the fact that each Landau level is \( N_B \) fold degenerate. In our case \( E \neq 0 \), but it is approximately constant, so the allowed values for \( y_k \) will be shifted by the constant \( \frac{eE(k_l)}{m\omega_C} \), but there are still \( N_B \) values for \( k \) such that \( y_k \) lies between 0 and \( W \).

Having computed the approximate spectrum of the Hamiltonian, we can now proceed to compute the conductance of a single Landau level. We will work in the Heisenberg picture.

For \( n \in \{1, \ldots, N\} \), the \( n \)-th Landau level contributes
\[ I_n = -e \sum_{k=k_{\text{min}}}^{k_{\text{max}}} \langle n,k| \frac{\dot{x}}{L} |n,k\rangle = -e \sum_{k=k_{\text{min}}}^{k_{\text{max}}} v^\prime_{x,n,k} \] (3.36)
to the current \( I = Ix \). In the above equation, we have defined \( v^\prime_{x,n,k} := \langle n,k| \dot{x} |n,k\rangle \) to be the expectation value of the \( x \)-component of the velocity of the \( (n,k) \)-th energy eigenstate \( \psi_{n,k} = |n,k\rangle \).

Now, the Heisenberg equation of motion for the operator \( \dot{x} \) is
\[ \dot{x} = i \hbar \{ H, x \} = i \hbar \frac{\partial H}{\partial p_x} [p_x, x] = \frac{1}{\hbar} \frac{\partial H}{\partial k} \frac{\partial p_x}{\partial k}. \]

Applying the above to the eigenfunction \( |n,k\rangle \) and using that \( L \gg 1 \) to approximate the derivative with a fraction, gives
\[ v^\prime_{x,n,k} = \frac{1}{\hbar} \frac{\partial \epsilon_{n,k}}{\partial k} \approx \frac{L}{2\pi \hbar} \frac{\Delta \epsilon_{n,m}}{\Delta m} = \frac{L}{\hbar} (\epsilon_{n,m+1} - \epsilon_{n,m}), \]
where \( \Delta m = 1 \). Using equation (3.35), we see that the velocity expectation value \( v^\prime_{x,n,k} \) of the state \( |n,k\rangle \) is given by
\[ v^\prime_{x,n,k} = \frac{L}{\hbar} (U(y_{m+1}) - U(y_m)), \]
where \( y_m \) is short for \( y_{k(m)} \).

Performing the sum over \( k \) in (3.36), which is actually a sum over \( m \), we see that all the intermediate terms cancel, leaving us with
\[ I_n = -e \frac{\hbar}{L} (\epsilon_{n,m_{\text{max}}} - \epsilon_{n,m_{\text{min}}}) = -e \frac{\hbar}{L} (U(y_{m_{\text{max}}}) - U(y_{m_{\text{min}}})) = e^2 \frac{\hbar}{L} V_H, \]
where \( V_H := -\frac{1}{L} (U(y_{m_{\text{max}}}) - U(y_{m_{\text{min}}})) \) is the Hall voltage difference between the lower \( (y = 0) \) and the upper \( (y = W) \) edges of the sample, which is perpendicular to the current flow in the \( x \)-direction.

We conclude that the \( n \)-th Landau level contributes one
\[ G_n = \frac{e^2}{h}. \]
quantum of conductance to the electronic transport, and this holds for each filled Landau level, independent of the value of $n$. Hence, the total conductivity $G$ is given by

$$G = \sum_{n=0}^{N-1} G_n = N e^2 / h.$$  

The resistivity is the inverse of the conductivity. Thus, the Hall resistivity assumes the value

$$\rho_H = \frac{1}{G} = \frac{1}{N e^2}$$

at integer filling factor $\nu = N$. As noted earlier, the above value coincides with the classical Hall resistivity at integer filling. Indeed, if we set $\nu = N$ in equation (3.18), we have $n_{el} = N n_B = \frac{NeB}{h}$, which implies that

$$\left(\rho_H\right)_{\text{classical}} = \frac{B}{en_{el}} = \frac{h}{Ne^2} = \left(\rho_H\right)_{\text{quantum}}.$$  

We conclude that a translationally invariant rectangular sample subject to an electric current in the $x$-direction generates an electric Hall potential $V_H$ such that, at integer filling factor, the corresponding quantum Hall resistivity $\rho_H$ equals the classical Hall resistivity. This is in agreement with Figure 2.2. The more intriguing part of the Hall resistance curve, however, is the part where the Hall curve deviates from the classical Hall curve, which happens when $\nu$ is not integer. We will discuss these deviations in the next section.

### 3.1.5 Quantization of the Hall Resistance

In order to truly understand the IQHE, we now consider non-interacting electrons moving in an actual physical sample. As before, we consider a sample of dimension $L$ times $W$, and inject a current $I$ in the $x$-direction. In order to prevent the electrons from leaving the sample at the top $y = W$ and bottom $y = 0$ edges, we add a confining potential $U_{\text{conf}}(y)$. Additionally, the electrons interact with the sample itself, which we model by an impurity potential $U_{\text{imp}}(x, y)$. Including the Hall potential from the previous section, which we now denote by $U_{\text{Hall}}(y)$, the total potential $U(x, y)$ is given by

$$U(x, y) = U_{\text{Hall}}(y) + U_{\text{conf}}(y) + U_{\text{imp}}(x, y).$$  

(3.37)

In Section (3.1.2), the Hamiltonian was translation invariant, and, as a consequence, the guiding centre coordinate $\mathbf{R}$ is a constant of motion. The appearance of $U$, which now also depends on $x$, breaks translational invariance, lifting the Landau level degeneracy. If the potential is too strong, as explained in the previous section, then the energy eigenstates get shifted by such amounts that the Landau level structure will disappear completely, thus preventing us from observing any kind of quantum Hall effect. We therefore assume that $\hbar \omega_C \gg l_B |\nabla U|$.

First, let us do a consistency check. Recall that $r = \mathbf{R} + \eta$. The Hamiltonian expressed in terms of $\eta$ instead of $\Pi$ — which are related by equation (3.9) — is given by

$$H = \frac{1}{2} m \omega_C^2 (\eta_x^2 + \eta_y^2) + U(\mathbf{R} + \eta).$$  

(3.38)

Also, using equation (3.5), we have

$$[\eta_x, \eta_y] = \frac{1}{(eB)^2} [\Pi_y, -\Pi_x] = -i l_B^2.$$  

(3.39)

Now, recall Heisenberg’s equation of motion for any operator $O$,

$$\dot{O} = \frac{i}{\hbar} [H, O].$$
Thus, using equations (3.38) and (3.39), we get the following equations of motion for the the electron coordinates $\eta$ relative to the guiding centre $R$,

$$
\dot{\eta}_x = \frac{i}{\hbar} \mathbb{E} \epsilon^{\eta y \eta z} \eta_y + \frac{i}{\hbar} \frac{\partial U}{\partial \eta_y} \eta_y, \\
\dot{\eta}_y = \frac{i}{\hbar} \mathbb{E} \epsilon^{\eta x \eta z} \eta_x + \frac{i}{\hbar} \frac{\partial U}{\partial \eta_x} \eta_x, \\
\dot{\eta}_z = \frac{i}{\hbar} \mathbb{E} \epsilon^{\eta x \eta y} \eta_x + \frac{i}{\hbar} \frac{\partial U}{\partial \eta_x} \eta_x.
$$

Indeed, in the limit $\hbar \omega_C \gg l_B |\nabla U|$, the second term is negligible compared to the first one, and we recover the classical cyclotron equation of motion. This means that locally, the electrons behave as if they do not feel the impurity potential.

Having concluded that the electrons approximately preform a cyclotron motion, let us investigate the dynamics of the guiding centres. Because we have assumed that the potential practically does not vary on length scales of the order of the cyclotron radius ($\propto l_B$), we can approximate $U(r) \approx U(R)$.

Hence, the equations of motion for $R$ become:

$$
\dot{X} = \frac{i}{\hbar} [U(R), X] = \frac{i}{\hbar} \frac{\partial U}{\partial Y} [Y, X] = \frac{\hbar}{l_B} \frac{\partial U}{\partial Y}, \\
\dot{Y} = \frac{i}{\hbar} [U(R), Y] = \frac{i}{\hbar} \frac{\partial U}{\partial X} [X, Y] = -\frac{\hbar}{l_B} \frac{\partial U}{\partial X}.
$$

Using $\frac{\hbar}{l_B} = \frac{1}{eB}$, we conclude that

$$
\dot{R} = \frac{1}{eB} \frac{\nabla U \times B}{B} = \frac{E \times B}{B^2}.
$$

The second equality sign follows from $U = -eV$ and $E = -\nabla V$.

Comparing to equation (2.11), we see that the guiding centre moves exactly as it does in the classical Hall effect: along equipotential lines of $U$, in such a way that is goes counter-clockwise around a positively charged potential hill. For this reason, the limit $\hbar \omega_C \gg l_B |\nabla U|$ is referred to as the semi-classical limit.

An example of the total potential $U$ that the electrons feel is depicted in Figure 3.1(a). In the bulk of the sample, $U_{imp}$ is the dominant term in (3.37), and the equipotential lines of $U$ encircle the impurities. Hence, electrons travelling along such equipotential lines are localized, and do not contribute to net electronic transport. Near the edges on the other hand, the confining potential $U_{conf}$ is dominant, which rises at the edges of the sample. Hence, electrons travelling at the edges can travel from left to right and contribute to the macroscopic current $I$ running through the sample. The corresponding states are called edge states. Moreover, due to (3.40), the edge currents are chiral because the impurity potential bends upwards near the edges. Thus, if an electron in an edge state hits an impurity, thereby forcing it to hop to a nearby available state, which is also an edge state, the electron keeps on moving in the same direction, because all the nearby available states “move” in the same direction. Hence, there is no backscattering. This explains why, at integer filling factor, the longitudinal resistance is zero.

We have now understood both the values of the Hall ($R_H = \frac{B}{en_el}$) and the longitudinal ($R_L = 0$) resistance at integer filling factor $\nu = N$, displayed in Figure 3.1(a). In order to understand why the quantum Hall resistivity curve (Figure 2.2) deviates from the classical Hall curve (which is a straight line), we will discuss what happens when the filling factor $\nu$ is perturbed around a some fixed integer $N$. We keep electron density $n_{el}$ fixed, and we adjust the filling factor by changing the magnetic field — see equation (3.18).

Suppose that the magnetic field $B$ is such that $\nu = N$. Reducing the magnetic field reduces the degeneracy of each Landau level, which forces some electrons in the fully occupied Landau levels to go into the empty $(N + 1)^{th}$ Landau level. The first couple of electrons promoted to the $(N + 1)^{th}$
3.1. THE INTEGER QUANTUM HALL EFFECT

Landau level will go into the potential valleys – see Figure 3.1(b): they will become localized electrons that do not contribute to the macroscopic current $I$. Thus, both the Hall resistivity as well as the longitudinal resistivity remain unchanged. As we decrease the magnetic field further, the electron puddles grow larger, because the equipotential lines that the newly promoted electrons follow encompass larger areas. Eventually, equipotential lines that connect the upper ($y = W$) and the lower ($y = 0$) end of the sample will become available, allowing conduction electrons at the edges to travel from the upper to the lower edge – see Figure 3.1(c). This allows for backscattering, and therefore simultaneously lowers the transverse Hall resistance and increases the longitudinal resistance, as shown the bottom graph of Figure 3.1(c).

Figure 3.1: A plot of the density of states (top), potential landscape as seen from above (middle) and the Hall and longitudinal resistance as a function of the magnetic field for constant electron density (bottom). In (a), the filling factor $\nu$ is an integer $n$. When the magnetic field is lowered, electrons are forced to populate the $(n+1)^{th}$ Landau level. The first excited electrons become localized (b), but when the magnetic field is decreased further (c), electrons are allowed to move from top to bottom through the sample, both lowering the Hall resistance and increasing the longitudinal resistance because the edge states at opposite edges become connected.

If, instead of lowering the magnetic field, we would have increased it, the same phenomena occurs in terms of electron holes instead of actual particles. This explains the plateaus in the quantum Hall resistances shown in Figure 2.2, which distinguish the quantum from the classical Hall effect.

The description above gives a very intuitive explanation of the peculiar shape of the Hall resistance curve. It is called the percolation picture. It explains the shape of the Hall resistivity curve around the points for which the filling factor is integer. However, a closer look at the Hall resistivity curve shown in Figure 2.2 shows that there are also plateaus at fractional filling factor. These plateaus go by the name fractional quantum Hall effect, which is the topic of the next section.
3.2 The Fractional Quantum Hall Effect

In the preceding section we have seen how, at integer $\nu = N$ filling factor, a Hall resistivity plateau is formed due to the fact that there is an energy gap between the filled $N^{th}$ Landau level and the empty $(N + 1)^{th}$ Landau level. However, in 1982, H.L. Stormer and D.C. Tsui discovered a Hall plateau at fractional filling factor $\nu = \frac{1}{3}$. The phenomena of the formation of Hall plateaus at fractional filling factor is called the fractional quantum Hall effect, or FQHE for short. From the theory that we have developed so far, this newly discovered plateau does not make any sense. When slightly perturbing the filling factor around $\nu = \frac{1}{3}$, there is no energy gap for electrons to cross: they can safely stay in the lowest Landau level, which is two-thirds empty. Hence, the reasoning we did in the previous section for the IQHE cannot hold for fractional filling factors. Besides, in the non-interacting limit the ground state at fractional filling factor is highly degenerate, because all available states in the fractionally filled Landau level have the same energy. Thus, there is no unique ground state to do perturbation theory on. Nevertheless, R.B. Laughlin [14] succeeded in 1983 to reproduce a wavefunction that resembles the numerically computed ground state wavefunction up to high precision. We will motivate Laughlin’s educated guess in the section below.

The FQHE is only observed in samples with very low impurity. Hence, the impurity potential is not expected to be the driving mechanism behind the FQHE. Throughout this section, we will assume

$$\hbar \omega_C \gg U_{\text{imp}}.$$ 

Also, as discussed above, the Landau level structure can by itself not be held accountable for the FQHE. There is one remaining mechanism that we have neglected when discussing the IQHE: that of the electron-electron interaction. Hence, we additionally assume that the interaction energy is of a order comparable to the kinetic energy,

$$\hbar \omega_C \approx U_{\text{int}}.$$ 

In the remainder of this chapter we will discuss Laughlin’s variational ground state wavefunction, which only holds for filling fractions of the form $\nu = \frac{1}{2s+1}$ with $s \in \mathbb{N}$; the anyonic single particle excitations of the FQHE fluid; and the theory of composite fermions which works for the more general fractions of the form $\nu = \frac{p}{2sp+1}$ with $s, p \in \mathbb{N}$. There are other fractions at which the FQHE is observed, like $\nu = \frac{5}{2}$, which are not of the form $\nu = \frac{p}{2sp+1}$. For a discussion on the non-composite fermion fractions, the readers is urged to look elsewhere.

3.2.1 The Ground State Wavefunction at $\nu = 1$

Before we discuss the Laughlin multi-particle wavefunction, let us briefly revisit the single-particle case.

Remark 3.6. From now on, we will work in the symmetric gauge.

It is easier to work in complex coordinates. We define: $z := x - iy$ and $\bar{z} = x + iy$, with corresponding partial derivative operators $\partial := (\partial_x + i\partial_y)/2$ and $\bar{\partial} := (\partial_x - i\partial_y)/2$. Most mathematicians might feel uneasy with this convention, but it is the one used in the literature, so we will stick to it.\footnote{We are using this notation to ‘compensate’ for the fact that the dynamical particles, in this case electrons, have a negative charge. Using our funny notation, the lowest Landau level will be described by holomorphic instead of anti-holomorphic functions.}

From equation (3.13), we see that the Hamiltonian is effectively a sum of two number operators: $a^\dagger a$ and $b^\dagger b$, where the $a$ and $b$ operators are given by equations (3.6) and (3.12) respectively. Recall that $a^\dagger$ raises the energy of a state, and $b^\dagger$ changes the position of the guiding centre. That is, $a$ and $a^\dagger$ are responsible for hopping between different Landau levels, while $b$ and $b^\dagger$ cause hopping within
a fixed Landau level. Using equations (3.2) and (3.10) for $\Pi$ and $R = (X, Y)$ respectively, we can express the ladder operators in terms of complex derivative and multiplication operators.

\begin{align}
    a &= -i\sqrt{2} \left( \frac{z}{4l_B} + l_B \partial \right), & a^\dagger &= i\sqrt{2} \left( \frac{\bar{z}}{4l_B} - l_B \partial \right), \\
    b &= -i\sqrt{2} \left( \frac{\bar{z}}{4l_B} + l_B \partial \right), & b^\dagger &= i\sqrt{2} \left( \frac{z}{4l_B} - l_B \partial \right). 
\end{align}

(3.41)  
(3.42)

A particle is in the lowest Landau level precisely when its corresponding wavefunction $\psi$ is in the kernel of $a$, that is $a\psi = 0$. The following lemma gives a concrete description of $\text{Ker}(a)$.

**Lemma 3.4.** $\text{Ker}(a) = \left\{ f(z) \exp \left( -\frac{|z|^2}{4l_B} \right) : f \text{ is holomorphic} \right\}$

**Proof.** We will prove two inclusions. First, suppose that $\psi \in \text{Ker}(a)$. Define $f(z, \bar{z}) := \psi(z, \bar{z}) \exp \left( -\frac{|z|^2}{4l_B} \right)$. Since $\psi$ is in the kernel of $a$, it is most certainly in the domain of $a$, which means that it is $\mathbb{R}$-differentiable, and therefore so is $f$. Moreover, using $a\psi = 0$, we get

\[ \bar{\partial} f(z, \bar{z}) = \left[ \partial \psi(z, \bar{z}) + \frac{z}{4l_B} \psi(z, \bar{z}) \right] \exp \left( \frac{|z|^2}{4l_B} \right) = \psi(z, \bar{z}) \left[ -\frac{z}{4l_B} + \frac{\bar{z}}{4l_B} \right] \exp \left( \frac{|z|^2}{4l_B} \right) = 0. \]

We conclude that $f$ is both $\mathbb{R}$-differentiable and satisfies the Cauchy-Riemann equations $\bar{\partial} f(z, \bar{z}) = 0$. Thus, $f$ is holomorphic, and we write $f(z, \bar{z}) = f(z)$. The other inclusion follows immediately by applying $a$ to any element of the set on the RHS of the equation above, which will give zero. \[\square\]

In exactly the same manner, we have the following lemma.

**Lemma 3.5.** $\text{Ker}(b) = \left\{ g(z) \exp \left( -\frac{|z|^2}{4l_B} \right) : g \text{ is anti-holomorphic} \right\}$

**Corollary 3.6.** Taking the intersection of $\text{Ker}(a)$ and $\text{Ker}(b)$ yields the one-dimensional subspace of functions $\{ \lambda \cdot \exp(-\frac{|z|^2}{4l_B}), \lambda \in \mathbb{C} \}$.

In particular, the $n = 0, m = 0$ wavefunction is the unique wavefunction in $\text{Ker}(a) \cap \text{Ker}(b)$ with norm 1,

\[ \psi_{0,0}(z) = \frac{1}{\sqrt{2\pi l_B}} \exp \left( -\frac{|z|^2}{4l_B} \right). \]

All other wavefunctions in $\text{Ker}(a)$ are obtained by applying $b^\dagger$ to $\psi_{0,0}$. Each successive application of $b^\dagger$ gives an extra factor of $z$. After normalization, we get

\[ \psi_{0,m}(z) = \frac{i^m}{\sqrt{2\pi l_B} m!} \left( \frac{z}{\sqrt{2l_B}} \right)^m \exp \left( -\frac{|z|^2}{4l_B} \right), \]

(3.43)

which is indeed of the form $f(z) \exp \left( -\frac{|z|^2}{4l_B} \right)$.

Next, we turn to the $N$-particle case. Instead of a function of one variable, the ground state wavefunction $\psi_N$ is now an anti-symmetric wavefunction of $N$ variables

\[ \psi_N(\{ z_i \}) \propto f_N(\{ z_i \}) \exp \left( -\sum_{i=1}^{N} \frac{|z_i|^2}{4l_B} \right), \]

(3.44)

where we again require $f_N(\{ z_i \})$ to be holomorphic in all $z_i$ variables. In the above equation, $\{ z_i \}$ stands for $z_1, ..., z_N$. There is an additional normalization factor, not included in $f_N$, that we will neglect for now.
Let us start by examining a case that we are already familiar with, the case of \( \nu = 1 \). At \( \nu = 1 \), the degeneracy of one Landau level \( N_B \) equals the total number of particles, i.e. \( N = N_B \). From equation (3.43), we see that a factor of \( z_i^{m_i} \) in front of the exponential means that the \( i \)-th particle has its guiding centre at position \( m_i \). Because the electrons are indistinguishable, the ground state wavefunction \( \psi_N(\{ z_i \}) \) for the case \( \nu = 1 \) must be a linear combination of wavefunctions representing all possible ways of positioning the different electrons at each of the possible \( m = 0, \ldots, N-1 \) guiding centre positions, in such a way that the total function is anti-symmetric with respect to the exchange of individual electron coordinates. There is exactly one function \( f_N \) that has the desired properties,

\[
f_N(\{ z_i \}) = l_B^{-N(N-1)/2} \det \begin{pmatrix} z^0_1 & \cdots & z^{N-1}_1 \\ \vdots & \ddots & \vdots \\ z^0_N & \cdots & z^{N-1}_N \end{pmatrix}.
\]

(3.45)

The above \( N \times N \) determinant is known as the Vandermonde determinant. Using a proof by induction, it can be shown that the above expression is equal to

\[
f_N(\{ z_i \}) = \prod_{i<j} \left( \frac{z_i - z_j}{l_B} \right). \tag{3.45}
\]

We have found the ground state wavefunction without solving any kind of Schrödinger equation. Of course, for non-interacting particles, this is not surprising, because there is a standard method for solving the \( N \)-particle case once the one-particle case has been solved: using anti-symmetrized single particle wavefunctions (in the case of fermions). However, Laughlin’s idea was to use the same properties of holomorphicity and anti-symmetry, together with his physical intuition, on his ansatz ground state wavefunction for other filling factors, namely those of the form \( 1/2s_{s+1} \), to come up with a trial wavefunction that turned out to describe the ground state extremely well. We will discuss Laughlin’s trial wavefunction in Section 3.2.3, but before we do so, let us pause for a second and try to develop some physical intuition for the \( N \)-particle ground state wavefunction \( \psi_N \) given by equation (3.44). Laughlin noticed that this wavefunction is related to the partition function of a well-known classical system.

### 3.2.2 Laughlin’s Plasma Analogy

There is a striking resemblance between ground state wavefunction (3.44) at \( \nu = 1 \), and the partition function of a classical 2D plasma. Let us define

\[
\phi(\{ z_i \}) := \prod_{i<j} \left( \frac{z_i - z_j}{l_B} \right) \exp \left( -\sum_{i=1}^{N} \frac{|z_i|^2}{4l_B^2} \right),
\]

to be the unnormalized wavefunction (so that we can stop writing the “\( \propto \)” symbol). Since \( |\phi(\{ z_i \})|^2 \) is proportional to the probability distribution \( |\psi_N(\{ z_i \})|^2 \), we can regard it as a classical Boltzmann weight

\[
|\phi(\{ z_i \})|^2 = \exp(-\beta U) \tag{3.46}
\]

for a suitably chosen potential \( U \). The corresponding partition function is

\[
Z = \int \prod_{i=1}^{N} dz_i |\phi(\{ z_i \})|^2 = \int \prod_{i=1}^{N} dz_i \exp(-\beta U(\{ z_i \})).
\]

Next, we choose \( \beta \) and \( U \) in such a way that we can interpret \( U \) as the potential for a 2D gas of interacting particles with charge \( q \) in a uniformly charged background. This requires that we set
$\beta = \frac{2}{q}$, and

$$U(\{z_i\}) = -q^2 \sum_{i<j} \ln \left| \frac{z_i - z_j}{l_B} \right| + q \sum_k \left| z_k \right|^2 \frac{l_B^2}{4l_B^2},$$

(3.47)

which indeed solves equation (3.46), assuming we take $q = 1$. The reason that we have introduced
the factor of $q$, is because it assumes the role of charge equation (3.47). At filling factor $\nu = 1$, the
charge has to be 1 in order to make sure that (3.46) is satisfied, so we could have left $q$ out of
the equation. However, it will turn out that the plasma analogy also works for other filling factors, for
which $q \neq 1$, so we will keep track of $q$ in our calculations below.

Before we continue, let us check that the above potential indeed corresponds to a 2D gas of
interacting charged particles in a uniformly charged background. In 2D, we have Gauss’ law in
integral form,

$$\oint E \cdot dn = 2\pi q,$$

(3.48)

where $q$ is the enclosed charge (modulo a constant, which we have introduced to make the formula’s
come out a little bit nicer). For a point charge $q$ at the origin, by the rotational symmetry of the
configuration, we deduce that $E(r) = \frac{q}{r} \hat{r}$. Hence, the corresponding electric potential is given by

$$V(r) = -q \ln \left( \frac{r}{l_B} \right),$$

which explains the first term in (3.47).

For the second term, which can be interpreted as $N$ charges $q$ each feeling a background potential
$V_B(z, \bar{z}) = \frac{|z|^2}{4l_B^2}$, we use Gauss’ law in differential form to relate the potential $V_B$ to some background
charge density $\rho_B$. We have

$$\nabla^2 V(r) = \nabla \cdot E(r) = -2\pi \rho_B(r).$$

Hence,

$$\rho_B = \frac{1}{2\pi} \nabla^2 \left( \frac{|z|^2}{4l_B^2} \right) = -\frac{1}{2\pi l_B^2} = -\frac{B}{\phi_0}.$$}

Indeed, the second term in (3.47) corresponds to the interaction between the charges at positions
$z_1, \ldots, z_N$ and a uniform background charge with a charge density $\rho_B$ that is equal to minus the
density of magnetic flux quanta $n_B$ – see equation (3.17).

To conclude, equation (3.46) allows us to regard $|\phi(\{z_i\})|^2$, which is proportional to the proba-

bility (distribution) that the $N$ electrons are found at positions $z_1, \ldots, z_N$, as a Boltzmann weight
exp($-\beta U$) of a 2D gas of particles of charge $q$ in a uniformly charged background. Hence, the proba-
nbility of finding the electrons in the quantum Hall fluid at positions $z_1, \ldots, z_N$ is proportional to the
probability of finding $N$ charge $q$ particles in a 2D charged background at the exact same positions.
Thus, we can now use our knowledge of the 2D plasma to gain insight in the quantum Hall fluid.

From electrostatics, it is known that charges in a uniformly charged background will arrange
themselves in such a way that the total charge is neutral, thereby minimizing $U$. This means that
the charges at the positions $z_1, \ldots, z_N$ will be distributed over the sample in such a way that, when
averaged over length scales much larger than the average electron distance, the electron charge
density $n_{el}$ satisfies

$$n_{el} q + \rho_B = 0.$$  

(3.49)

Since $q = 1$ at $\nu = 1$, we require $n_{el} = -\rho_B = \frac{1}{2\pi l_B^2}$, which agrees with (3.18) for $\nu = 1$. Moreover,
we know that the wavefunction will be very close to zero whenever the $n_{el}$ does not satisfy the charge
neutralit y condition in (3.49), because in that case the Boltzmann factor will be negligibly small. This means that the $\nu = 1$ quantum Hall system is a strongly correlated system, where the electrons arrange themselves in such a way that the electron density $n_{el}$ is approximately constant $\frac{1}{2\pi l_B^2}$ on length scales larger than $l_B$.

Now that we have established Laughlin’s plasma analogy, we are ready to discuss Laughlin’s ground state wavefunctions.

### 3.2.3 Laughlin’s Ground State Wavefunction

The charge neutrality condition (3.49) relates the imaginary charge $q$ to the filling factor. Indeed, combining equations (3.18) and (3.49) yields

$$q = -\frac{\rho_B}{n_{el}} = -\frac{1}{n_{el} 2\pi l_B^2} = \frac{n_B}{n_{el}} = \frac{1}{\nu}.$$  

In the previous subsection, we had $q = 1$. If, instead, we keep $q$ arbitrary, then equations (3.46) and (3.47) suggest that we take

$$\phi_q(z_i) := \left[ \prod_{i<j} (\frac{z_i - z_j}{l_B})^q \right] \exp \left( -\sum_{i=1}^{N} \frac{|z_i|^2}{4l_B^2} \right),$$

as a trail unnormalized ground state wavefunction for a filling factor of $\nu = 1/q$. This is Laughlin’s trial wavefunction. Comparing to the non-interacting ground state wavefunction (3.44), we require that the prefactor is holomorphic, and locally integrable, which means that $q$ has to be a positive integer. Moreover, $\phi$ has to be antisymmetric with respect to the the exchange of particle coordinates, implying that $q$ has to be odd. This means that the wavefunction in (3.50) serves as a trail ground state wavefunction for filling factors of the form

$$\nu = \frac{1}{q} = \frac{1}{2s + 1},$$

with $s \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots \}$. Note that taking $s = 0$ corresponds to the familiar case of $\nu = 1$.

Now that we have obtained a trail ground state wavefunction by an educated guess, let us try to show that this trial wavefunction indeed describes the ground state of the quantum Hall system at filling factor $\nu = 1/(2s + 1)$. In order to do so, we will artificially construct a different kind of interaction potential of which $\phi_q$ is the exact ground state, and then we will argue that this artificial potential has a 99% overlap with the Coulomb potential for as far as the ground state wavefunction is concerned. We will focus on the case $s > 0$, since the $s = 0$ case corresponds to $\nu = 1$, for which we have already shown that (3.50) describes the ground state.

At fractional filling factor $\nu = 1/(2s + 1)$ with $s > 0$, only the lowest Landau level is partially filled, and the higher Landau levels are completely empty. Thus, the dynamics of the system occur within the lowest Landau level, meaning that all electrons have the same kinetic energy, which we can therefore neglect. Put differently, the only energy relevant for this system is the interaction energy due to the mutual Coulomb repulsion of the electrons. Before we look at the $N$-particle case, let us first focus on the case of $N = 2$ electrons.

Because the electrons have equal charge, the two-particle Coulomb interaction potential $U_C$ should normally speaking not have any bound states: two electrons that feel each others repulsive Coulomb force simply accelerate away from each other, exchanging potential for kinetic energy. However, since both electrons have fixed kinetic energy, they cannot exchange potential for kinetic energy, and instead rotate around each other, as they would do classically due to the external perpendicular magnetic field. The wavefunction of two electrons orbiting each other with an angular
momentum $\hbar m$ has a prefactor of the form $(z_1 - z_2)^m$. The possible energy eigenstates of the two-electron system are the different bound states labeled by the exponent $m$. As a consequence, when acting on the ground state, the potential $U_C$ can be expanded as follows

$$U_C = \sum_{m=0}^{\infty} u_m^C P_m,$$

where $P_m$ is the operator that projects onto the $m$th bound state, and $u_m^C$ is the corresponding eigenvalue of $U_C$, which, because the average distance between two electrons in the ground state with relative angular momentum $m$ is approximately $l_B\sqrt{2m}$, is equal to

$$u_m^C \approx U_C(z = l_B\sqrt{2m}). \tag{3.51}$$

The above values $u_m^C$ are called the Haldane’s pseudopotentials.

As with the two-particle case, the $N$-particle interaction potential $U$ can be decomposed into Haldane’s pseudopotentials,

$$U = \sum_{i<j} \sum_{m=0}^{\infty} u_m P_m(i,j),$$

where now $P_m(i,j)$ projects onto the $(i,j)$th two-particle state with relative angular momentum $m$. The Coulomb potential $U_C$ determines the values for $u_m$ through formula (3.51). Likewise, any arbitrary choice of $u_m$’s determines an interaction potential $U$.

We will now construct an artificial interaction potential $U_A$ by choosing specific values for the $u_m$’s such that $\phi^A$ is the exact ground state wavefunction of $U_A$. We define

$$u_m^A := \begin{cases} u_m^C & \text{for } m < q \\ 0 & \text{for } m \geq q \end{cases}. \tag{3.52}$$

With the above choice of $u_m$’s, it easily follows that $U_A\phi^A = 0$. In order to see this, note that for each pair $(i,j)$, $\phi^A$ contains a prefactor $(z_i - z_j)^q$, which means that each pair has a relative momentum of at least $q$. Thus $P_m(i,j)\phi^A = 0$ for $m < q$, and therefore $U_A\phi^A = 0$. Moreover, if $\phi$ is any other wavefunction that describes the system in the same filling factor $\nu = 1/q$ as $\phi^A$, then $\phi$ can be expressed in terms of the standard exponential times a polynomial prefactor that can be decomposed into powers of the form $\prod_{i<j} (z_i - z_j)^{m_{ij}}$. Now, if $m_{ij} \neq q$ for any pair of $(i,j)$, then, in order to keep the filling factor constant, at least one other pair has to have an $m_{ij'} < q$. This increases the potential energy of the state $\phi$ by $u_{m_{ij'}} = u_{m_{ij'}}^C > 0$. Therefore, $\phi^A$ is indeed the lowest energy state for the artificial interaction potential $U_A$. Moreover, we conclude that there is a strictly positive energy gap between the ground state Laughlin wavefunction and the excited states. This means that we have at least one of the required ingredients for the formation of a plateau of the Hall resistance at fractional $\nu = 1/q$ filling$^3$.

What are we to make of this? After all, we are interested in the Coulomb potential, not in the artificial potential constructed above. It turns out that, for particle numbers where numerical approximation of the ground state (using exact diagonalization) is viable, the Laughlin ground state wavefunction $\phi^A$ agrees with over 99% with the numerically computed ground state wavefunction of the quantum Hall fluid at filling factor $\nu = 1/q$. The reason for this is that, when computing Haldane’s pseudopotentials $u_m^C$ for the Coulomb potential, one obtains a series of numbers that decreases with $m$, because of equation (3.51) and the fact that the Coulomb potential decreases

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$^4$This value can be obtained by going to the centre of mass and relative coordinates, effectively reducing the two-particle problem to a one-particle problem, and then using the fact that the expectation value of the radius of the guiding centre for a single particle with guiding centre eigenvalue $m$ is $l_B\sqrt{2m}$. See Goerbig [7] p.41 for details.

$^5$Note that this energy gap is also present when using the Coulomb potential, because the numbers $u_m^C$ decrease discretely with $m$. 
as $1/r$. Combined with the fact that the actual ground state wavefunction contains but a small contribution of high relative angular momentum states, the interaction potential from (3.52) is a good approximation to the Coulomb potential for as far as the ground state wavefunction is concerned.

### 3.2.4 Elementary Excitations

In the previous section we have seen that there is an energy gap between the Laughlin ground state wavefunction and the excited states. In order to understand the occurrence of the QHE at filling factors of the form $\nu = \frac{1}{2s+1}$, we have to establish the nature of the excitations. We are looking for effectively non-interacting charged single particle excitations, that, when created, due to the presence of an energy gap become localized in the potential valleys of the impurity potential, causing the formation of a plateau in the Hall resistance exactly as in the IQHE – see Section 3.1.5.

It turns out that there are two kinds of excitations relative to the ground state: elementary excitations that involve changing the filling factor, and collective charge-density wave excitations that keep the overall filling factor fixed. We will only focus on the first kind of excitations, because these are specific to the FQHE. Regarding the charge-density wave excitations, we can mention that the dispersion relation has a minimum at some non-zero wavevector, suggesting that for the low energy excitations the electrons arrange themselves in a regular pattern such as a Wigner crystal; but we will not go into detail here.

The elementary excitations are created by changing the filling factor through the addition of a degeneracy $\text{deg}$ we conclude that the function $\phi$ becomes

$$\phi_{qh}(z_0, \{z_i\}) := \prod_{j=1}^{N} \left( \frac{z_j - z_0}{l_B} \right) \phi^q(\{z_i\}). \quad (3.53)$$

The subscript $qh$ stands for quasi-hole, which we will elaborate on below.

What influence does the extra prefactor have on the ground state? Since original prefactor of $\phi^q$ is holomorphic in all its arguments, we can expand it around the point $z_0$,

$$\phi^q(\{z_i\}) = \sum_{m_1, \ldots, m_N = 0}^{\infty} \left[ c_{m_1, \ldots, m_N} \left( \frac{z_1 - z_0}{l_B} \right)^{m_1} \cdots \left( \frac{z_N - z_0}{l_B} \right)^{m_N} \right] \exp \left( \sum_{j=1}^{N} \frac{|z_j|^2}{4l_B^2} \right),$$

Thus, $\phi_{qh}$ becomes

$$\phi_{qh}(z_0, \{z_i\}) = \sum_{m_1, \ldots, m_N = 0}^{\infty} \left[ c_{m_1, \ldots, m_N} \left( \frac{z_1 - z_0}{l_B} \right)^{m_1+1} \cdots \left( \frac{z_N - z_0}{l_B} \right)^{m_N+1} \right] \exp \left( -\sum_{j=1}^{N} \frac{|z_j|^2}{4l_B^2} \right). \quad (3.54)$$

From equation (3.43) we see that the exponent of the factor $(z_i - z_0)$ is related to the position of the guiding centre of $z_i$ relative to $z_0$. Recall that there are at most $N_B = m_{\text{max}}$ possible positions for the guiding centre in the sample. Now, equation (3.54) says that adding the extra factor $\prod_j (z_j - z_0)/l_B$ to $\phi_{qh}$ increases all the powers $m_i$ by one. This means that relative to $z_0$, all the particles have their guiding centre increased by one, leaving a hole at the center $z_0$. Therefore, we call excitations of the form $\phi_{qh}$ quasi-holes. Moreover, since all the $m_i$ are increased by one, so is $m_{\text{max}} = N_B$. Hence, we conclude that the function $\phi_{ext}$ in (3.53) corresponds to an exited state where the Landau level degeneracy $N_B$, and therefore the total flux penetrating the system, has increased.

In order to determine the charge of a quasi-hole, we follow Murthy and Shankar [19] (see p.9) and observe what happens when adiabatically adding a flux tube, i.e. a localized quantum of flux $\phi_0 = \frac{\Phi}{\epsilon}$.
— the flux associated to a quasi-hole — to the system. Let $\Sigma$ be a disc of radius $r$ centered at the position $z_0$ of the quasi-hole, which we take to be origin for convenience. Increasing the magnetic flux locally by an amount $\phi_0$ in a time interval $[0,T]$ generates an azimuthal electric field $E$ due to Faraday’s law of Induction: $E(r,t) = E_\theta(r,t)e_\theta$. Because of the Hall effect, this electric field generates a radial current $j_r(r,t) = j_\nu(r,t)e_r$ with radial component $j_r = \frac{E_\theta}{r}$, where $\rho_H$ is the Hall resistivity. Using the fact that, at the Laughlin filling fractions, the quantum Hall resistivity equals the classical Hall resistivity (2.20), we have

$$\rho_H = \frac{B}{c\nu e_{cl}} = \frac{B}{c\nu e B\nu} = \frac{2\pi l_B^2 B}{e\nu} = \frac{\hbar}{e^2 \nu}. $$

Hence, radial component $j_r$ of the current $j$ is given by

$$j_r(r,t) = \frac{e^2 \nu}{\hbar} E_\theta(r,t).$$

The total charge entering the disc $\Sigma$ per unit time is given by $\dot{Q} = -j_r(2\pi r)$. Hence, the total charge accumulated in the time interval $[0,T]$ wherein the flux tube is inserted is equal to

$$Q = -\int_0^T j(r,t)2\pi r dt = -\frac{e^2 \nu}{\hbar} \int_0^T E_\theta(r,t)2\pi r dt. \quad (3.55)$$

Using Faraday,

$$E_\theta(r,t)2\pi r = \oint_{\partial \Sigma} E(r,t) \cdot dr = -\frac{d}{dt} \int_{\Sigma} Bd^2 r = -\frac{d\Phi}{dt}, \quad (3.56)$$

where $\Phi(t)$ is the flux penetrating the disc at time $t$. Combining (3.55) and (3.56) gives that the total charge $Q$ accumulated in the time interval $[0,T]$ is equal to

$$Q = \frac{e^2 \nu}{\hbar} \Phi(T) = \frac{e^2 \nu}{\hbar} \phi_0 = e\nu.$$ 

Hence, by taking $r \to 0$, adiabatically inserting the flux tube results in the accumulation of $e\nu$ charge at $z_0$. We conclude that the quasi-hole carries a fractional charge of $e\nu$.

In addition to being fractionally charged, the quasi-holes also obey so called fractional statistics, meaning that the total wavefunction describing multiple quasi-holes picks up a complex phase $e^{i\alpha \pi}$ when quasi-holes are exchanged, where $\alpha$ is not necessarily an integer. Particles obeying fractional statistics are also called anyons. We will discuss anyons detail in Chapter 6. Specifically, the anyonic nature of the quasi-holes will be explained in Section 6.1.1.

In addition to quasi-holes, we can also create quasi-particles by multiplying the Laughlin wavefunction with a prefactor $\prod_j (\bar{z}_j - z_0)/l_B$. However, doing so means that the total prefactor is no longer analytic, implying that we are no longer describing a state in the lowest Landau level. To remedy this situation, we have to orthogonally project onto the lowest Landau level after adding the prefactor. That is,

$$\phi_{qp}(z_0, \{z_i\}) := P_{LLL} \prod_{j=1}^N \left( \frac{\bar{z}_j - z_0}{l_B} \right) \phi^q(\{z_i\}),$$

where $P_{LLL}$ denotes the orthogonal projection operator that projects onto the lowest Landau level.

To summarize, we have found wavefunctions that give excellent approximations to the ground state wavefunctions at fractional filling factor $\nu = \frac{1}{2s+1}$. We then identified the single particle ground state excitations to be quasi-holes and quasi-particles, carrying fractional charge and one quantum of flux. When created, the quasi-holes and quasi-particles get pinned to the equipotential lines of the impurity potential, which, by the same reasoning as for the IQHE, cause the Hall resistance to form plateaus around filling factors of the form $\nu = \frac{1}{2s+1}$.

As mentioned at the beginning of this chapter, not all filling factors at which the FQHE is observed are of the form $\nu = \frac{1}{2s+1}$. In order to explain the FQHE for non-Laughlin fractions, we turn to composite fermion theory.


### 3.2.5 Jain’s Composite Fermion Theory

The Laughlin wavefunctions describe the FQHE ground states at filling factors of the form \( \nu = \frac{1}{2s+1} \), where \( s \) is a non-zero integer. However, plateaus in the Hall resistance have been observed at other filling factors. We will discuss those which can be described by *composite fermion theory*, a theory proposed by Jain in 1989 [11] which concerns filling factors of the form

\[
\nu = \frac{p}{2ps + 1}, \tag{3.57}
\]

where both \( p \) and \( s \) are non-negative integers. Experimentally observed factors not of the form (3.57), such as the 5/2 and 7/2 states, will not be discussed.

The theory of composite fermions is based on the idea of *flux attachment*, which we illustrate with the following example. Take the Laughlin ground state wavefunction (3.50), where \( q = 2s + 1 \) is an odd integer. We can split \( \phi^{2s+1} \) into a product of two other functions,

\[
\phi^{2s+1}(\{z_i\}) = \left[ \prod_{i<j} \left( \frac{z_i - z_j}{l_B} \right)^{2s+1} \right] \exp \left( -\sum_{i=1}^{N} \frac{|z_i|^2}{4l_B^2} \right)
\]

When going from the second to the third line, we have recognized the \( \nu = 1 \) ground state wavefunction (3.44), denoted by \( \xi^1 \), as a part of the Laughlin \( \nu = \frac{1}{2s+1} \) wavefunction. As explained in Section 3.2.4, the quasi-hole-like prefactor means that we have attached 2\( s \) quasi-holes, each carrying one flux quantum, to each of the electrons located at the positions \( z_1, ..., z_N \). This means we can interpret the \( \nu = \frac{1}{2s+1} \) Laughlin ground state wavefunction as a \( \nu^* = 1 \) ground state wavefunction for composite particles, where each composite particle consists of one electron with 2\( s \) flux quanta attached to it. The star indicates that we are not talking about the actual filling factor, which is still \( \frac{1}{2s+1} \), but about the composite fermion filling factor. Indeed, after attaching 2\( s \) flux quanta to each electron, the remaining flux quanta to composite fermion ratio is indeed 1 : 1, agreeing with \( \nu^* = 1 \) found above.

Jain’s idea is to extend the splitting of the wavefunctions illustrated above to more general filling factors of the form \( \frac{p}{2ps+1} \), as follows. At \( \nu = \frac{1}{2s+1} \) we showed above how to factorize the ground state wavefunction into a quasi-hole factor and the \( \nu^* = 1 \) wavefunction \( \xi^1 \). Similarly, we can factorize a \( \nu = \frac{p}{2sp+1} \) ground state wavefunction into a quasi-hole part and a \( \nu^* = p \) ground state wavefunction, the latter of which we denote by \( \xi^p(\{z_i, \bar{z}_i\}) \). The wavefunction \( \xi^p(\{z_i, \bar{z}_i\}) \) consists of the standard exponential multiplied by a prefactor of the form (3.45), where we now take the determinant of a \( pN_B \) times \( pN_B \) matrix, a fixed row of which contains all the possible polynomial prefactors associated to the \( pN_B \) available single-particle eigenstates. The newly obtained composite fermion ground state wavefunction for \( \nu^* = p \) takes the form

\[
\phi^{s,p}_L(\{z_i\}) = P_{LLL} \left[ \prod_{i<j} \left( \frac{z_i - z_j}{l_B} \right)^{2s} \right] \xi^p(\{z_i, \bar{z}_i\}),
\]

where we have projected onto the lowest Landau level because the electron filling factor \( \nu = \frac{p}{2sp+1} \) < 1, and therefore \( \phi^{s,p}_L \) has to be a lowest Landau level wavefunction, which it is not initially because \( \xi^p \) contains anti-holomorphic terms.
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The fact that the composite fermion filling factor $\nu^*$ is equal to $p$ can be seen as follows. Suppose that at $\nu = \frac{p}{2p+1}$ we have an electron density of $n_{el}$ and a flux quanta density of $n_B = \frac{1}{2\pi\phi_0} = \frac{B}{\phi_0}$.

Recall that the filling factor is given by equation (3.18), i.e.

$$\nu = \frac{n_{el}}{n_B}.$$

Once we attach $2s$ flux quanta carrying $\phi_0 = \frac{e}{h}$ flux each to every electron, the remaining magnetic field $B^*$ that acts on the composite fermions is given by

$$B^* = B - 2s\phi_0 n_{el},$$

and the corresponding flux quanta density is equal to

$$n_B^* = n_B - 2sn_{el}.$$

We have as many composite fermions as electrons, Hence,

$$\nu^* = \frac{n_{el}}{n_B^*} = \frac{n_{el}}{n_B - 2sn_{el}} = \frac{\nu}{1 - 2s\nu} = p,$$

where we have used $\nu = \frac{p}{2p+1}$ for the last equality sign.

In short, at filling factor $\nu = \frac{p}{2p+1}$, we can view our system of electrons as a system of composite fermions at integer filling factor $\nu^* = p$. The formation of a plateau in the Hall resistance can be explained in terms of the IQHE, but now applied to the composite fermions.

This concludes our general discussion on the quantum Hall effects. We will now proceed to the first topic of interest, which is that of Casimir operators in the FQHE.
Chapter 4

The FQHE on the Infinite Plane

From now on, we will restrict our attention to the FQHE. The goal of this chapter is to gain a better understanding of the FQHE fluid in the low temperature limit, where the effective dynamics occur within the fractionally filled Landau level. The physics of this fractionally filled Landau level is completely captured by the projected density operators, which form an algebra, called the projected density operator algebra. Understanding this algebra is therefore of prime importance to understanding the FQHE. In order to gain a better understanding of the algebra of projected density operators, we will try to find and compute certain Casimir operators, which are specific operators that commute with all the elements of the algebra.

In Section 4.1, we will set up the formalism of second quantization and show that the projected density operators satisfy the projected density operator algebra. Once the groundwork has been laid out, in Section 4.2 we will turn our attention towards finding and computing Casimir operators of this same algebra.

4.1 The Projected Density Operator Algebra

We take our 2D system of strongly correlated spinless electrons in a uniform perpendicular magnetic field to be at fractional filling factor, and we assume that it is in the low temperature limit. As mentioned at the beginning of Section 3.2, we neglect the impurities in the sample, but we do include both the kinetic and the interaction term in the Hamiltonian.

Let $n_0$ denote the (fractionally filled) highest non-empty Landau level. The low energy excitations of our system are intra-Landau level excitations within the $n_0$th Landau level, because they do not involve enough energy to have an electron jump to the next unoccupied Landau level. This situation is best described with the language of second quantization.

4.1.1 Second Quantization

Because we will be working in the language of second quantization, we allow the number of particles to fluctuate. That is, we introduce Fock space $\mathcal{F}$ as the direct sum of all $n$-particle states $F_n$,

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} F_n,$$

where $F_n$ is the anti-symmetrized (since the dynamical particles are electrons, which are fermions) tensor product of $n$ copies of the single particle Hilbert space $\mathcal{H}$ defined in Section 2.2.4: $F_n := \mathcal{H} \wedge \mathcal{H} \wedge \ldots \wedge \mathcal{H}$ ($n$-times), and $F_0 := \mathbb{C}$. Recall from Section 2.2.4 that the single particle space $\mathcal{H}$ is
the space of square-integrable sections $L^2(\mathbb{R}^2, \mathcal{L})$ of the line bundle $\mathcal{L}$ over $\mathbb{R}^2$. We extend the inner product $\langle -, - \rangle$ on $\mathcal{H}$ to an inner product $\langle -, -\rangle_F$ on $\mathcal{F}$ as follows,

$$\langle \phi_1 \wedge \ldots \wedge \phi_k, \phi'_1 \wedge \ldots \wedge \phi'_l \rangle_F := \left\{ \begin{array}{ll}
\det(\langle \phi_i, \phi'_j \rangle) & \text{if } k = l \\
0 & \text{otherwise}
\end{array} \right.. \quad (4.1)
$$

As explained in Corollary 3.5, we have a basis of single particle energy eigensections $\phi_{n,m} \in \mathcal{H}$, where $n$ refers to the Landau level and $m$ to the guiding centre. Corresponding to these eigensections, we define the ladder operators $c_{n,m}$ and $c^\dagger_{n,m}$, which are each other's Hermitian conjugates. By definition, $c^\dagger_{n,m}$ acts on an arbitrary state $\phi_1 \wedge \ldots \wedge \phi_k \in F_k$ as follows,

$$c^\dagger_{n,m}(\phi_1 \wedge \ldots \wedge \phi_k) := \phi_{n,m} \wedge \phi_1 \wedge \ldots \wedge \phi_k. \quad (4.2)
$$

That is, $c^\dagger_{n,m}$ creates a particle in the eigenstate $\phi_{n,m}$. Now, $c_{n,m}$ is defined to be the adjoint of $c^\dagger_{n,m}$ with respect to the inner product given by (4.1). Explicitly, for any basis state $\phi_{n_1,m_1} \wedge \ldots \wedge \phi_{n_k,m_k}$,

$$c_{n,m}(\phi_{n_1,m_1} \wedge \ldots \wedge \phi_{n_k,m_k}) := \sum_{i=1}^{k} (-1)^{i-1} \delta_{(n,m),(n_i,m_i)} \phi_{n_1,m_1} \wedge \ldots \wedge \widehat{\phi_{n_i,m_i}} \wedge \ldots \wedge \phi_{n_k,m_k},
$$

where the hat indicates that the term underneath it should be omitted. We see that $c_{n,m}$ destroys a particle in the state $|n,m\rangle$, if present. Hence, the ladder operators are commonly referred to as creation ($c^\dagger$) and annihilation ($c$) operators.

Using the newly defined ladder operators $c_{n,m}$ and $c^\dagger_{n,m}$, we can define the singular operators $\psi(r)$ and $\psi^\dagger(r)$

$$\psi(r) := \sum_{n,m} \phi_{n,m}(r)c_{n,m} \quad (4.3)
$$

$$\psi^\dagger(r) := \sum_{n,m} \phi^\dagger_{n,m}(r)c^\dagger_{n,m}, \quad (4.4)
$$

that create and destroy particles that are Dirac delta centered at the position $r$. Since the Dirac delta function is not an element of the one particle Hilbert space $\mathcal{H}$, the operator $\psi^\dagger(r)$ only makes sense when integrated over. As before, $\psi(r)$ is the adjoint of $\psi^\dagger(r)$.

The operators defined above satisfy the following anti-commutation relations:

$$\{c_{n,m}, c^\dagger_{n',m'}\} = \delta_{n,n'}\delta_{m,m'}, \quad \{\psi(r), \psi^\dagger(r')\} = \delta(r-r'), \quad (4.5)
$$

$$\{c_{n,m}, c_{n',m'}\} = 0, \quad \{\psi(r), \psi(r')\} = 0, \quad (4.6)
$$

$$\{c^\dagger_{n,m}, c^\dagger_{n',m'}\} = 0, \quad \{\psi^\dagger(r), \psi^\dagger(r')\} = 0. \quad (4.7)
$$

The model that we are considering is that of interacting spinless electrons confined to a two-dimensional plane, subject to a perpendicular magnetic field $\mathbf{B} = Be_z$. The Hamiltonian $H$ that corresponds to this system consists of a kinetic and an interaction term. In the language of second quantization, $H$ is given by the following formula,

$$H = H_{kin} + H_{int} = \int_{\mathbb{R}^2} dr \psi^\dagger(r)\frac{1}{2m}(\Pi_x^2 + \Pi_y^2)\psi(r) + \frac{1}{2} \int_{\mathbb{R}^4} dr \, dr' \psi^\dagger(r')U(r'-r)\psi(r)\psi(r'). \quad (4.8)
$$

Here, $\Pi$ is one-particle mechanical momentum operator defined by equation (3.2), which in terms of operators acting on the one-particle Hilbert space $\mathcal{H}$ reads

$$\Pi = -ih\nabla + eA(r), \quad (4.9)$$
and $U$ is the Coulomb interaction potential, 
\[
U(r) = \frac{e^2}{4\pi\varepsilon_0 |r|}.
\] (4.10)

We will massage the interaction term into a form that is more usefull for our purposes. From equations (4.5) – (4.7), we get
\[
H_{\text{int}} = \frac{1}{2} \int_{\mathbb{R}^4} dr\, dr' \, \psi'^\dagger(r') \psi^\dagger(r) U(r' - r) \psi(r') \psi(r)
\]
\[
= -\frac{1}{2} \int_{\mathbb{R}^4} dr\, dr' \, \psi'^\dagger(r') \psi^\dagger(r') U(r' - r) \psi(r) \psi(r')
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^4} dr\, dr' \, \psi(r) U(r' - r) \psi'(r') \psi(r') - \frac{1}{2} \int_{\mathbb{R}^4} dr\, dr' \, \psi'^\dagger(r') U(r' - r) \psi(r') \delta(r - r')
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^4} dr\, dr' \, \rho(r) U(r' - r) \rho(r') - \frac{U(0)}{2} \int_{\mathbb{R}^4} dr\, \rho(r).
\]

where we have defined the density operator
\[
\rho(r) := \psi^\dagger(r) \psi(r).
\]

The extra term $\frac{U(0)}{2} \int_{\mathbb{R}^2} dr\, \rho(r)$ that we have picked up through our manipulations, albeit infinite, is merely a constant, since the integral $\int_{\mathbb{R}^2} dr\, \rho(r)$ is the operator that counts the number of electrons in the system, which is fixed for our purposes. Because it is the energy differences that determine the dynamics of our system, we will ignore this infinite but constant term.

In short, the Hamiltonian that we will be working with is given by
\[
H = \int_{\mathbb{R}^2} dr\, \psi^\dagger(r) \left( \frac{\hbar^2}{2m} \left( \frac{\nabla_x^2}{4} + \frac{\nabla_y^2}{4} \right) + \frac{1}{2} \int_{\mathbb{R}^4} dr\, dr' \, \rho(r) U(r - r') \rho(r') \right).
\] (4.11)

### 4.1.2 Projection onto the Fractionally Filled Landau Level

Since all the effective dynamics occur within the $n_0^{\text{th}}$ Landau level, we want to work with operators that act only on the $n_0^{\text{th}}$ Landau level. The most obvious way to achieve this is to restrict ourselves to using only ladder operators $c_{n,m}$ and $c_{n,m}^\dagger$ with $n = n_0$. Hence, we define
\[
\psi_{n_0}(r) := \sum_m \phi_{n_0,m}(r)c_{n_0,m},
\]
\[
\psi_{n_0}^\dagger(r) := \sum_m \phi_{n_0,m}^\dagger(r)c_{n_0,m}^\dagger.
\]

In the $n_0^{\text{th}}$ Landau level, the kinetic energy is constant $\hbar\omega_C(n_0 + 1/2)$, and can therefore be ignored. Thus, the effective Hamiltonian consists only of the interaction term in equation (4.11), which, after projecting to the $n_0^{\text{th}}$ Landau level, takes the form
\[
H_{\text{eff}} = \frac{1}{2} \int_{\mathbb{R}^4} dr\, dr' \, \psi_{n_0}^\dagger(r) \psi_{n_0}(r) U(r - r') \psi_{n_0}^\dagger(r') \psi_{n_0}(r')
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^4} dr\, dr' \, \rho_{n_0}(r) U(r - r') \rho_{n_0}(r'),
\] (4.12)

where we have defined the projected density operator
\[
\rho_{n_0}(r) := \psi_{n_0}^\dagger(r) \psi_{n_0}(r).
\]
It is the projected density operators $\rho_{n_0}$ that we are interested in, because, in the $n_0^b$ Landau level, the only relevant type of energy is the electron-electron interaction energy, which depends on the electron density distribution that is described by the projected density operators. Indeed, the effective Hamiltonian (4.12) is completely described in terms of the projected density operators. The remainder of Section 4.1 deals with the projected density operators, and the algebra that they form.

For computational convenience, we will describe our system in Fourier space. Fourier transforming the projected density operators yields for the effective Hamiltonian

$$
H_{\text{eff}} = \frac{1}{2} \int \, d\mathbf{r} \, dr' \int \frac{dq}{(2\pi)^2} \rho_{n_0}(\mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{r}} \int \frac{dq'}{(2\pi)^2} \rho_{n_0}(\mathbf{q}') e^{i\mathbf{q}' \cdot \mathbf{r}'} U(\mathbf{r} - \mathbf{r}')
$$

$$
= \frac{1}{2} \int \frac{dq}{(2\pi)^2} \frac{dq'}{(2\pi)^2} \rho_{n_0}(\mathbf{q}) \rho_{n_0}(\mathbf{q}') \int drdr' e^{i(\mathbf{q} \cdot \mathbf{r} + \mathbf{q}' \cdot \mathbf{r}')} U(\mathbf{r} - \mathbf{r}')
$$

$$
= \frac{1}{2} \int \frac{dq}{(2\pi)^2} \frac{dq'}{(2\pi)^2} \rho_{n_0}(\mathbf{q}) \rho_{n_0}(\mathbf{q}') \int \frac{dsdt}{2} e^{\frac{1}{2}(\mathbf{q} \cdot (s+t) + \mathbf{q}' \cdot (s-t))} U(t)
$$

$$
= \frac{1}{2} \int \frac{dq}{(2\pi)^2} \frac{dq'}{(2\pi)^2} \rho_{n_0}(\mathbf{q}) \rho_{n_0}(\mathbf{q}') (2\pi)^2 \delta(\mathbf{q} + \mathbf{q}') \int dt e^{\frac{1}{2}\mathbf{q} \cdot t} U(t)
$$

$$
= \frac{1}{2} \int \frac{dq}{(2\pi)^2} \rho_{n_0}(\mathbf{q}) \rho_{n_0}(-\mathbf{q}) \int dt e^{i\mathbf{q} \cdot t} U(t)
$$

$$
= \frac{1}{2} \int \frac{dq}{(2\pi)^2} U(\mathbf{q}) \rho_{n_0}(\mathbf{q}) \rho_{n_0}(-\mathbf{q}),
$$

where $U(\mathbf{q}) = \frac{e^q}{2\pi} \frac{1}{|q|}$ is the Fourier transform of the Coulomb potential (4.10) — see Appendix A.2 for details — and,

$$
\rho_{n_0}(\mathbf{q}) := \int \, d\mathbf{r} \, e^{-i\mathbf{q} \cdot \mathbf{r}} \rho_{n_0}(\mathbf{r}).
$$

We adopt the convention that the factors of $\frac{1}{2\pi}$ always accompany the $q$-integrals.

Because we have projected to a single Landau level, we prefer to express our operators in terms of the energy eigenbasis $\{ |n, m| \}$ instead of the position basis $\{ |\mathbf{r}| \}$. We have,

$$
\rho_{n_0}(\mathbf{q}) = \int \, d\mathbf{r} \, e^{-i\mathbf{q} \cdot \mathbf{r}} \psi^\dagger_{n_0}(\mathbf{r}) \psi_{n_0}(\mathbf{r})
$$

$$
= \sum_{m,m'} \int \, d\mathbf{r} \, \phi_{n_0,m}^*(\mathbf{r}) e^{-i\mathbf{q} \cdot \mathbf{r}} \phi_{n_0,m'}(\mathbf{r}) c^\dagger_{n_0,m} c_{n_0,m'}
$$

$$
= \sum_{m,m'} \langle n_0, m | e^{-i\mathbf{q} \cdot \mathbf{r}} | n_0, m' \rangle c^\dagger_{n_0,m} c_{n_0,m'},
$$

where, going from the second to the third line, we observed that $\int \, d\mathbf{r} \, \phi_{n_0,m}^*(\mathbf{r}) e^{-i\mathbf{q} \cdot \mathbf{r}} \phi_{n_0,m'}(\mathbf{r})$ is the expectation value $\langle n_0, m | e^{-i\mathbf{q} \cdot \mathbf{r}} | n_0, m' \rangle$ of the operator $e^{-i\mathbf{q} \cdot \mathbf{r}}$.\footnote{We will refrain from putting hats on $\mathbf{r}$ whenever it represents the position operator instead of the position coordinate, as it should be clear from the context which of the two it is meant to represent.}

The next step is to simplify the matrix coefficients $\langle n_0, m | e^{-i\mathbf{q} \cdot \mathbf{r}} | n_0, m' \rangle$ of $\rho_{n_0}(\mathbf{q})$. In doing so, we can make use of the fact that the one-particle Hilbert space $\mathcal{H}$ is a tensor product of two harmonic oscillator Hilbert spaces: $\mathcal{H} \cong \mathcal{H}_0 \otimes \mathcal{K}er(a)$ — see Corollary 3.5 — and write the position operator $\mathbf{r}$ as a sum of the guiding centre operator $\mathbf{R}$ and the cyclotron radius operator $\eta$: $\mathbf{r} = \mathbf{R} + \eta$. Recall that $\eta$ is related to the mechanical momentum $\mathbf{P}$ by (3.9). Because $\Pi_x$ and $\Pi_y$ commute with $X$ and $Y$ — see Appendix equations (A.3) and (A.4), so do $\eta_x$ and $\eta_y$. Hence, we can split $e^{-i\mathbf{q} \cdot \mathbf{r}}$ into two,

$$
e^{-i\mathbf{q} \cdot \mathbf{r}} = e^{-i\eta \cdot \mathbf{q}_x} e^{-i\mathbf{q} \cdot \mathbf{R}}.
$$
Now, using that $\eta$ only acts on the $\mathcal{H}_0$ part and $R$ only acts on the $\text{Ker}(a)$ part of $\mathcal{H}$, we get
\[
\langle n_0, m | e^{-i q \cdot R} | n_0, m' \rangle = \langle n_0 | e^{-i q \eta} | n_0 \rangle \langle m | e^{-i q R} | m' \rangle.
\] (4.17)

The first term in the above expression is called the form factor:
\[
F_{n_0}(q) := \langle n_0 | e^{-i q \eta} | n_0 \rangle.
\] (4.18)

With the above definition, equation (4.15) becomes
\[
\rho_{n_0}(q) = F_{n_0}(q) \bar{\rho}_{n_0}(q),
\] (4.19)

where, by definition,
\[
\bar{\rho}_{n_0}(q) := \sum_{m, m'} \langle m | e^{-i q R} | m' \rangle c_{n_0, m}^\dagger c_{n_0, m'},
\] (4.20)

which we will also refer to as a projected density operator. Before we proceed to discuss the projected density operator algebra, we will first compute the form factor in (4.18).

**Remark 4.1.** Instead of projecting to the $n_0^{th}$ Landau level by restricting the sums over $n$ in (4.3) and (4.4) to $n = n_0$, we could have also first projected the first quantized single particle operator $\sigma(q) := e^{-i q \cdot r}$ to an intra-Landau level operator, and then proceeded to second quantization. Indeed, using (4.16) - (4.18), projecting $\sigma(q)$ to the $n_0^{th}$ Landau level means
\[
\sigma(q) = e^{-i q \cdot r} \mathcal{P}_{n_0} F_{n_0}(q)e^{-i q R} = F_{n_0}(q) \tilde{\sigma}(q),
\]

where $\tilde{\sigma}(q) := e^{-i q \cdot R}$ is a projected density operator that acts only within each single Landau level.

This is the approach that we will take in the next chapter, when we consider a finite sample with periodic boundary conditions.

### 4.1.3 Computation of the Form Factor

In order to compute the form factor, we will follow the reasoning of Lederer and Goerbig [15]. To start with, let us express the operator $e^{-i q \eta}$ in terms of the single-particle ladder operators $a$ and $a^\dagger$. In the computation below, we distinguish between the vector $q = (q_x, q_y) \in \mathbb{R}^2$ and the complex number $q := q_x + i q_y$. Now, from equation (A.7),

\[
q \cdot \eta = q_x \eta_x + q_y \eta_y = \frac{i l_B}{\sqrt{2}} (-a^\dagger + a) q_x - \frac{l_B}{\sqrt{2}} (a^\dagger + a) q_y
= \frac{l_B}{\sqrt{2}} \left[ (-i q_x - q_y) a^\dagger + (i q_x - q_y) a \right]
= \frac{l_B}{\sqrt{2}} \left[ -i (q_x - i q_y) a^\dagger + i (q_x + i q_y) a \right]
= \frac{i l_B}{\sqrt{2}} (-q^* a^\dagger + qa).
\]

Hence, by the Baker-Campbell-Hausdorff formula
\[
e^{A+B} = e^A e^B e^{-\frac{1}{2} [A, B]},
\] (4.21)

which holds whenever the commutator $[A, B]$ commutes with both $A$ and $B$, we have,
\[
e^{-i q \cdot \eta} = e^{\frac{i l_B}{\sqrt{2}} (-q^* a^\dagger + qa)} = e^{-\frac{l_B}{\sqrt{2}} q^* a^\dagger} e^{i \frac{l_B}{\sqrt{2}} q a} e^{-\frac{l_B}{2} q^* a^\dagger [-q^* a^\dagger, qa]} = e^{-\frac{l_B}{\sqrt{2}} q^* a^\dagger} e^{i \frac{l_B}{\sqrt{2}} qa} e^{-\frac{i l_B}{2} q^2 |q|^2}.
\]
Back to the form factor $F(q)$. Recall that, because, in equation (4.17), we have split the state $|n, m\rangle$ into an $|n\rangle$ and an $|m\rangle$ part, we are only acting on the energy part of the Hilbert space, denoted by $H_0$ (see Corollary 3.5). The idea is to insert the identity operator $\sum_k |k\rangle \langle k|$ in the middle and pull out the sum over $k$. Here, $|k\rangle$ refers to the energy eigenstates of the non-degenerate Harmonic oscillator $H_0$. This yields

$$F_{n_0}(q) = \langle n_0 | e^{-i q \hat{\eta}^\dagger} | n_0 \rangle = e^{-\frac{i q^2}{2}} \langle n_0 | e^{-\frac{i q \hat{\eta}^\dagger} n_0\rangle e^{\frac{i q}{\sqrt{2}}} q a^\dagger | n_0 \rangle$$

$$= e^{-\frac{i q^2}{2}} \sum_{k=0}^{\infty} \langle n_0 | e^{-\frac{i q \hat{\eta}^\dagger} k \rangle \langle k | e^{\frac{i q}{\sqrt{2}}} q a^\dagger | n_0 \rangle$$

$$= e^{-\frac{i q^2}{2}} \sum_{k=0}^{\infty} \langle n_0 | e^{-\frac{i q \hat{\eta}^\dagger} k \rangle \langle n_0 | e^{\frac{i q}{\sqrt{2}}} q a^\dagger | k \rangle^*.$$  

For fixed $k$, let us examine the term $\langle n_0 | e^{-\frac{i q \hat{\eta}^\dagger} k \rangle \langle k |$. Since the set of all $|k\rangle = \frac{1}{\sqrt{k!}} (a^\dagger)^k |0\rangle$ is an orthonormal basis of $H_0$, at most one term in the exponential survives. That is, assuming $k \leq n_0$, we have,

$$\langle n_0 | e^{-\frac{i q \hat{\eta}^\dagger} k \rangle \langle k | = \langle n_0 | \frac{1}{(n-k)!} \left( \frac{l_B}{\sqrt{2}} q a^\dagger \right)^{n_0-k} | k \rangle$$

$$= \frac{1}{(n_0-k)!} \sqrt{\frac{n_0!}{k!}} \left( -\frac{l_B q^*}{\sqrt{2}} \right)^{n_0-k} \langle n_0 | | n_0 \rangle$$

$$= \frac{1}{(n_0-k)!} \sqrt{\frac{n_0!}{k!}} \left( -\frac{l_B q^*}{\sqrt{2}} \right)^{n_0-k},$$  

and for $k > n_0$, we get,

$$\langle n_0 | e^{-\frac{i q \hat{\eta}^\dagger} k \rangle \langle k | = 0.$$  

Finally, the form factor can be computed.

$$F_{n_0}(q) = e^{-\frac{i q^2}{2}} \sum_{k=0}^{n_0} \frac{1}{(n_0-k)!} \sqrt{\frac{n_0!}{k!}} \left( -\frac{l_B q^*}{\sqrt{2}} \right)^{n_0-k} \frac{1}{(n_0-k)!} \sqrt{\frac{n_0!}{k!}} \left( \frac{l_B q}{\sqrt{2}} \right)^{n_0-k}$$

$$= e^{-\frac{i q^2}{2}} \sum_{k=0}^{n_0} \frac{n_0!}{(n_0-k)!k!} \frac{1}{(n_0-k)!} \left( \frac{-l_B^2 |q|^2}{2} \right)^{n_0-k}$$

$$= e^{-\frac{i q^2}{2}} \sum_{i=0}^{n_0} \frac{n_0!}{i!(n_0-i)!} \frac{1}{i!} \left( \frac{-l_B^2 |q|^2}{2} \right)^i.$$

Let us summarize the result of this section in the following Corollary.

**Corollary 4.1.** The form factor $F_{n_0}(q) = \langle n_0 | e^{-i q \hat{\eta}^\dagger} | n_0 \rangle$ corresponding to the $n_0^{th}$ Landau level is given by

$$F_{n_0}(q) = e^{-\frac{i q^2}{2}} L_{n_0} \left( \frac{l_B^2 |q|^2}{2} \right),$$  

where $L_n$ is the $n^{th}$ Laguerre polynomial, which is defined by

$$L_n(x) := \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} (-x)^i \frac{1}{i!}.$$  

Having computed the form factor, we now turn to the projected density operators.
4.1.4 Derivation of the Algebra

For two projected density operators $\tilde{\rho}_{n_0}(q)$ and $\tilde{\rho}_{n_0}(s)$, given by equation (4.20),

$$\tilde{\rho}_{n_0}(q) = \sum_{m,m'} \langle m| e^{-iq \cdot R} | m' \rangle e^{\dagger}_{m,n_0} c_{m,n_0} e_{m',n_0},$$

both $\tilde{\rho}_{n_0}(q) + \tilde{\rho}_{n_0}(s)$ and $\tilde{\rho}_{n_0}(q)\tilde{\rho}_{n_0}(s)$ make sense as operators acting on the $n_0^{\text{th}}$ Landau level.

Definition 4.1. Let $A_{pd}$ be the unital associative algebra over $\mathbb{C}$ generated by all the projected density operators $\tilde{\rho}_{n_0}(q)$ with $q \in \mathbb{R}^2$. We can add a Lie algebra structure to $A_{pd}$ by defining the Lie bracket of two elements $\tilde{\rho}_{n_0}(q), \tilde{\rho}_{n_0}(s) \in A_{pd}$ to be

$$[\tilde{\rho}_{n_0}(q), \tilde{\rho}_{n_0}(s)] := \tilde{\rho}_{n_0}(q)\tilde{\rho}_{n_0}(s) - \tilde{\rho}_{n_0}(s)\tilde{\rho}_{n_0}(q).$$

The topic of this section is to compute the Lie bracket on $A_{pd}$.

Before we focus on the Lie algebra structure of the (second quantized) projected density operators $\tilde{\rho}_{n_0}(q)$, we will first look into the (first quantized) single particle projected density operators $\tilde{\sigma}(q)$ introduced in Remark 4.1.

Lemma 4.1. The single particle projected density operators $\tilde{\sigma}(q) = e^{-iq \cdot R}$ form a unital associative algebra over $\mathbb{C}$ for which the product has the following simple expression,

$$\tilde{\sigma}(q)\tilde{\sigma}(s) = \exp \left( -\frac{i l_B^2}{2} q \times s \right) \tilde{\sigma}(q + s).$$

(4.24)

Notation: for vectors $q, s \in \mathbb{R}^2$, $q \times s$ means $q_1 s_2 - q_2 s_1$.

Proof. The fact that the operators form a unital associative algebra over $\mathbb{C}$ is obvious. Now, by equation (3.11),

$$[q \cdot R, s \cdot R] = il_B^2 q \times s,$$

which commutes with both $q \cdot R$ and $s \cdot R$. Hence, using Baker-Campbell-Hausdorff (4.21),

$$\tilde{\sigma}(q)\tilde{\sigma}(s) = e^{-iq \cdot R} e^{-is \cdot R} = e^{\frac{i}{2} [q \cdot R, -is \cdot R]} e^{-i(q + s) \cdot R} = \exp \left( -\frac{i l_B^2}{2} q \times s \right) \tilde{\sigma}(q + s).$$

□

Corollary 4.2. The single-particle projected density operators $\tilde{\sigma}(q) = e^{-iq \cdot R}$ satisfy the following Lie bracket,

$$[\tilde{\sigma}(q), \tilde{\sigma}(s)] = -2i \sin \left( \frac{l_B^2}{2} q \times s \right) \tilde{\sigma}(q + s).$$

(4.25)

The associative algebra in (4.24) is called the magnetic translation algebra\(^2\) because the same algebra is satisfied by the magnetic translation operators, which we will discuss in Section 5.1.2.

With the help of Corollary 4.2, we can derive an explicit formula for the Lie bracket between two second quantized projected density operators $\tilde{\rho}_{n_0}(q)$. But, before we do, let us make a notational remark.

\(^2\)The magnetic translation algebra is usually written with a plus sign in the exponent. In our case, the minus sign can be traced back to the negative charge of the electrons. Some authors prefer to work with a positive sign in the algebra, and choose to have the magnetic field point in the $-z$ direction to compensate for the negative electron charge, but we will not do so.
4.1. THE PROJECTED DENSITY OPERATOR ALGEBRA

Remark 4.2. To avoid cluttering, we will drop the $n_0$ index in all formulas form now on. However, it should be kept in mind that everything that follows refers to the $n_0^{th}$ Landau level. In particular, $\psi(r)$ actually means $\psi_{n_0}(r)$, and, likewise, $c_m$ refers to $c_{n_0,m}$.

First, we compute the commutator $[c_m^\dagger c_{m'}, c_{l'}^\dagger c_l]$, for which we require the following two identities:

$$[AB, C] = A[B, C]_\pm - [C, A]_\pm B,$$  \hspace{1cm} (4.26)

$$[A, BC] = [A, B]_\pm C - B[C, A]_\pm,$$  \hspace{1cm} (4.27)

which hold in any associative algebra. Using equations (4.26) and (4.27), we get

$$[c_m^\dagger c_{m'}, c_{l'}^\dagger c_l] = c_m^\dagger [c_{m'}, c_l^\dagger c_l] - [c_l^\dagger c_l', c_m]c_{m'},$$  

$$= c_m^\dagger (c_{m'}c_l^\dagger - c_l^\dagger 0) - (c_l^\dagger c_l, c_m) - 0 \cdot c_l^\dagger c_m'$$  

$$= c_m^\dagger c_l^\dagger \delta_{m',l} - c_l^\dagger c_m' \delta_{l',m}. \hspace{1cm} (4.28)$$

Lemma 4.2. The Lie bracket between two projected density operators $\bar{\rho}(q)$ and $\bar{\rho}(s)$ is given by

$$[\bar{\rho}(q), \bar{\rho}(s)] = -2i \sin \left[\frac{(q \times s)^2 B}{2}\right] \bar{\rho}(q + s). \hspace{1cm} (4.29)$$

Proof. By equation (4.28),

$$[\bar{\rho}(q), \bar{\rho}(s)] = \sum_{m,m',l,l'} \langle m | e^{-iq \cdot R} | m' \rangle \langle l | e^{-is \cdot R} | l' \rangle [c_m^\dagger c_{m'}, c_{l'}^\dagger c_l]$$  

$$= \sum_{m,m',l,l'} \langle m | e^{-iq \cdot R} | m' \rangle \langle m' | e^{-is \cdot R} | l' \rangle c_m^\dagger c_{m'} c_l^\dagger c_l - \sum_{m',l,l'} \langle l | e^{-is \cdot R} | l' \rangle \langle l' | e^{-iq \cdot R} | m' \rangle c_{m'}^\dagger c_m$$  

$$= \sum_{m,m',l} \langle m | e^{-iq \cdot R} e^{-is \cdot R} | m' \rangle c_m^\dagger c_{m'} - \sum_{m',l} \langle l | e^{-is \cdot R} e^{-iq \cdot R} | m' \rangle c_{m'}^\dagger c_m$$  

$$= \sum_{m,m',l} \langle m | e^{-iq \cdot R} e^{-is \cdot R} | m' \rangle c_m^\dagger c_{m'}. \hspace{1cm} (4.30)$$

Now use Corollary 4.2, which says that

$$[e^{-iq \cdot R}, e^{-is \cdot R}] = -2i \sin \left[\frac{(q \times s)^2 B}{2}\right] e^{-i(q+s) \cdot R},$$

and then pull the sine factor out of the sum over $m$ and $m'$ in (4.30). \hfill \square

The algebra of projected density operators (4.29) was first identified by Girvin, MacDonald and Platzman [5], and is commonly referred to as the GMP algebra.

Remark 4.3. Despite the fact that the Fock space projected density operators $\bar{\rho}(q)$ inherit the Lie algebra structure (4.25) of the single particle projected density operators $\hat{\rho}(q)$, the associative algebra structure is completely different; that is, the $\bar{\rho}(q)$’s do not satisfy (4.24). In the next section, we will find out that this makes computing the Casimir operators, which are trivial in case of the single particle operators, a lot more difficult.

Let us summarize Section 4.1. The dynamics of the electrons in the partially filled Landau level are determined by an algebra of projected density operators $\bar{\rho}(q)$ labelled by vectors $q \in \mathbb{R}^2$. The commutator of two projected density operators is given by

$$[\bar{\rho}(q), \bar{\rho}(s)] = -2i \sin \left[\frac{(q \times s)^2 B}{2}\right] \bar{\rho}(q + s).$$
4.2 CASIMIR OPERATORS

The effective Hamiltonian that acts on the electrons in the $n_0^{th}$ Landau level can be expressed in terms of the projected density operators: from equation (4.13),

$$H_{\text{eff}} = \frac{1}{2} \int \frac{dq}{(2\pi)^2} \bar{U}(q) \rho(q) \bar{\rho}(-q),$$

where we have absorbed the form factors (4.19) into the Coulomb potential,

$$\bar{U}(q) := U(q) F_{n_0}(q) F_{n_0}(-q) = \frac{e^2}{2\epsilon_0 q} e^{-\frac{l^2 B^2}{2q^2}} \left[ L_{n_0} \left( \frac{l^2 B^2}{2q^2} \right) \right]^2.$$

In the last equality sign above we have used $U(q) = \frac{e^2}{2\epsilon_0 q}$ where $e$ is the electron charge and not Euler’s number, and equation (4.23) for the form factor.

As it stands, we have a complete description of the mathematical landscape that we are about to explore, and in the next section, we set about doing exactly that.

4.2 Casimir Operators

In an attempt to get a better understanding of the algebra of projected density operators, we will try to find Casimir operators of the algebra.

Casimir operators are specific operators that lie in the center of the algebra, meaning that they commute with all the other elements. More precisely, in a general Lie algebra $g$ with a countable basis $\{X_i\}$, where $i$ runs over some countable index set $I$, the second order Casimir $\zeta$ is the element of the universal enveloping algebra $U(g)$ of $g$ given by $\zeta = \sum_{i \in I} X_i X_i$, where $X_i$ is the dual vector of $X_i$ with respect to a fixed invariant bilinear form. There are additional Casimir elements for each order, for which we will give explicit formulas in the section below.

Casimir operators are quite common in physics. By Dixmier’s lemma\(^3\), when acting on an irreducible vector space, Casimir operators are represented by scalar multiples of the identity. These scalars sometimes carry physical significance. For example, the total angular momentum and the total spin operator are Casimir operators, and the corresponding scalars that represent them are the total angular momentum and total spin respectively. Keeping this in mind, we now endeavor to find and compute Casimir operators of the algebra of projected density operators, in the hope that they too, carry physical significance.

4.2.1 Construction of the Casimir Operators

The first Casimir operator that we find is quite similar to the effective Hamiltonian itself: it differs only in the fact that the Coulomb potential is absent.

**Theorem 4.1.** the operator

$$\zeta := \int_{\mathbb{R}^2} \frac{dq}{(2\pi)^2} \bar{\rho}(q) \bar{\rho}(-q)$$

commutes with all elements of the associative algebra $\mathcal{A}_{pd}$ generated by the $\bar{\rho}(q)$’s.

**Proof.** We will prove the theorem by hand using the ladder operator expression of the projected density operators (4.20). In order to prove the above lemma, we have to show that $\zeta$ commutes with each of the $\bar{\rho}(q)$’s. Explicitly, $\zeta$ is given by

$$\zeta = \int \frac{dq}{(2\pi)^2} \sum_{m,m',l,l'} \langle m|e^{-i q \cdot R}|m'\rangle \langle l|e^{i q \cdot R}|l'\rangle c_m^\dagger c_m c_{m'}^\dagger c_{l'}^\dagger. \quad (4.32)$$

\(^3\)An infinite dimensional generalization of Schur’s lemma for associative algebras over an uncountable and algebraically closed field, which says that any central element of the algebra is represented by a scalar when the algebra acts irreducibly on some vector space.
Now, for fixed \( s \in \mathbb{R}^2 \), we have
\[
[\zeta, \bar{\rho}(s)] = \int \frac{dq}{(2\pi)^2} \sum_{m,m',l,l'} \langle m | e^{-iq \cdot R} | m' \rangle \langle l | e^{ip \cdot R} | l' \rangle \sum_{p,p'} \langle p | e^{-is \cdot R} | p' \rangle [c^\dagger_m c_{m'} c^\dagger_{p'} c_{p'}],
\]
where, using (4.26) – (4.28),
\[
[c^\dagger_m c_{m'} c^\dagger_{p'} c_{p'}] = c^\dagger_m c_{m'} [c^\dagger_{p'} c_{p'}] + [c^\dagger_m c_{m'} c^\dagger_{p'} c_{p'}] c^\dagger_{p'} c_{p'}
\]
\[
= c^\dagger_m c_{m'} (c^\dagger_{p'} c_{p'} \delta_{p',p} - c^\dagger_{p'} c_{p'} \delta_{p',l}) + (c^\dagger_m c_{m'} \delta_{m',p} - c^\dagger_m c_{m'} \delta_{m',l}) c^\dagger_{p'} c_{p'}.
\]
Thus, by (4.25)
\[
[\zeta, \bar{\rho}(s)] = \int \frac{dq}{(2\pi)^2} \left\{ \sum_{m,m',l,l'} \langle m | e^{-iq \cdot R} | m' \rangle \langle l | e^{ip \cdot R} e^{-is \cdot R} | l' \rangle c^\dagger_m c_{m'} c^\dagger_{p'} c_{p'} \\
- \sum_{m,m',l',p} \langle m | e^{-iq \cdot R} | m' \rangle \langle l | e^{ip \cdot R} e^{-is \cdot R} | l' \rangle c^\dagger_m c_{m'} c^\dagger_{p'} c_{p'} \\
+ \sum_{m,l,l',p} \langle l | e^{iq \cdot R} | l' \rangle \langle m | e^{-is \cdot R} | m' \rangle c^\dagger_m c_{m'} c^\dagger_{p'} c_{p'} \\
- \sum_{m,l,l',p} \langle l | e^{iq \cdot R} | l' \rangle \langle m | e^{-is \cdot R} | m' \rangle c^\dagger_p c_{m'} c^\dagger_l c_{l'} \right\}
\]
\[
= \int \frac{dq}{(2\pi)^2} \left\{ \sum_{m,m',l,l'} \langle m | e^{-iq \cdot R} | m' \rangle \langle l | e^{ip \cdot R} | l' \rangle c^\dagger_m c_{m'} c^\dagger_{p'} c_{p'} \\
+ \sum_{m,m',l,l'} \langle l | e^{iq \cdot R} | l' \rangle \langle m | e^{-is \cdot R} | m' \rangle c^\dagger_m c_{m'} c^\dagger_{p'} c_{p'} \right\}
\]
\[
= \int \frac{dq}{(2\pi)^2} \left[ 2i \sin \left( \frac{(q \times s)^2}{2B} \right) \right] \left( \bar{\rho}(q) \bar{\rho}(-q) + \bar{\rho}(q + s) \bar{\rho}(q) \right)
\]
\[
= \int \frac{dq}{(2\pi)^2} \left[ 2i \sin \left( \frac{(q \times s)^2}{2B} \right) \right] \left( \bar{\rho}(q + s) \bar{\rho}(-q) - \bar{\rho}(q + s) \bar{\rho}(q) \right)
\]
\[
= 0.
\]
Going to the last equality sign above, we relabeled \( q \mapsto q + s \) in the first term. Note that \((q + s) \times s = q \times s\).

It turns out that the above computation can be shortened significantly by exploiting the commutation relation (4.29) directly. We have,
\[
[\zeta, \rho(s)] = \int \frac{dq}{(2\pi)^2} \left[ \bar{\rho}(q) \bar{\rho}(-q), \bar{\rho}(s) \right]
\]
\[
= \int \frac{dq}{(2\pi)^2} \left( \bar{\rho}(q)[\bar{\rho}(-q), \bar{\rho}(s)] + [\bar{\rho}(q), \bar{\rho}(s)] \bar{\rho}(-q) \right)
\]
\[
= \int \frac{dq}{(2\pi)^2} \left[ 2i \sin \left( \frac{(q \times s)^2}{2B} \right) \right] \left( \bar{\rho}(q) \bar{\rho}(-q) - \bar{\rho}(q + s) \bar{\rho}(-q) \right)
\]
\[
= 0.
\]
That is, the element $\zeta$ is a central element of the abstract algebra $A_{pl}$ itself, by which we mean that we forget about the action of $A_{pl}$ on Fock space.

The reasoning above can be generalized to Casimir operators of higher order. The expressions for the higher order Casimir operators are obtained from Fairly, Fletcher and Zachos [3], but the proof below is not.

**Theorem 4.2.** For each $r \in \mathbb{N}$, we have a Casimir invariant, given by,

$$
\zeta_{r+1} := \int \frac{dq_1}{(2\pi)^2} \cdots \frac{dq_r}{(2\pi)^2} \left( \prod_{i<j} e^{-\frac{i}{2} B_{ij} q_i \cdot q_j} \right) \rho(q_1) \cdot \cdots \cdot \rho(q_r) \rho(-q_1 - \cdots - q_r).
$$

**Remark 4.4.** The index $\mathcal{R} := r + 1$ refers to the amount of projected density operators that occur in the expression for the Casimir. It is called the order of the Casimir operator. Note that $\zeta_2 = \zeta$, since the cross-product in the exponential vanishes.

**Proof.** We will prove that $\zeta_{r+1}$ is central in $A_{pl}$.

Equation (4.26) for commutators can be generalized to an arbitrary product of $n$ operators:

$$
[A_1 \cdot \cdots \cdot A_n, B] = \sum_{i=1}^{n} A_1 \cdot \cdots \cdot A_{i-1} \cdot [A_i, B] \cdot A_{i+1} \cdot \cdots \cdot A_n.
$$

(4.34)

To verify the above equation, simply expand the RHS and note that all terms except the first and the last cancel, or prove by induction using (4.26). Now, by (4.34),

$$
[\zeta_{r+1}, \rho(s)] = \int \frac{dq_1}{(2\pi)^2} \cdots \frac{dq_r}{(2\pi)^2} \left( \prod_{i<j} e^{-\frac{i}{2} B_{ij} q_i \cdot q_j} \right) \rho(q_1) \cdots \rho(q_r) \rho(-q_1 - \cdots - q_r), \rho(s)
$$

$$
= \int \frac{dq_1}{(2\pi)^2} \cdots \frac{dq_r}{(2\pi)^2} \left( \prod_{i<j} e^{-\frac{i}{2} B_{ij} q_i \cdot q_j} \right)
$$

$$
\times \left\{ \sum_{r=1}^{r} \rho(q_1) \cdots \rho(q_{r-1}) \left[ \rho(q_r), \rho(s) \right] \rho(q_{r+1}) \cdots \rho(q_r) \rho(-q_1 - \cdots - q_r)
$$

$$
+ \rho(q_1) \cdots \rho(q_r) \rho(-q_1 - \cdots - q_r), \rho(s) \right\}
$$

$$
= \sum_{i=1}^{r} \left\{ - \int \frac{dq_1}{(2\pi)^2} \cdots \frac{dq_r}{(2\pi)^2} \left( \prod_{i<j} e^{-\frac{i}{2} B_{ij} q_i \cdot q_j} \right) 2i \sin \left( \frac{\ell_B^{2} q_i \times s}{2} \right)
$$

$$
\times \rho(q_1) \cdots \rho(q_{i-1}) \rho(q_i + s) \rho(q_{i+1}) \cdots \rho(q_r) \rho(-q_1 - \cdots - q_r)
$$

$$
+ \int \frac{dq_1}{(2\pi)^2} \cdots \frac{dq_r}{(2\pi)^2} \left( \prod_{i<j} e^{-\frac{i}{2} B_{ij} q_i \cdot q_j} \right) 2i \sin \left( \frac{\ell_B^{2} (q_1 + \cdots + q_r) \times s}{2} \right)
$$

$$
\times \rho(q_1) \cdots \rho(q_r) \rho(-q_1 - \cdots - q_r + s) \right\}.
$$

(4.35)

Let us focus on the $l$-th term in the sum above. We will perform a change of variables to the new variable $\tilde{q}_l := q_l + s$, and then remove the tilde afterwards, which amounts to replacing $q_l$ by $q_l - s$ in the expression above. This change of variables leaves the measure unchanged, but we do pick up extra terms from the exponentials

$$
\left( \prod_{i<j} e^{-\frac{i}{2} B_{ij} q_i \cdot q_j} \right) \mapsto \left( \prod_{i<j} e^{-\frac{i}{2} B_{ij} q_i \cdot q_j} \right) \left( \prod_{i=1}^{l} e^{\frac{i}{2} B_{il} q_i \times s} \right) \left( \prod_{l<j} e^{\frac{i}{2} B_{lj} q_l \times q_j} \right).
$$
while the sine term remains unaltered. Rewriting the sine term as a difference of two exponentials
\[-2i \sin \left( \frac{t_B^2 q_i \times s}{2} \right) = \exp \left( -i \frac{t_B^2 q_i \times s}{2} \right) - \exp \left( i \frac{t_B^2 q_i \times s}{2} \right),\]
we have
\[- \left( \prod_{i<j} e^{-\frac{i}{2} t_B^2 q_i \times q_j} \right) 2i \sin \left( \frac{t_B^2 q_i \times s}{2} \right) \]
\[
\mapsto \left( \prod_{i<j} e^{-\frac{i}{2} t_B^2 q_i \times q_j} \right) \left[ \left( \prod_{i<l} e^{\frac{i}{2} t_B^2 q_i \times s} \right) \left( \prod_{l<j} e^{-\frac{i}{2} t_B^2 q_j \times s} \right) - \left( \prod_{i<l} e^{\frac{i}{2} t_B^2 q_i \times s} \right) \left( \prod_{l<j} e^{-\frac{i}{2} t_B^2 q_j \times s} \right) \right] \]
under \( q_l \mapsto q_l - s \).

Hence, performing the transformation \( q_l \mapsto q_l - s \) in each term of the sum over \( l \) in (4.35) separately gives
\[
[\zeta_{r+1}, \rho(s)] = \sum_{l=1}^{r} \left\{ \int \frac{dq_1}{(2\pi)^2} \cdots \frac{dq_r}{(2\pi)^2} \left( \prod_{i<j} e^{-\frac{i}{2} t_B^2 q_i \times q_j} \right) \right. \\
\times \left[ \left( \prod_{i<l} e^{\frac{i}{2} t_B^2 q_i \times s} \right) \left( \prod_{l<j} e^{-\frac{i}{2} t_B^2 q_j \times s} \right) - \left( \prod_{i<l} e^{\frac{i}{2} t_B^2 q_i \times s} \right) \left( \prod_{l<j} e^{-\frac{i}{2} t_B^2 q_j \times s} \right) \right] \\
\left. \times \rho(q_1) \ldots \rho(q_r) \rho(-q_1 - \ldots - q_r + s) \right\} \\
+ \int \frac{dq_1}{(2\pi)^2} \cdots \frac{dq_r}{(2\pi)^2} \left( \prod_{i<j} e^{-\frac{i}{2} t_B^2 q_i \times q_j} \right) 2i \sin \left( \frac{t_B^2 (q_1 + \ldots + q_r) \times s}{2} \right) \\
\times \rho(q_1) \ldots \rho(q_r) \rho(-q_1 - \ldots - q_r + s) \\
= \int \frac{dq_1}{(2\pi)^2} \cdots \frac{dq_r}{(2\pi)^2} \left( \prod_{i<j} e^{-\frac{i}{2} t_B^2 q_i \times q_j} \right) \rho(q_1) \ldots \rho(q_r) \rho(-q_1 - \ldots - q_r + s) \\
\times \left\{ \sum_{l=1}^{r} \left[ \left( \prod_{i<l} e^{\frac{i}{2} t_B^2 q_i \times s} \right) \left( \prod_{l<j} e^{-\frac{i}{2} t_B^2 q_j \times s} \right) - \left( \prod_{i<l} e^{\frac{i}{2} t_B^2 q_i \times s} \right) \left( \prod_{l<j} e^{-\frac{i}{2} t_B^2 q_j \times s} \right) \right] \\
+ 2i \sin \left( \frac{t_B^2 (q_1 + \ldots + q_r) \times s}{2} \right) \right\} \\
= 0,
\]
since
\[
\sum_{l=1}^{r} \left[ \left( \prod_{i<l} e^{\frac{i}{2} t_B^2 q_i \times s} \right) \left( \prod_{l<j} e^{-\frac{i}{2} t_B^2 q_j \times s} \right) - \left( \prod_{i<l} e^{\frac{i}{2} t_B^2 q_i \times s} \right) \left( \prod_{l<j} e^{-\frac{i}{2} t_B^2 q_j \times s} \right) \right] \\
= \left( \prod_{j=1}^{r} e^{-\frac{i}{2} t_B^2 q_j \times s} \right) - \left( \prod_{i=1}^{r} e^{\frac{i}{2} t_B^2 q_i \times s} \right) = -2i \sin \left( \frac{t_B^2 (q_1 + \ldots + q_r) \times s}{2} \right).
\]

Now that we have found a Casimir operator \( \zeta_{\mathcal{R}} \) of the algebra \( \mathcal{A}_{pd} \) for each order \( \mathcal{R} \in \{2, 3, 4, \ldots\} \), let us try to rewrite the abstract expression (4.33) into a concrete expressions for \( \zeta_{\mathcal{R}} \) in terms of ladder operators acting on Fock space, to see if there is any physically relevant information to be obtained from the Casimir operators.
4.2.2 Computation of the Second Order Casimir

We have the following explicit formula for the second order Casimir operator \( \zeta = \zeta_2 \), given by equation (4.32), in terms of the ladder operators \( c_{m'} \) and \( c_m^\dagger \),

\[
\zeta = \sum_{m,m',l,l'} \int \frac{dq}{(2\pi)^2} \langle m | e^{-iq R} | m' \rangle \langle l | e^{iq R} | l' \rangle c_{m'} c_m^\dagger c_{l'} c_l.
\]  

(4.36)

In this section, we will try to simplify the above expression by computing the prefactors \( \langle m | e^{-iq R} | m' \rangle \).

We will proceed in the same manner as we did in Section 4.1.3. From equation (A.5), we have

\[
q \cdot R = q_x X + q_y Y = \frac{l_B}{\sqrt{2}} \left[ q_x (b^\dagger + b) + iq_y (b^\dagger - b) \right] = \frac{l_B}{\sqrt{2}} (qb^\dagger + q^* b),
\]

where, as before, we define \( q := q_x + iq_y \in \mathbb{C} \). Hence, by Baker-Campbell-Hausdorff (4.21),

\[
e^{-iq R} = e^{-\frac{i l_B}{\sqrt{2}} (qb^\dagger + q^* b)} = e^{-\frac{i l_B q_b^*}{\sqrt{2}} b^\dagger e^{-\frac{i l_B q^* b}{\sqrt{2}}} e^\frac{i l_B q^* b}{\sqrt{2}} e^{-\frac{i l_B q b^*}{\sqrt{2}}} e^{-\frac{i l_B q_b}{\sqrt{2}}} \sqrt{2} |q|^2}.
\]

Following the same reasoning as in equation (4.22), we have

\[
\langle m | e^{-iq R} | m' \rangle = e^{-\frac{\xi^2}{2} |q|^2} \sum_{k=0}^{\infty} \langle m | e^{-\frac{i l_B q_b^*}{\sqrt{2}} b^\dagger} | k \rangle \langle k | e^{-\frac{i l_B q^* b}{\sqrt{2}}} | m' \rangle
\]

\[
= e^{-\frac{\xi^2}{2} |q|^2} \sum_{k=0}^{M} \sqrt{\frac{m!}{k!}} \frac{1}{(m-k)!} \left( -\frac{i l_B q}{\sqrt{2}} \right)^{m-k} \sqrt{\frac{m!}{k!}} \frac{1}{(m-k)!} \left( \frac{i l_B q^*}{\sqrt{2}} \right)^{m'-k},
\]

where \( M := \min\{m, m'\} \). Keep in mind that, although the mathematical manipulations are similar to those in Section 4.1.3, the labels \( m \) and \( k \) now refer to the guiding center instead of the energy.

Contrary to the computation in Section 4.1.3, \( m \) is in general not equal to \( m' \). Hence, we distinguish between the cases \( m \geq m' \) and \( m \leq m' \). In case \( m \geq m' \), we have \( M = m' \), and therefore

\[
\langle m | e^{-iq R} | m' \rangle = e^{-\frac{\xi^2}{2} |q|^2} \sqrt{\frac{m!}{m!}} \left( -\frac{i l_B q}{\sqrt{2}} \right)^{m - m'} \sum_{k=0}^{m'} \frac{m!}{(m-k)!} \left( -\frac{i l_B q}{\sqrt{2}} \right)^{m'-k} \left( \frac{i l_B q^*}{\sqrt{2}} \right)^{m'-k}
\]

\[
= e^{-\frac{\xi^2}{2} |q|^2} \sqrt{\frac{m!}{m!}} \left( -\frac{i l_B q}{\sqrt{2}} \right)^{m - m'} L_{m'}^{(m-m')} \left( \frac{i l_B q^2}{2} \right),
\]

(4.37)

where \( L_m^{(n)} \) is the \( n \)th generalized Laguerre polynomial,

\[
L_n^{(\alpha)}(x) := \sum_{i=0}^{n} \frac{(n + \alpha)!}{(n - i)! i!} (-x)^i = \sum_{i=0}^{n} \frac{(n + \alpha)!}{(n - i)! (\alpha + i)!} \frac{(-x)^i}{i!}.
\]

Similarly, in case \( m \leq m' \), we get

\[
\langle m | e^{-iq R} | m' \rangle = e^{-\frac{\xi^2}{2} |q|^2} \sqrt{\frac{m!}{m!}} \left( -\frac{i l_B q^*}{\sqrt{2}} \right)^{m' - m} L_{m}^{(m' - m)} \left( \frac{i l_B |q|^2}{2} \right).
\]

(4.38)

The next step towards finding a simplified expression for \( \zeta \) is computing the matrix coefficients \( \int dq (\langle m | e^{-iq R} | m' \rangle \langle l | e^{iq R} | l' \rangle) \). From equations (4.37) and (4.38), we see that the differences \( m' - m \) and \( l' - l \) determine the outcome. For example, if \( m = m' \) and \( l = l' \), then, by orthogonality of the Laguerre polynomials (see Abramowitz and Stegun [1], Chapter 22 on Orthogonal Polynomials, p.775),

\[
\int_0^\infty dr r^\alpha e^{-\rho r} L_n^{(\alpha)}(r)L_n^{(\alpha)}(r) = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{n,m},
\]

(4.39)
which holds for $\alpha > -1$, and $\Gamma$ denotes the gamma function

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^t,$$

we have

$$\int \frac{dq}{(2\pi)^2} \langle m | e^{-i q \cdot R} | m \rangle \langle l | e^{i q \cdot R} | l \rangle = \int \frac{dq}{(2\pi)^2} e^{-\frac{i q^2}{2}} L_m \left( \frac{t_B^2 |q|^2}{2} \right) L_l \left( \frac{t_B^2 |q|^2}{2} \right)$$

$$= \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^\infty dr r e^{-\frac{i q^2}{2} r^2} L_m \left( \frac{t_B^2 r^2}{2} \right) L_l \left( \frac{t_B^2 r^2}{2} \right)$$

$$= \frac{1}{2\pi t_B^2} \int_0^\infty e^{-s} L_m(s) L_l(s)$$

$$= \frac{1}{2\pi t_B^2} \delta_{m,l}.$$

When going from the second to the third line above, we substituted $s = \frac{t_B^2 r^2}{2}$.

In general, $m$ and $m'$ are not equal, and neither are $l$ and $l'$. Define $a := m' - m$ and $a + b := l - l'$ to be their respective differences. Before we proceed to the most general case, let us examine the case that $b = 0$ and $a > 0$, i.e. $m' = m + a$ and $l = l' + a$. Why we have chosen $a$ and $b$ in such a way that the order of the primes is reversed should become clear in the next few lines. Now,

$$\int \frac{dq}{(2\pi)^2} \langle m | e^{-i q \cdot R} | m + a \rangle \langle l' + a | e^{i q \cdot R} | l' \rangle$$

$$= \int \frac{dq}{(2\pi)^2} e^{-\frac{i q^2}{2} |q|^2} \sqrt{\frac{m!}{(m + a)!}} \sqrt{\frac{l'!}{(l' + a)!}} \left( -i l_B q^* \right)^a \left( i l_B q \right)^a L_m^{(a)} \left( \frac{t_B^2 |q|^2}{2} \right) L_{l'}^{(a)} \left( \frac{t_B^2 |q|^2}{2} \right)$$

$$= \sqrt{\frac{m!}{(m + a)!}} \sqrt{\frac{l'!}{(l' + a)!}} \frac{1}{2\pi t_B^2} \int_0^\infty ds e^{-s} s^a L_m^{(a)}(s) L_{l'}^{(a)}(s)$$

$$= \frac{m!}{(m + a)!} \frac{1}{2\pi t_B^2} \delta_{m,l'} = \frac{m!}{m!} \frac{1}{2\pi t_B^2} \delta_{m,l'},$$

where we have used the same substitution as before followed by the orthogonality of the Laguerre polynomials (4.39). The case $a < 0$ goes analogously. Note that the condition $\alpha > -1$ mentioned above is never violated, because, as can be seen from equations (4.37) and (4.38), it is always the absolute value $|a|$ of the difference $m' - m$ that occurs in expressions above.

Finally, we can tackle the remaining case, that of non-zero $b$. It turns out that this is actually the most trivial case of all. Note that, in the above computation, the fact that $m' = m + a = l - l'$ guarantees that the exponents of $q$ and $q^*$ are equal, meaning that any polar angle dependence cancels out. Whenever we take $b \neq 0$, there will be a $\theta$ dependence in the expression of the form $e^{i b \theta}$ which integrates to zero. For example, let us consider the case $a, b > 0$.

$$\int \frac{dq}{(2\pi)^2} \langle m | e^{-i q \cdot R} | m + a \rangle \langle l' + a + b | e^{i q \cdot R} | l' \rangle$$

$$= \int \frac{dq}{(2\pi)^2} e^{-\frac{i q^2}{2} |q|^2} \sqrt{\frac{m!}{(m + a)!}} \sqrt{\frac{l'!}{(l' + a + b)!}} \left( -i l_B q^* \right)^{a+b} \left( i l_B q \right)^{a+b} L_m^{(a+b)} \left( \frac{t_B^2 |q|^2}{2} \right) L_{l'}^{(a+b)} \left( \frac{t_B^2 |q|^2}{2} \right)$$

$$= \sqrt{\frac{m!}{(m + a)!}} \sqrt{\frac{l'!}{(l' + a + b)!}} \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta e^{i b \theta} \int_0^\infty dr r e^{-\frac{i q^2}{2} r^2} \left( -i l_B q^* \right)^{2a+b} \left( i l_B q \right)^{2a+b} L_m^{(a+b)} \left( \frac{t_B^2 r^2}{2} \right) L_{l'}^{(a+b)} \left( \frac{t_B^2 r^2}{2} \right)$$

$$= 0,$
because \( \int_0^{2\pi} d\theta e^{ib\theta} = 0 \) and the fact that the \( r \)-integral is bounded due to the Gaussian factor. Any other choice of \( a \) and \( b \) such that \( b \neq 0 \) gives zero in a similar fashion.

Let us summarize the obtained results in the following corollary.

**Corollary 4.3.** For \( m, l', a \) and \( b \) integers such that \( m, m + a, l' + a + b, l' > 0 \), we have

\[
\int \frac{dq}{(2\pi)^2} \langle m | e^{-iq \cdot R} | m + a \rangle \langle l' + a + b | e^{iq \cdot R} | l' \rangle = \frac{1}{2\pi l_B^2} \delta_{m,l'} \delta_{b,0}.
\]

In particular, the above expression is independent of \( a \).

Equipped with the above corollary, we can simplify the expression (4.36) for \( \zeta \) significantly.

Originally, the sum in (4.36) goes over all non-negative integers. Rearranging the sum such that it goes over \( m, l', a \) and \( b \) in such a way that \( m, m + a, l' + a + b, l' > 0 \), we get

\[
\zeta = \sum_{m, l', a, b} \int \frac{dq}{(2\pi)^2} \delta_{m, l'} \delta_{b, 0} \delta_m c_m^\dagger c_m^a c_m^b c_{l'}^b c_{l'}^a.
\]

Let us summarize our result.

**Corollary 4.4.** The second order Casimir operator \( \zeta \) defined in (4.31) is given by the following explicit formula:

\[
\zeta = \frac{1}{2\pi l_B^2} \sum_{m=0}^\infty \sum_{a=-m}^\infty c_m^a c_m^\dagger. \tag{4.41}
\]

We have tried to apply a similar reasoning to the higher order Casimir operators, but the effort turned out to be fruitless. The difficulty lies in the fact that, for as far as our knowledge goes, formula (4.39) regarding the orthogonality of the Laguerre polynomials does not have a higher order counterpart.

With the above formula for \( \zeta \) at our disposal, we can now try to find out what the physical interpretation of the second order Casimir \( \zeta \) is. This is the topic of the next section.

### 4.2.3 Physical Interpretation of the Second Order Casimir

In order to give physical meaning to the Casimir operator \( \zeta \), we have to normal order the formula for \( \zeta \) in equation (4.41). Before we do so, let us briefly review the mechanism of the ladder operators and their normal ordering.

For starters, we write an arbitrary \( n_0^{th} \) Landau level multi-particle basis state in terms of creation operators,

\[
\phi_{m_1} \wedge \ldots \wedge \phi_{m_k} = c_{m_1}^\dagger \cdots c_{m_k}^\dagger |0\rangle, \tag{4.42}
\]

where \( |0\rangle \in \mathcal{F}_0 \) denotes the vacuum. Recall that we are suppressing the index \( n_0 \) that labels the energy level from the notation. As an illustrative example, using the fermionic anti-commutation relations (4.5) – (4.7), we will compute the action of the operator \( c_m^\dagger c_m^b \) on the basis state (4.42). The strategy is always the same: (anti)-commute the lowering operators past all the creation operators in (4.42) until they annihilate the vacuum. This is the reason that we want our operators to be *normal ordered*, which means that the creation operators are always to the left of the annihilation operators.
We have,
\[
c_{m}^{\dagger}c_{m}\cdots c_{m k}^{\dagger}|0\rangle = c_{m}^{\dagger}\sum_{i=1}^{k}(-1)^{i-1}\delta_{m,m_{i}}c_{m_{i}}^{\dagger}\cdots c_{m k}^{\dagger}|0\rangle = \sum_{i=1}^{k}\delta_{m,m_{i}}c_{m_{i}}^{\dagger}\cdots c_{m k}^{\dagger}|0\rangle = \left(\sum_{i=1}^{k}\delta_{m,m_{i}}\right)c_{m_{1}}^{\dagger}\cdots c_{m k}^{\dagger}|0\rangle.
\]

The hat above means that the operator underneath it does not occur in the sum. We conclude that every state of the form \(c_{m}^{\dagger}c_{m}\cdots c_{m k}^{\dagger}|0\rangle\) is an eigenstate of \(c_{m}^{\dagger}c_{m}\) with eigenvalue
\[
\sum_{i=1}^{k}\delta_{m,m_{i}} = \begin{cases} 1 & \text{if } m \in \{m_{1},\ldots,m_{k}\} \\ 0 & \text{if } m \notin \{m_{1},\ldots,m_{k}\} \end{cases}.
\]

That is, \(c_{m}^{\dagger}c_{m}\) checks whether \(\phi_{m}\) occurs in (4.42) or not, i.e whether the state \(m\) is occupied by an electron. We refer to \(c_{m}^{\dagger}c_{m}\) as the number operator.

**Definition 4.2.** We write \(n_{m}\) for the number operator \(c_{m}^{\dagger}c_{m}\).

In a similar fashion, we compute that every state of the form (4.42) is an eigenstate of \(c_{m}^{\dagger}c_{m}^{\dagger}c_{l}c_{l}c_{m}\) with eigenvalue
\[
\sum_{i,j=1}^{k}\delta_{m_{i},m_{j}}\delta_{l_{i},l_{j}} = \begin{cases} 1 & \text{if } m,l \in \{m_{1},\ldots,m_{k}\} \\ 0 & \text{otherwise} \end{cases}.
\]

That is, \(c_{m}^{\dagger}c_{m}^{\dagger}c_{l}c_{l}c_{m}\) counts whether the pair \((m,l)\) is occupied by electrons or not.

**Definition 4.3.** We write \(n_{m,l} := c_{m}^{\dagger}c_{m}^{\dagger}c_{l}c_{l}c_{m}\).

Finally, we come to the second order Casimir \(\zeta\). We have,
\[
\zeta = \frac{1}{2\pi l_{B}^{2}}\sum_{m,l=0}^{\infty}n_{m,l}c_{m}^{\dagger}c_{l}^{\dagger}c_{m} = \frac{1}{2\pi l_{B}^{2}}\left( -\sum_{m,l=0}^{\infty}c_{m}^{\dagger}c_{l}^{\dagger}c_{l}c_{m} + \sum_{m,l=0}^{\infty}c_{m}^{\dagger}c_{m} \right) = \frac{1}{2\pi l_{B}^{2}}\left( -\sum_{m,l=0}^{\infty}n_{m,l} + \sum_{m=0}^{\infty}n_{m} \right) .
\]

The effect of normal ordering is that, for fixed \(m\), we get a number operator for every \(l\). Thus, we get infinitely many number operators\(^{5}\). This means the operator \(\zeta\) returns the number \(\infty\) whenever there is at least one particle present. We conclude that, despite the fact that the constituents in the final expression (4.43) for \(\zeta\) have a physical meaning, \(\zeta\) itself does not.

The most logical step to take in order to overcome the infinities arising in the final expression (4.43) for \(\zeta\) is to truncate the sums over \(m\) and \(l\) at some arbitrary integer \(M\), which would correspond

\(^{4}\)The number operator \(n_{m}\) counts the number of particles in the state \(m\), hence the name.

\(^{5}\)Whenever \(m = l\) there is also a number operator hidden in \(n_{m,l}\), but this only gives one extra number operator \(n_{m}\) for each \(m\), which cannot possibly compensate for the infinitely many number operators that the second term in (4.43) contains.
to restricting the number of available guiding center positions to be finite. This is exactly what happens on a finite sample. However, in order to model a finite sample correctly, we have to include boundary conditions, which come with difficulties of their own. How to deal with the boundary of a finite sample is the subject of the next chapter.
Chapter 5

The FQHE on the Torus

The second order Casimir operator constructed in the previous chapter suffers from infinities arising from the fact that each Landau level is infinitely degenerate, which is a consequence of the fact that we were working on an infinite plane. These infinities make it difficult to give physical meaning to the second order Casimir operator. Moreover, the approach we took in computing the second order Casimir does not seem to generalize to higher order Casimir operators.

To overcome these problems, we want to have finitely degenerate Landau levels. This can be achieved by working on a torus, i.e. a finite sample with periodic boundary conditions. In Section 5.1, we will construct the line bundle over the torus and set up the appropriate mathematical formalism to describe electrons in a uniform perpendicular magnetic field moving on a torus. Next, in Section 5.2, we will redo the computation for the Casimir operators found in Theorem 4.2, and give physical meaning to them.

5.1 Periodic Boundary Conditions

In this section, we will construct the mathematical model that we use to describe the FQHE on the torus.

**Definition 5.1.** Let $\mathbb{T}$ denote the torus of dimension $a$ times $b$. That is, $\mathbb{T} := (\mathbb{R}/a\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$, where $a$ and $b$ are positive real numbers.

Moreover, we use $p : \mathbb{R}^2 \rightarrow \mathbb{T}$ to denote the projection that maps $r \in \mathbb{R}^2$ to its corresponding equivalence class $\bar{r} \in \mathbb{T}$.

We apply a constant magnetic field $B = Be_z$ that is everywhere perpendicular to the surface of the torus.

5.1.1 Construction of the One-Particle Hilbert Space

We will now construct the one-particle Hilbert space of sections over $\mathbb{T}$. In doing so, we can copy the construction given in Section 2.2.4 word for word, keeping the following remark in mind.

**Remark 5.1.** In the part denoted by “notation” just below equation (2.44), it not clear what is meant by the “domain enclosed by two given curves”, as we now have a choice between two possible domains. We require that both domains give the same phase factor, which is to say that the integral over the difference between the two domains, i.e. the whole torus $\mathbb{T}$, gives a total phase factor of $2\pi$. In other words, we require

$$1 = \exp \left( 2\pi i k \int_{\mathbb{T}} B \right) = \exp (2\pi i k Bab)$$
Recalling that \( \kappa = -\frac{\phi}{\pi} = -\frac{1}{\phi_0} \), see Example 2.2, where \( \phi_0 \) is the flux quantum, the above equation restricts the magnetic field \( B \) to take only specific values such that the total magnetic flux \( \Phi = Bab \) through the torus is given by
\[
\Phi = N_B \phi_0.
\]
That is, the total flux is an integer \( N_B \) times the flux quantum \( \phi_0 \). Also, comparing to equation (3.16), we conclude that \( N_B \), being equal to
\[
N_B = \frac{\Phi}{\phi_0} = \frac{eBab}{h} = \frac{\text{Area}(\mathbb{T})}{2\pi l_B^2},
\]
is also equal to the degeneracy of each Landau level.

Repeating all the steps in Sections 2.2.4 and 2.2.5, we have now constructed a line bundle \( \mathcal{L}_T \) over \( \mathbb{T} \). However, working on a torus has its disadvantages. Notably, the singular homology, and therefore also the de-Rham cohomology groups of the torus, given by
\[
H^k_{dR}(\mathbb{T}) := \frac{\text{Ker}(d_k)}{\text{Im}(d_{k-1})} = \begin{cases} 
\mathbb{R} & \text{for } k = 0, 2 \\
\mathbb{R}^2 & \text{for } k = 1 \\
0 & \text{otherwise}
\end{cases},
\]
are less trivial than those of the infinite plane. As a consequence, a given 2-form \( B \in \Omega^2(\mathbb{T}) \) can in general not be written as \( B = \nabla \times A \) on all of \( \mathbb{T} \) (but it is possible to write \( B \) as the rotation of a vector potential locally; not that this is of any use to us)
\(^1\). In order to circumvent this drawback, we will work with the universal covering space of \( \mathbb{T} \), which is \( \mathbb{R}^2 \) itself. Of course, we do want to keep the periodicity of the torus, so we equip \( \mathbb{R}^2 \) with a line bundle that is the pull-back of \( \mathcal{L}_T \).

**Definition 5.2.** Let \( \mathcal{L}^* := p^*(\mathcal{L}_T) \) be the pull-back bundle over \( \mathbb{R}^2 \), where \( p : \mathbb{R}^2 \to \mathbb{T} \) is the canonical projection from Definition 5.1.

For a fixed point \( r \in \mathbb{R}^2 \), the fibre of \( \mathcal{L}^* \) at the point \( r \) is given by \( \mathcal{L}^*|_r = \mathcal{L}_T|_{p(r)} \). That is, we have extended \( \mathcal{L}_T \) periodically over \( \mathbb{R}^2 \).

We can describe sections of \( \mathcal{L}^* \) as sections of the original line bundle \( \mathcal{L} \) constructed in Section 2.2.4, with an extra periodicity condition that we will specify below. That is, we can embed \( \Gamma(\mathcal{L}^*) \) into \( \Gamma(\mathcal{L}) \). As a consequence, we are now back to the more familiar situation of \( \mathcal{L} \) over \( \mathbb{R}^2 \). In particular, the magnetic field \( B \) that causes the QHE takes the familiar form \( B = Be_z \). We can view this magnetic field as the curvature 2-form \( B = dA \) of a connection 1-form \( A \), exactly as described in Section 2.2.5.

In order to specify the periodicity condition mentioned above, we have to make sense of what it means to translate a section \( f \in \Gamma(\mathcal{L}) \).

**Definition 5.3.** Let \( \tau^u_1 \) and \( \tau^v_2 \) be the unique operators that, when acting on a section \( f \in \Gamma(\mathcal{L}) \),
\[(a) \text{ translate a distance } u \text{ in the } x\text{-direction and a distance } v \text{ in the } y\text{-direction respectively, and,}
(b) \text{ commute with the connection } \nabla \text{ defined in Section 2.2.5.}
\]

To make the above definition more concrete, let us choose a gauge and compute the aforementioned translation operators in this particular gauge.

**Remark 5.2.** From now on, we will work in the symmetric gauge.

In order for the operators \( \tau^u_1 \) and \( \tau^v_2 \) to commute with the connection, it is necessary that, in addition to translating, \( \tau^u_1 \) and \( \tau^v_2 \) also multiply with a prefactor to compensate for the extra term coming from the magnetic vector potential \( A \), which plays the role of the connection 1-form.

\(^1\)By rotation, we mean the two-dimensional version \( \nabla \times A = \partial_x A_y - \partial_y A_x \).
To illustrate this concept, let us try the following ansatz for $\tau_1^u$ and $\tau_2^v$. For an arbitrary section $f \in \Gamma(\mathcal{L})$, write
\begin{align*}
\tau_1^u(f)(x, y) &= F_1(x)G_1(y)f(x - u, y), \\
\tau_2^v(f)(x, y) &= F_2(x)G_2(y)f(x, y - v),
\end{align*}
where $F_1(x)G_1(y)$ and $F_2(x)G_2(y)$ are still to be determined prefactors. We require that the translation operators commute with $\nabla$. Because $\Pi_x$ and $\Pi_y$ are (proportional to) the covariant derivative operators — see Remark 2.9 — this amounts to,
\begin{align*}
[\Pi_x, \tau_1^u] &= 0, & [\Pi_y, \tau_1^u] &= 0, \\
[\Pi_x, \tau_2^v] &= 0, & [\Pi_y, \tau_2^v] &= 0.
\end{align*} \tag{5.2}
\tag{5.3}
Now, the first equation of (5.2) applied to a section $f \in \Gamma(\mathcal{L})$ reads
\begin{equation}
(-i\hbar\partial_x - \frac{eB}{2}y)F_1(x)G_1(y)f(x - u, y) = F_1(x)G_1(y)\left(-i\hbar\partial_x - \frac{eB}{2}y\right)f(x - u, y),
\end{equation}
That is, $F_1(x) = 1$. The second equation of (5.2) with $F_1(x) = 1$ gives
\begin{equation}
(-i\hbar\partial_y + \frac{eB}{2}x)G_1(y)f(x - u, y) = G_1(y)\left(-i\hbar\partial_y + \frac{eB}{2}(x - u)\right)f(x - u, y),
\end{equation}
which is solved by the less trivial $G_1(y) = \exp\left(-\frac{ieBu}{2\hbar}\right) = \exp\left(-\frac{iuv}{2\hbar}\right)$. Analogously, equation (5.3) gives $F_2(x) = \exp\left(\frac{iuv}{2\hbar}\right)$ and $G_2(y) = 1$.

\textbf{Corollary 5.1.} The translation operators $\tau_1^u$ and $\tau_2^v$ take the following form when expressed in the symmetric gauge:
\begin{align*}
\tau_1^u(f)(x, y) &= \exp\left(-\frac{iuy}{2\hbar}\right)f(x - u, y), \tag{5.4} \\
\tau_2^v(f)(x, y) &= \exp\left(\frac{iux}{2\hbar}\right)f(x, y - v). \tag{5.5}
\end{align*}
where $f \in \Gamma(\mathcal{L})$ is arbitrary.

Having defined the translation operators $\tau_1^u$ and $\tau_2^v$, let us summarize in the corollary below which sections of $\mathcal{L}$ correspond to sections of $\mathcal{L}^*$.

\textbf{Corollary 5.2.} The space of sections $\Gamma(\mathcal{L}^*)$ of the periodic line bundle $\mathcal{L}^*$ over $\mathbb{R}^2$ is isomorphic to the subspace $\{f \in \Gamma(\mathcal{L}) : f = \tau_1^u(f) = \tau_2^v(f)\} \subset \Gamma(\mathcal{L})$.

Note that the sections $f \in \Gamma(\mathcal{L}^*)$ correspond to actual periodic sections of $\mathcal{L}$, despite the fact that the equation $f = \tau_1^u(f) = \tau_2^v(f)$ seems to imply otherwise due to the extra phase factors involved. The additional phase factors in (5.4) and (5.5) appear because of the fact that we are working with a specific trivialization of the line bundle, which is done in such a way that two neighboring fibres are rotated with respect to one another.

\textbf{Remark 5.3.} At this point, one of the main differences between the infinite plane and the torus becomes apparent. Because all square-integrable sections on the plane are rapidly-decreasing functions, no momentum or position operator (or function thereof) can map wavefunctions out of the space of square-integrable sections. In the case of the torus, however, because of the extra periodicity condition $f = \tau_1^u(f) = \tau_2^v(f)$ on sections $f \in \Gamma(\mathcal{L}^*)$, it is possible that some operators are no longer well-defined on $\Gamma(\mathcal{L}^*)$ because they do not preserve periodicity.
Fortunately, the covariant derivative operators commute with the translation operators (by construction), so the mechanical momentum and therefore also the ladder operators $a$ and $a^\dagger$ are well-defined on sections of $\mathcal{L}^*$. Hence, we can still use the ladder operators $a$ and $a^\dagger$ to split up the one-particle Hilbert space — defined below — as we did on the infinite plane. The guiding centre operators on the other hand, will turn out to be less fortunate.

**Definition 5.4.** The one-particle Hilbert space, which we denote by $\mathcal{H}^*$, is the space of measurable square-integrable sections of $\Gamma(\mathcal{L}^*)$ modulo the equivalence relation that identifies to sections whenever the set on which they disagree has measure zero. Instead of integrating over all of $\mathbb{R}^2$, we now only require for each $\psi \in \mathcal{H}^*$ that

$$\int_T d\mathbf{r} |\psi(\mathbf{r})|^2 < \infty. \quad (5.6)$$

Note that we are back to the situation of a uniform perpendicular magnetic field on an infinite plane: the only difference between $\mathcal{H}$ and $\mathcal{H}^*$ is that we now have a weaker integrability condition, but an additional periodicity condition. Hence, we can re-use part of the theory constructed in Chapter 4. In particular, the free Hamiltonian takes the same form at it did in Chapter 4, namely that of equation (3.19),

$$H_{\text{kin}} = \frac{1}{2m} (\Pi_x^2 + \Pi_y^2) = \hbar \omega_C \left( a^\dagger a + \frac{1}{2} \right), \quad (5.7)$$

where the ladder operators $a$ and $a^\dagger$ are defined by equation (3.6). As a consequence, we can use Corollary 3.1 to split the one-particle Hilbert space into a tensor product.

**Corollary 5.3.** The one-particle Hilbert space $\mathcal{H}^*$ splits into a tensor product

$$\mathcal{H}^* \cong \mathcal{H}_0 \otimes \text{Ker}(a)^*, \quad (5.8)$$

where $\mathcal{H}_0$ is given by Definition 3.2. Equivalently, since $\mathcal{H}_0 = \bigoplus_{n=0}^\infty V_n$ where each of the $V_n$’s is a one-dimensional complex vector space, we can also write $\mathcal{H}$ as a direct sum

$$\mathcal{H}^* = \bigoplus_{n=0}^\infty L_n^*$$

where each of the Landau levels $L_n^*$ is Ker$(a)^*$.

**Remark 5.4.** We put a star on Ker$(a)^* \subset \mathcal{H}^*$ to distinguish it from Ker$(a) \subset \mathcal{H}$. Note that, Ker$(a)^*$ is not the same as all periodic sections of Ker$(a)$ because the integrability condition (5.6) for $\mathcal{H}^*$ is much weaker than the integrability condition for $\mathcal{H}$.

Additionally, as noted just below equation (5.1), there is one crucial difference with the infinite plane case discussed in Chapter 4.

**Remark 5.5.** Every Landau level $L_n^*$ has finite dimension $\dim(\text{Ker}(a)) = N_B$.

To summarize, we have constructed a line bundle $\mathcal{L}_T$ over $T$, and pulled it back to a line bundle $\mathcal{L}^*$ over $\mathbb{R}^2$ to return to the more familiar situation of sections of $\mathcal{L}$ over $\mathbb{R}^2$. We then defined the one-particle Hilbert space $\mathcal{H}^*$ as the space of square-integrable sections of $\mathcal{L}^*$, which, compared to $\mathcal{H}$ for the infinite plane has extra periodicity conditions and a weaker integrability condition. The main differences between the infinite plane and the torus are that, for the case of the torus, (a) the magnetic field $B$ is restricted to values such that the total flux $\Phi = Bab$, where $ab = \text{Area}(T)$, through the torus is an integer $N_B$ times the flux quantum $\phi_0 = \hbar e$, and (b) each Landau level has a degeneracy of this same integer $N_B$.

Our next task is to gain a better understanding of the separate Landau levels. This is what the next section is about.

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2The fact that the guiding centre operators are no longer well-defined is actually a good thing, because they are the cause of the infinite degeneracy of each Landau level, see Corollary 3.2.

3To distinguish it from the Hilbert space $\mathcal{H}$ of not necessarily periodic sections — see Definition 2.6.
5.1.2 The Projected Density Operators

We want to construct and compute the Casimir operators defined in the previous Chapter for the case of periodic sections of over $\mathbb{R}^2$. However, as mentioned in Remark 5.3, we do run in some serious problems if we try to blindly repeat the analysis of Chapter 4. For example, for $f \in \Gamma(\mathcal{L}^*)$, $xf$ is not necessarily in $\Gamma(\mathcal{L}^*)$. As a consequence, something as elementary as the position operators $x$ and $y$, and therefore also the guiding centre coordinate operators $X$ and $Y$, are no longer well-defined endomorphisms of the space of periodic sections $\Gamma(\mathcal{L}^*)$. Without the guiding centre operators, we lose the eigenbasis $\{ |m \} \}$ for each Landau level, and therefore the whole analysis of Chapter 4 falls apart.

Fortunately, the situation is not as bad as it seems. For example, any T-periodic function of $x$ and $y$ is a well-defined operator on $\Gamma(\mathcal{L}^*)$. Moreover, it turns out that the single particle projected density operators $\bar{\sigma}(\mathbf{q}) = e^{-i\mathbf{q} \cdot \mathbf{R}}$ do conserve the periodicity of sections of $\mathcal{L}^*$ for specific values of $\mathbf{q}$. But, before we make this claim more specific, let us first try to understand the meaning of the operators $\bar{\sigma}(\mathbf{q})$ and their relation to the translation operators $\tau_1^u$ and $\tau_2^v$.

Recall that $\mathbf{R} = (X,Y)$, which in the symmetric gauge is given by

$$X = x - \frac{\Pi_y}{eB} = il_B^2 \partial_y + \frac{x}{2} = -il_B^2 \left( -\partial_y + \frac{ix}{2il_B^2} \right),$$

$$Y = y + \frac{\Pi_x}{eB} = -il_B^2 \partial_x + \frac{y}{2} = il_B^2 \left( -\partial_x - \frac{iyl_B^2}{2} \right).$$

Comparing to equations (5.4) and (5.5), it looks like $X$ and $Y$ are closely related to the generators of the translation operators $\tau_1^u$ and $\tau_2^v$.

**Lemma 5.1.** The operators $-iY/l_B^2$ and $iX/l_B^2$ generate the translations in the $x$ and $y$ direction respectively. That is, for $u, v \in \mathbb{R}$,

$$\tau_1^u = \exp \left( -iY \frac{u}{l_B^2} \right),$$

$$\tau_2^v = \exp \left( iX \frac{v}{l_B^2} \right).$$

**Proof.** A straightforward computation shows that for arbitrary $f \in \Gamma(\mathcal{L})$,

$$\left[ \exp \left( -iY \frac{u}{l_B^2} \right) f \right](x,y) = \left[ \exp \left( -u \partial_x - \frac{iuy}{2l_B^2} \right) f \right](x,y) = \exp \left( -iuy \frac{2l_B^2}{2l_B^2} \right) f(x-u,y) = \tau_1^u(f)(x,y),$$

and likewise for $\tau_2^v$.

**Remark 5.6.** We see that the single particle projected density operators $\bar{\sigma}(\mathbf{q})$ are nothing but magnetic translation operators that translate over a (rotated$^4$) wavevector $\mathbf{q}$. It is therefore not surprising that, when imposing periodic boundary conditions, only a subset of all $\bar{\sigma}(\mathbf{q})$’s respect the periodicity. This also explains the fact that the Lie bracket of two single-particle projected density operators, given by (4.25),

$$[\bar{\sigma}(\mathbf{q}), \bar{\sigma}(\mathbf{s})] = -2i \sin \left( \frac{l_B^2}{2} \mathbf{q} \times \mathbf{s} \right) \bar{\sigma}(\mathbf{q} + \mathbf{s}).$$

is known as the magnetic translation algebra, as mentioned in Section 4.1.4. Using (5.9) and (5.10), we can write the translation operators $\tau_1^u$ and $\tau_2^v$ as

$$\tau_1^u = \bar{\sigma}(0,u/l_B^2) \quad \text{and} \quad \tau_2^v = \bar{\sigma}(-v/l_B^2,0).$$

---

$^4$because $-Y$ corresponds to translations in the $x$ direction, and $X$ corresponds to translations in the $y$ direction.
5.1. PERIODIC BOUNDARY CONDITIONS

Having established the relation between the magnetic translation operators and the single-particle projected density operators, we can prove the following lemma.

**Lemma 5.2.** Write \( q = (q_1, q_2) = (2\pi k_1/a, 2\pi k_2/b) \). Now, we have the following equivalence:

\[
\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2 \iff [\bar{\sigma}(q), \tau_1^a] = [\bar{\sigma}(q), \tau_2^b] = 0.
\]

**Proof.** Using (5.11), for \( \tau_1^a \) we have,

\[
[\bar{\sigma}(q), \tau_1^a] = [\bar{\sigma}(q), \bar{\sigma}(0, a/l_B^2)] = -2i \sin \left( \frac{q_1 a}{2} \right) \bar{\sigma} \left( q + \frac{a}{l_B^2} e_y \right).
\]

Indeed, we see that the RHS equals zero if and only if \( q_1 = 2\pi k_1/a \) for some \( k_1 \in \mathbb{Z} \). Likewise, \([\bar{\sigma}(q), \tau_2^b] = 0\) if and only if \( q_2 = 2\pi k_2/b \) for some \( k_2 \in \mathbb{Z} \).

Finally, we can make precise the statement made at the beginning of this section.

**Lemma 5.3.** As before, write \( q = (q_1, q_2) = (2\pi k_1/a, 2\pi k_2/b) \). The following two statements are equivalent:

\[
\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2 \iff \bar{\sigma}(q) \in \text{End}(\Gamma(\mathcal{L}^*)).
\]

In words, the above statement says that \( \bar{\sigma}(q) \) maps periodic sections of \( \mathcal{L} \) to periodic sections of \( \mathcal{L} \) if and only if \( q = (2\pi k_1/a, 2\pi k_2/b) \) is such that \( \mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2 \).

**Proof.** First, suppose that \( \mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2 \). We have to prove that \( \bar{\sigma}(q) \) maps \( \Gamma(\mathcal{L}^*) \) back into itself. Pick \( f \in \Gamma(\mathcal{L}) \) satisfying \( f = \tau_1^a(f) = \tau_2^b(f) \). We have to show that \( \bar{\sigma}(q)f \in \Gamma(\mathcal{L}) \) also satisfies \( \bar{\sigma}(q)f = \tau_1^a \bar{\sigma}(q)f = \tau_2^b \bar{\sigma}(q)f \). By Lemma 5.2, we have

\[
\bar{\sigma}(q)f = \bar{\sigma}(q)\tau_1^a(f) = \tau_1^a \bar{\sigma}(q)f,
\]

and, similarly, \( \bar{\sigma}(q)f = \tau_2^b \bar{\sigma}(q)f \).

For the converse, suppose that \( \bar{\sigma}(q) \in \text{End}(\Gamma(\mathcal{L}^*)) \). Now, By equation (5.13), for every \( f \in \Gamma(\mathcal{L}^*) \) we have,

\[
\bar{\sigma}(q)f = \tau_1^a \bar{\sigma}(q)f = \bar{\sigma}(q)\tau_1^a f + 2i \sin \left( \frac{q_1 a}{2} \right) \bar{\sigma} \left( q + \frac{a}{l_B^2} e_y \right) f
\]

\[
= \bar{\sigma}(q)f + 2i \sin \left( \frac{q_1 a}{2} \right) \bar{\sigma} \left( q + \frac{a}{l_B^2} e_y \right) f
\]

Hence, we conclude that for every \( f \in \Gamma(\mathcal{L}^*) \),

\[
2i \sin \left( \frac{q_1 a}{2} \right) \bar{\sigma} \left( q + \frac{a}{l_B^2} e_y \right) f = 0.
\]

Since \( \bar{\sigma} \left( q + \frac{a}{l_B^2} e_y \right) \) is a combination of translations, it maps any nonzero \( f \) to a nonzero section. Hence, the only possibility for (5.14) to hold is whenever \( k_1 \in \mathbb{Z} \). The claim that \( k_2 \in \mathbb{Z} \) follows analogously.

The above Lemma implies that the only operators relevant for our purposes are the \( \bar{\sigma}(q) \)'s for which \((q_1, q_2) = (2\pi k_1/a, 2\pi k_2/b) \). Hence the following definition.

**Definition 5.5.** From now on, we will use the notation \( \bar{\sigma}(\mathbf{k}) \) for \( \bar{\sigma} \left( \frac{2\pi k_1}{a}, \frac{2\pi k_2}{b} \right) \).

For convenience, we mention the associative (4.24) and Lie algebra (4.25) formulas satisfied by the single-particle projected density operators in terms of the integer vectors \( \mathbf{k}, \mathbf{l} \in \mathbb{Z}^2 \) below.
Lemma 5.4. The single particle projected density operators $\tilde{\sigma}(k)$ satisfy
\[
\tilde{\sigma}(k)\tilde{\sigma}(l) = \exp\left(-\frac{i\pi k \times l}{N_B}\right)\tilde{\sigma}(k + l),
\]
and also
\[
[\tilde{\sigma}(k), \tilde{\sigma}(l)] = -2i \sin\left(\frac{\pi k \times l}{N_B}\right)\tilde{\sigma}(k + l).
\]
\[(5.15)\]
\[(5.16)\]

Proof. By equation (5.1),
\[
\frac{i l^2_B q \times s}{2} = \frac{2i \pi l^2_B k \times l}{ab} = \frac{i\pi k \times l}{N_B}
\]
But, there is more.

Lemma 5.5. When acting on $\Gamma(L^*)$, we have the following operator identity
\[
\tilde{\sigma}(N_B k) = e^{i\pi k_1 k_2 N_B} = \pm 1
\]
\[(5.17)\]
for any $k \in \mathbb{Z}^2$.

Proof. Using Baker-Campbell-Hausdorff (4.21), equation (5.1) and $[X, Y] = il_B^2$, we have
\[
\tilde{\sigma}(N_B k) = \exp\left(-2\pi i N_B \left[ \frac{k_1 X}{a} + \frac{k_2 Y}{b} \right]\right) = \exp\left(-i \frac{k_1 b X + k_2 a Y}{l_B^2}\right)
\]
\[
= \exp\left(-i k_1 b X\right) \exp\left(-i k_2 a Y\right) \exp\left(\frac{-k_1 k_2 a b}{2l_B^4} [X, Y]\right)
\]
\[
= (\tau_2^{-b})^{k_1} (\tau_1^a)^{k_2} \exp(i\pi k_1 k_2 N_B).
\]
Now, for any section $f \in \Gamma(L^*)$, $f = \tau_1^a(f) = \tau_2^b(f)$. Hence,
\[
\tilde{\sigma}(N_B k) f = e^{i\pi k_1 k_2 N_B} f.
\]
\[
(5.18)
\]

Corollary 5.4. For $k, l \in \mathbb{Z}^2$,
\[
\tilde{\sigma}(k + N_B l) = e^{-i\pi k \times l} e^{i\pi l_1 l_2 N_B} \tilde{\sigma}(k) = \pm \tilde{\sigma}(k)
\]
when acting on $\Gamma(L^*)$. As a consequence, we effectively only have $N_B^2$ operators of the form $\tilde{\sigma}(k)$ acting on $\Gamma(L^*)$. This means that any sum over $q$ or $k$ need only go over a Brillouin zone that contains $N_B^2$ adjacent values of $q$ or $k$.

Proof. Using equation (5.15) and Lemma 5.5, we have
\[
\tilde{\sigma}(k + N_B l) = e^{-i\pi k \times l} \tilde{\sigma}(k)\tilde{\sigma}(N_B l) = e^{-i\pi k \times l} e^{i\pi l_1 l_2 N_B} \tilde{\sigma}(k)
\]
\[
= \pm \tilde{\sigma}(k).
\]

Definition 5.6. The Brillouin zone $BZ$ over which we will implicitly sum whenever there is a sum over $k$ space is the set of integers $\{0, 1, 2, \ldots, N_B - 1\}^2$.

We conclude that, despite the fact that something as fundamental as the position operator does not make sense when working with periodic boundary conditions, we do have a subalgebra of the single-particle projected density operators at our disposal, namely those $\tilde{\sigma}(k)$’s for which $k \in BZ$. Using these operators, we can construct the effective Hamiltonian and properly describe the FQHE on the torus. However, before we do, there is one more thing that we have to discuss.
5.1.3 An $N_B$-Dimensional Representation of the Magnetic Translation Algebra

In the previous section, we constructed a subalgebra of single particle projected density operators, each of which is an endomorphism of every Landau level. We would like to know explicitly how a given single particle projected density operator acts on the fractionally filled $n$th Landau level (or any other Landau level for that matter). What we will do is guess a finite dimensional matrix representation of the algebra spanned by the $\tilde{\sigma}(k)$'s. We will then assume that there actually exists an orthonormal basis of $L_{n_0}$ such that the operators $\tilde{\sigma}(k)$ take the form of the matrices that we will construct, and then we will use these matrices to compute the Casimir operators $\zeta_r$.

Recall from Lemma 5.4 that the single-particle projected density operators $\tilde{\sigma}(k)$ satisfy

$$\tilde{\sigma}(k)\tilde{\sigma}(l) = \exp\left(-\frac{i\pi N_B}{k \times l}\right)\tilde{\sigma}(k + l) \quad (5.19)$$

in terms of the integer vectors $k, l \in \mathbb{Z}^2$. We are looking for $N_B \times N_B$ matrices $M_k$ that satisfy the same relation.

Before we continue, anticipating the fact that we will later on use the relations derived in this section extensively in the computation of the Casimir operators, we will switch notation. In the expressions for the Casimir operators, the integer vector $k$ will be summed over, and instead of one or two, we will have $r$ such summation indices $k_1, \ldots, k_r$. Because each $k_i$ also has an $x$ and a $y$-component, which up to now we have labelled with a 1 and a 2 index, we want to come up with a different way of labelling the $x$ and $y$-components than using the indices 1 and 2.

**Notation:** For a vector $k \in \mathbb{Z}^2$, from now on, we will use a prime to distinguish the components from each other. That is,

$$k = (k, k') \in \mathbb{Z}^2.$$

So the $x$-component of $k$ is $k$, and the $y$-component of $k$ is $k'$.

Now, (5.19) in terms of the to be constructed $N_B \times N_B$ matrices $M_k$ and $M_l$ reads

$$M_k M_l = \exp\left(-\frac{i\pi N_B}{k \times l}\right)M_{k+l}. \quad (5.20)$$

**Definition 5.7.** The complex root of unity $\alpha := \exp\left(-\frac{2\pi i}{N_B}\right)$ will occur quite often in the remainder of this chapter. That often, that we have decided to reserve the name$^5$ for it.

$$\alpha := \exp\left(-\frac{2\pi i}{N_B}\right)$$

**Definition 5.8.** The orthonormal basis of $\text{Ker}(a)$ in which we want to express the matrices $M_k$ will be labeled by the $N_B$ complex roots of unity

$$\{\alpha^0, \alpha^1, \ldots, |\alpha^{N_B-1}\rangle\}.$$ 

We choose this kind of labelling of the basis states over the more conventional $|0\rangle, |1\rangle, \ldots, |N_B-1\rangle$ because $\alpha^m$ is periodic in $m$, which will be convenient when constructing the matrices $M_k$.

Next, we will define the matrices $M_k$ by their action on the basis $\{\alpha^m\}$.

$^5$The complex root of unity $\alpha$ is completely unrelated to the statistical parameter $\alpha$ of the anyons in the next chapter, which is a real number.
Educated guess:
Quoting my supervisor Andre Henriques: “there are not too many things you can try” after a failed first attempt which we will not go into, let us try the following expression\(^6\) for \(\tilde{M}_k\),
\[
\tilde{M}_k|\alpha^m\rangle := \alpha^{-mk'}|\alpha^{m+k}\rangle,
\]
where, we recall that
\[
k = (k, k').
\]
In other words, the \(m\)th basis vector gets mapped \((m + k)\)th basis vector \((\text{mod } N_B)\), and picks up an additional phase \(\alpha^{-mk'}\). The tilde is there because we anticipate that this matrix might not work out perfectly, and possibly will require further adjustments. The above definition leads to the following multiplication formula
\[
\tilde{M}_k\tilde{M}_l|\alpha^m\rangle = \alpha^{-ml'}|\alpha^{m+l}\rangle = \alpha^{-k'l}\alpha^{-m(k'+l')}|\alpha^{m+k+l}\rangle = \alpha^{-k'l}\tilde{M}_{k+l}|\alpha^m\rangle. \tag{5.21}
\]
This looks similar to equation (5.20), apart from the prefactor \(\alpha^{-ml'}\). Let us multiply the matrices \(\tilde{M}_k\) with a yet to be determined factor \(c(k)\) in an attempt to get the correct prefactor in equation (5.21). That is, define \(M_k := c(k)\tilde{M}_k\).

By equation (5.21), the \(M_k\)'s satisfy the algebra
\[
M_kM_{l'} = \alpha^{-k'l}\frac{c(k)c(l)}{c(k+l)} M_{k+l}. \tag{5.22}
\]
For (5.22) to equal (5.20), we require
\[
\alpha^{-k'l}\frac{c(k)c(l)}{c(k+l)} = \alpha^{\frac{1}{2}(kl' - k'l')}.
\]
That is,
\[
c(k)c(l) = c(k+l)\alpha^\frac{1}{2}(kl' + k'l').
\]
The above equation is solved by \(c(k) = \alpha^{-\frac{1}{2}kk'}\).

Let us summarize our result in the following corollary.

**Corollary 5.5.** There exists an \(N_B\)-dimensional representation of the algebra of operators \(\tilde{\sigma}(k)\) given by
\[
\tilde{\sigma}(k) \mapsto M_k, \tag{5.23}
\]
where the matrix \(M_k\) is defined by its action on the basis \(|\alpha^0\rangle, |\alpha^1\rangle, \ldots, |\alpha^{N_B-1}\rangle\) through
\[
M_k|\alpha^m\rangle := \alpha^{-\frac{1}{2}kk'}\alpha^{-mk'}|\alpha^{m+k}\rangle. \tag{5.24}
\]

---

\(^6\)The suggested form of the matrices \(M_k\) should be credited entirely to A.G. Henriques. The author claims no originality on his part. Moreover, it was later pointed out by V. Gritsev that the matrices constructed in this section are also discussed in Fairly, Fletcher and Zachos [3].
Proof. Indeed, from equation (5.24), we see that
\[ M_{k+N_B} = e^{-i\mathbf{k} \times \mathbf{a}} M_k e^{i\mathbf{a} N_B} = \pm M_k. \] (5.25)

As stated at the beginning of this section, we will assume that there exist an orthonormal basis of \( L^*_\alpha \cong \text{Ker}(a)^* \), which we label by \( \{\alpha^m\} : m = 0, \ldots, N_B - 1 \) such that the operators \( \hat{\sigma}(\mathbf{k}) \) act on this basis via the identification (5.23). We now have an explicit formula for the matrices \( \hat{\sigma}(\mathbf{k}) \) that we will use to compute explicit expressions for the Casimir operators \( \zeta_r \) in terms of ladder operators acting on Fock space.

### 5.1.4 The FQHE on the Torus

As we did in Chapter 4, we will work in the language of second quantization. That is, we define the Fock space
\[ \mathcal{F}^* := \bigoplus_{n=0}^{\infty} F^*_n, \]
where \( F^*_n \) is the \( n \)-particle Hilbert space \( \mathcal{H}^* \wedge \ldots \wedge \mathcal{H}^* \) (\( n \)-times). Next, we choose a basis \( \{\phi_{n,m}\} \) of \( \mathcal{H}^* \) in accordance with the isomorphism in equation (5.8), where \( n \in \mathbb{N} \) labels the Landau level, and \( m \in \{0, \ldots, N_B - 1\} \) refers to the basis vectors of \( \text{Ker}(a)^* \) constructed in the previous section, and define corresponding creation and annihilation operators \( c_{n,m}^\dagger \) and \( c_{n,m} \) acting on \( \mathcal{F}^* \) by
\[
c_{n,m}^\dagger (\phi_{n_1,m_1} \wedge \ldots \wedge \phi_{n_k,m_k}) := \phi_{n,m} \wedge \phi_{n_1,m_1} \wedge \ldots \wedge \phi_{n_k,m_k},
\]
\[
c_{n,m} (\phi_{n_1,m_1} \wedge \ldots \wedge \phi_{n_k,m_k}) := \sum_{i=1}^{k} \delta(n,n_i) \phi_{n_1,m_1} \wedge \ldots \wedge \phi_{n_i-1,m_{i-1}} \wedge \phi_{n_{i+1},m_{i+1}} \wedge \ldots \wedge \phi_{n_k,m_k}.
\]

Next, using the ladder operators \( c_{n,m} \) and \( c_{n,m}^\dagger \), we define the periodic analogue of the projected density operators \( \hat{\rho}_{\alpha}(\mathbf{k}) \) used in Chapter 4 — see equation (4.20) — by extending the single-particle projected density operators \( \hat{\rho}_{\alpha}(\mathbf{k}) \) to act on all of Fock space,
\[
\hat{\rho}_{\alpha}(\mathbf{k}) = \sum_{m,m'} \langle m|\hat{\sigma}(\mathbf{k})|m'\rangle c_{n_0,m}^\dagger c_{n_0,m'}.
\] (5.26)

In terms of the basis \( \{\alpha^m\} \) described in Section 5.1.3, \( \hat{\sigma}(\mathbf{k}) = M_k \), and the matrix coefficients \( \langle m|\hat{\sigma}(\mathbf{k})|m'\rangle \) can be readily computed.

We shall write
\[
(M_k)_{m,m'} := \langle \alpha^m|M_k|\alpha^{m'} \rangle
\]
for the \((m,m')\)th matrix element of \( M_k \) in the \( |\alpha^m\rangle \) basis.

**Definition 5.9.** Let \( \delta_{m,m'} \) denote the delta function that takes its arguments mod \( N_B \). That is
\[
\delta_{m,m'} := \begin{cases} 
1 & \text{if } m = m' \mod N_B \\
0 & \text{otherwise}
\end{cases}
\]

Using equation (5.24), we can compute the matrix elements of \( M_k \) in the \( |\alpha^m\rangle \) basis without too much difficulty.
Lemma 5.7. The matrix elements of $M_k$ are given by

$$ (M_k)_{m,m'} = \alpha^{-\frac{1}{2} k k'} \alpha^{-m' k'} \delta_{m,m'+k}. $$

(5.27)

Proof. By equation (5.24), we have

$$ (M_k)_{m,m'} = \langle \alpha^m | M_k | \alpha^{m'} \rangle = \alpha^{-\frac{1}{2} k k'} \alpha^{-m' k'} \langle \alpha^m | \alpha^{m'+k} \rangle, $$

where, because of the orthonormality of the basis $\{ | \alpha^m \rangle \}$,

$$ \langle \alpha^m | \alpha^{m'+k} \rangle = \delta_{\alpha^m, \alpha^{m'+k}} = \delta_{m,m'+k}. $$

Corollary 5.6. We can rewrite equation (5.26) as

$$ \tilde{\rho}_{n_0}(k) = \alpha^{-\frac{1}{2} k k'} \sum_{m,m'} \alpha^{-m' k'} \delta_{m,m'+k} \epsilon^\dagger_{n_0,m} c_{n_0,m'}. $$

(5.28)

Lemma 5.8. The projected density operators $\tilde{\rho}_{n_0}(k)$ acting on $\mathcal{F}^*$ satisfy the same Lie algebra as the projected density operators from Chapter 4 that act on $\mathcal{F}$:

$$ [\tilde{\rho}_{n_0}(k), \tilde{\rho}_{n_0}(l)] = -2i \sin \left( \frac{\pi k \times l}{N_B} \right) \tilde{\rho}_{n_0}(k + l). $$

(5.29)

Proof. Using Lemma 5.4, we can copy the proof of Lemma 4.2.

Remark 5.7. As for the FQHE on the infinite plane, the projected density operators $\tilde{\rho}_{n_0}(k)$ only inherit the Lie algebra structure (5.16) of the single particle projected density operators $\sigma(k)$, but not the associative algebra structure (5.15).

Remark 5.8. The projected density operators $\tilde{\rho}_{n_0}(k)$ are also periodic up to a sign in $k$.

$$ \tilde{\rho}_{n_0}(k + N_b l) = e^{-i \pi k \times l} e^{i \pi N_b l'} \tilde{\rho}_{n_0}(k) = \pm \tilde{\rho}_{n_0}(k). $$

(5.30)

This follows directly from equation (5.25).

Finally, we can discuss the FQHE on the torus. Going to the low temperature limit, where all the effective dynamics occur within the highest non-empty (fractionally filled) Landau level labelled by $n_0$, we proceed as in Section 4.1.2, neglecting the kinetic energy term which is an irrelevant constant. The effective Hamiltonian is now given by the periodic analogue of equation (4.13),

$$ H_{eff} = \frac{1}{2ab} \sum_{k \in BZ} U(k) \tilde{\rho}_{n_0}(k) \tilde{\rho}_{n_0}(-k), $$

(5.31)

where $U(k)$ is the Fourier transform of $U(r)$.

We have now set up all the necessary mathematical formalism to describe the FQHE on the torus. In the next section, we will attempt to compute Casimir operators of the Lie algebra (5.29) satisfied by the projected density operators.

---

7There is a caveat in the above expression, which does not really matter for our present purpose: the interaction potential $U(r)$ cannot be equal to the Coulomb potential, because it has to be $T$-periodic. We will not discuss the specific form of $U(r)$. All that matters is that $U$ is periodic, has the correct short distance behavior, and is Fourier transformable. For more details, see Haldane [10].
5.2 Casimir Operators on the Torus

In this section we will redo the computations of Section 4.2 for the case of the torus. The computations will turn out to be less complicated because, when working with periodic boundary conditions, the partially filled $n_0^{th}$ Landau level is merely a finite dimensional vector space of dimension $N_B$.

**Notation:** as we did in Chapter 4, from now on, we will suppress the index $n_0$ that refers to the fractionally filled Landau level in the notation.

### 5.2.1 Construction of the $R^{th}$ Order Casimir

According to Theorem 4.2, we have Casimir operator $\zeta_r$ for each $r \in \mathbb{N}$, given by

$$\zeta_{r+1} := \frac{1}{(ab)^r} \sum_{q_1, \ldots, q_r} \left( \prod_{i<j} \exp \left(-\frac{i\ell^2 q_i \times q_j}{2} \right) \right) \bar{\rho}(q_1) \cdots \bar{\rho}(q_r) \bar{\rho}(-q_1 - \cdots - q_r)$$

$$= \frac{1}{(ab)^r} \sum_{k_1, \ldots, k_r} \left( \prod_{i<j} \exp \left(-i\pi k_i \times k_j \right) \right) \bar{\rho}(k_1) \cdots \bar{\rho}(k_r) \bar{\rho}(-k_1 - \cdots - k_r).$$

It turns out that the proof of Theorem 4.2 requires some modifications in order to work on the torus. The reason is, that in the proof of Theorem 4.2 we use a shift in the integration variable $q$, which, as equation (5.30) shows, introduces extra minus signs possibly leading to not all terms cancelling out properly. Hence, we will carefully repeat the proof of Theorem 4.2 for the case of the torus while keeping (5.30) in mind.

**Theorem 5.1.** For each $r \in \mathbb{N}$, the operator $\zeta_{r+1}$, given by

$$\zeta_{r+1} = \frac{1}{(ab)^r} \sum_{k_1, \ldots, k_r} \left( \prod_{i<j} e^{-\frac{i\pi}{N_B} k_i \times k_j} \right) \bar{\rho}(k_1) \cdots \bar{\rho}(k_r) \bar{\rho}(-k_1 - \cdots - k_r),$$

is a central operator of the projected density operator algebra.

**Proof.** Following the reasoning of (4.35), we obtain for arbitrary $l \in \mathbb{Z}^2$,

$$[\zeta_{r+1}, \bar{\rho}(l)] = \sum_{m=1}^{r} \left\{ - \sum_{k_1, \ldots, k_r} \left( \prod_{i<j} e^{-\frac{i\pi}{N_B} k_i \times k_j} \right) 2i \sin \left( \frac{\pi k_m \times l}{N_B} \right) \right\}$$

$$\times \bar{\rho}(k_1) \cdots \bar{\rho}(k_{m-1}) \bar{\rho}(k_{m} + l) \bar{\rho}(k_{m+1}) \cdots \bar{\rho}(k_r) \bar{\rho}(-k_1 - \cdots - k_r)$$

$$+ \sum_{k_1, \ldots, k_r} \left( \prod_{i<j} e^{-\frac{i\pi}{N_B} k_i \times k_j} \right) 2i \sin \left( \frac{\pi (k_1 + \cdots + k_r) \times l}{N_B} \right)$$

$$\times \bar{\rho}(k_1) \cdots \bar{\rho}(k_r) \bar{\rho}(-k_1 - \cdots - k_r + l).$$

As in the proof of Theorem 4.2, we now focus on the $m^{th}$ term in the sum above. Keep in mind that all of the $k_i$‘s above run over the Brillouin zone $BZ := \{0, \ldots, N_B - 1\}^2$. We do a change of summation variables to the new variable $k_m := k_m + l$, and then remove the tilde afterwards, which effectively amounts to two things: (1) in the expression above we replace $k_m$ by $k_m - l$, and (2) the new variable $k_m$ now runs over the shifted Brillouin zone $BZ + l$. We will discuss (2) below. Plugging (1) into (5.33) means that we substitute

$$\left( \prod_{i<j} e^{-\frac{i\pi}{N_B} k_i \times k_j} \right) \mapsto \left( \prod_{i<j} e^{-\frac{i\pi}{N_B} k_i \times k_j} \right) \left( \prod_{i=m} e^{\frac{i\pi}{N_B} k_i \times l} \right) \left( \prod_{m<j} e^{-\frac{i\pi}{N_B} k_j \times l} \right).$$
On to (2). Having done the change of variables describe above, the \( m \)-th term in the sum in (5.33) becomes

\[
- \sum_{k_1, \ldots, k_r} \left( \prod_{i<j} e^{-\frac{\pi i}{N_B} k_i \cdot k_j} \right) \left( \prod_{i<m} e^{-\frac{\pi i}{N_B} k_i \cdot t} \right) \left( \prod_{m<j} e^{-\frac{\pi i}{N_B} k_j \cdot t} \right) 2i \sin \left( \frac{\pi k_m \cdot l}{N_B} \right) \times \tilde{\rho}(k_1) \ldots \tilde{\rho}(k_r) \tilde{\rho}(-k_1 - \ldots - k_r + l),
\]

where the tilde on the sum means that the \( m \)-th variable \( k_m \) runs over the shifted Brillouin zone \( BZ + l \), while the other variables run over the ordinary Brillouin zone \( BZ \). We would like to shift the summation over \( k_m \) back to a summation over \( BZ \). However, such a shift might give rise to extra signs due to (5.30), which could spoil the argument we used in proving Theorem 4.2. Let us investigate this further.

Each \( k_m \) that does not lie in \( BZ \) we write as \( \hat{k}_m + zN_B \), with \( \hat{k}_m \in BZ \), for a unique integer valued vector \( z \in \mathbb{Z}^2 \). As before, we drop the tilde, which means that, effectively, we have to replace \( k_m \) by \( k_m + zN_B \) for some terms in (5.34). This is what happens if we do so for a single term in (5.34). Using (5.30), we get

\[
\left( \prod_{i<m} e^{-\frac{\pi i}{N_B} k_i \cdot z} \right) \left( \prod_{m<j} e^{-i\pi z \cdot k_j} \right) \prod_{m<j} e^{-i\pi z \cdot k_j} = \prod_{i<m} e^{-2i\pi k_i \cdot z} e^{2i\pi N_B z'}. \]

That is, replacing \( k_m \) by \( k_m + zN_B \) yields a total extra factor of

\[
\prod_{i<m} e^{-2i\pi k_i \cdot z} e^{2i\pi N_B z'} = 1.
\]

Hence, we can safely change the domain of summation for \( k_m \) in (5.34) to \( BZ \) without any consequences. Note that, what we have actually shown is that (5.34) without the summation sign in front, is perfectly periodic in each of the variables \( k_i \), despite the fact that \( \tilde{\rho}(k_i) \) is not. We conclude that, after changing the variable \( k_m \mapsto k_m - l \), (5.34) becomes a sum over the unshifted Brillouin zone for all summation variables:

\[
- \sum_{k_1, \ldots, k_r} \left( \prod_{i<j} e^{-\frac{\pi i}{N_B} k_i \cdot k_j} \right) \left( \prod_{i<m} e^{-\frac{\pi i}{N_B} k_i \cdot t} \right) \left( \prod_{m<j} e^{-\frac{\pi i}{N_B} k_j \cdot t} \right) 2i \sin \left( \frac{\pi k_m \cdot l}{N_B} \right) \times \tilde{\rho}(k_1) \ldots \tilde{\rho}(k_r) \tilde{\rho}(-k_1 - \ldots - k_r + l).
\]

We perform the change of variables \( k_m \mapsto k_m - l \) for all \( m \in \{1, \ldots, r\} \), and then we proceed analogously to the proof of Theorem 4.2: we split the sine above into a sum of exponentials and
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then plug (5.35) into (5.33) to get

\[
\begin{align*}
[\zeta_{r+1}, \tilde{\rho}(l)] &= \sum_{k_1, \ldots, k_r} \left( \prod_{i<j} e^{-\frac{i\pi}{N_B} k_i \times k_j} \right) \tilde{\rho}(k_1) \cdots \tilde{\rho}(k_r) \tilde{\rho}(-k_1 - \cdots - k_r + l) \\
&\quad \times \left\{ \sum_{m=1}^{r} \left( \prod_{i<m} e^{\frac{i\pi}{N_B} k_i \times l} \right) \left( \prod_{m<j} e^{-\frac{i\pi}{N_B} k_j \times l} \right) \right. \\
&\quad \left. + 2i \sin \left( \frac{\pi(k_1 + \cdots + k_r) \times l}{N_B} \right) \right\} \\
&= 0.
\end{align*}
\]

5.2.2 Computation of the \( R^\text{th} \) Order Casimir

Contrary to the case of the infinite plane, we can now compute all the Casimir operators \( \zeta_{r+1} \) for \( r \in \mathbb{N} \) in one go.

**Theorem 5.2.** For \( r \in \mathbb{N} \), the \((r + 1)^{\text{th}}\) order Casimir operator (5.32) takes the following form

\[
\zeta_{r+1} = \frac{1}{(2\pi l_B^2)^r} \sum_{m_1, m_1', \ldots, m_{r+1}, m_{r+1}'} \left[ \prod_{i=1}^{r} \delta_{m_i, m_{r+1}} \prod_{i=1}^{r} \delta_{m_i', m_{r+1}'} \right] c_{m_1}^\dagger c_{m_2}^\dagger c_{m_2'} c_{m_3}^\dagger \cdots c_{m_{r+1}}^\dagger c_{m_{r+1}'} (5.36)
\]

when acting on the Fock space \( \mathcal{F}^* \) through the representation given by equation (5.28).

**Proof.** We start with the expression given by equation (5.32), and simply plug in equation (5.28).

\[
\zeta_{r+1} = \frac{1}{(ab)^r} \sum_{k_1, \ldots, k_r} \left[ \prod_{i<j} \exp \left( -\frac{i\pi k_i \times k_j}{N_B} \right) \right] \tilde{\rho}(k_1) \cdots \tilde{\rho}(k_r) \tilde{\rho}(-k_1 - \cdots - k_r) \\
= \frac{1}{(2\pi l_B^2 N_B)^r} \sum_{k_1, \ldots, k_r} \left\{ \left[ \prod_{i<j} \alpha^{-\frac{i}{2} k_i \times k_j} \right] \left[ \prod_{i=1}^{r} \alpha^{-\frac{i}{2} k_i k_i'} \right] \prod_{i=1}^{r} \alpha^{-\frac{i}{2} \sum_{i, j} k_i k_j'} \\
\times \sum_{m_1, m_1', \ldots, m_{r+1}, m_{r+1}'} \left[ \prod_{i=1}^{r} \alpha^{-m_i m_i'} \prod_{i=1}^{r} \delta_{m_i, m_{r+1}} \prod_{i=1}^{r} \delta_{m_i', m_{r+1}'} \right] \prod_{i=1}^{r} \delta_{m_i, m_{r+1} + k_i} \prod_{i=1}^{r} \delta_{m_i', m_{r+1}' - k_i - \cdots - k_r} \\
\times c_{m_1}^\dagger c_{m_2}^\dagger c_{m_2'} c_{m_3}^\dagger \cdots c_{m_{r+1}}^\dagger c_{m_{r+1}'} \right\} \\
= \frac{1}{(2\pi l_B^2 N_B)^r} \sum_{m_1, m_1', \ldots, m_{r+1}, m_{r+1}'} \left\{ \left[ \prod_{k_1, \ldots, k_r} \right] \right. \\
\times \alpha^{-\sum_{i=1}^{r} m_i k_i'} \sum_{k_1, \ldots, k_r} \left. \left[ \prod_{i=1}^{r} \delta_{m_i, m_{r+1} + k_i} \prod_{i=1}^{r} \delta_{m_i', m_{r+1}' - k_i - \cdots - k_r} \right] \right. \\
\times c_{m_1}^\dagger c_{m_2}^\dagger c_{m_2'} c_{m_3}^\dagger \cdots c_{m_{r+1}}^\dagger c_{m_{r+1}'} \right) (5.37)
\]
Let us focus on the part in curly brackets in the two lines above (5.37) for now, keeping all the \(m\)'s fixed. We will denote this part by the symbol (*)..

\[
(*) := \sum_{k_1, \ldots, k_r} \alpha^{\frac{1}{2} \sum_{i<j} (k_i k'_j - k'_i k_j)} \alpha^{\frac{1}{2} \sum_{i<j} k_i k'_j - \frac{1}{2} \sum_{i<j} k_i k_j} \alpha^{- \sum m_i' k_i + m_{r+1} \sum k'_i} \times \left( \prod_{i=1}^{r} \delta_{m_i, m_{r+1} + k_i} \right) \delta_{m_{r+1}, m_{r+1} - k_1 - \ldots - k_r}.
\]

We will rewrite the first exponent in (5.38) as

\[
\alpha^{\frac{1}{2} \sum_{i<j} (k_i k'_j - k'_i k_j) - \sum_{i<j} k_i k'_j - \frac{1}{2} \sum_{i<j} k_i k_j} = \alpha^{\frac{1}{2} \sum_{i<j} k_i k'_j - \frac{1}{2} \sum_{i<j} k_i k_j} \alpha^{- \sum m_i' k_i + m_{r+1} \sum k'_i}
\]

Next, we will perform the sums over \(k_1, \ldots, k_r\) in (*). Each Dirac delta \(\delta_{m_i, m_i' + k_i}\) forces \(k_i\) take any value satisfying \(m_i = m_i' + k_i (mod N_B)\). Because each \(k_i\) runs over the set \(\{0, \ldots, N_B - 1\}\), there is exactly one value for \(k_i\) that solves \(m_i = m_i' + k_i (mod N_B)\), say \(k_i = m_i - m_i' + z_i N_B\) for some integer \(z_i \in \mathbb{Z}\). Plugging this equation for \(k_i\) back into the exponents yields

\[
(A) = \alpha^{- \sum_{i<j} k'_i (m_i - m_i' + z_i N_B + \sum_{i<j} [m_j - m_j' + z_j N_B])} = \alpha^{- \sum_{i<j} k'_i (m_i - m_i' + \sum_{i<j} [m_j - m_j'])} \alpha^{- N_B \sum k'_i (z_i + \sum_{i<j} z_j)} = \alpha^{- \sum_{i<j} k'_i (m_i - m_i' + \sum_{i<j} [m_j - m_j'])} =: (B)
\]

because \(\alpha^{N_B} = 1\). Plugging (B) back into (*) gives

\[
(*) = \sum_{k_1, \ldots, k_r} \alpha^{- \sum_{i<j} k'_i (m_i - m_i' + \sum_{i<j} [m_j - m_j'])} \delta_{m_{r+1}, m_{r+1} - \sum [m_j - m_j']}
\]

\[
= \prod_{i=1}^{r} \left( \sum_{k'_i} \alpha^{- k'_i (m_i - m_i' + \sum_{i<j} [m_j - m_j'])} \delta_{m_{r+1}, m_{r+1} - \sum [m_j - m_j']} \right) \delta_{m_{r+1}, m_{r+1} - \sum [m_j - m_j']}.
\]

Now, observe that for any integer \(n \in \mathbb{Z}\),

\[
\sum_{k'=0}^{N_B-1} \alpha^{-k'n} = \begin{cases} N_B & \text{for } n = 0 \ (mod \ N_B) \\ \frac{1}{1 - \alpha^{-n}} & \text{for } n \neq 0 \ (mod \ N_B) \\ 0 \end{cases}
\]

\[
= N_B \delta_{n,0}.
\]

Applying (5.40) to the term within brackets in (5.39) yields

\[
(*) = (N_B)^r \left( \prod_{i=1}^{r} \delta_{m_i, m_i' + \sum_{i<j} [m_j - m_j'], 0} \delta_{m_{r+1}, m_{r+1} - \sum [m_j - m_j']} \right) \delta_{m_{r+1}, m_{r+1} - \sum [m_j - m_j']}
\]

We conclude that \(\zeta_{r+1}\) is given by

\[
\zeta_{r+1} = \frac{1}{(2\pi l_B^2)^r} \sum_{m_1, m_1', \ldots, m_{r+1}, m_{r+1}'} \left\{ \left( \prod_{i=1}^{r} \delta_{m_i, m_i' + \sum_{i<j} [m_j - m_j'], 0} \right) \delta_{m_{r+1}, m_{r+1} - \sum [m_j - m_j']} \times c_{m_1} c_{m_1'} c_{m_2} c_{m_2'} \cdots c_{m_{r+1}} c_{m_{r+1}'} \right\}.
\]
5.2. CASIMIR OPERATORS ON THE TORUS

In order to get to equation (5.36), we will have to simplify the product of Dirac delta’s. Each Dirac delta represents an equation, so the product represents a system of equations. Hence, in order to simplify the product of Dirac delta’s, all we have to do is simplify the corresponding system of equations. We have (everything \(\text{mod } N_B\)),

\[
\begin{align*}
    m'_{r+1} &= m_1 + \sum_{1<j}^r (m_j - m'_j) \\
    m'_{r+1} &= m_2 + \sum_{2<j}^r (m_j - m'_j) \\
    &\vdots \\
    m'_{r+1} &= m_{r-1} + (m_r - m'_r) \\
    m'_{r+1} &= m_r \\
    m'_{r+1} &= m_{r+1} + \sum_{j=1}^r (m_j - m'_j)
\end{align*}
\]

Next, we plug the last equation into all the other ones, and then subtract \(\sum_{j=1}^r (m_j - m'_j)\) on both sides.

\[
\begin{align*}
    m_{r+1} &= m_1 - (m_1 - m'_1) \\
    m_{r+1} &= m_2 - (m_1 - m'_1) - (m_2 - m'_2) \\
    &\vdots \\
    m_{r+1} &= m_{r-1} - (m_1 - m'_1) - (m_2 - m'_2) - \ldots - (m_{r-1} - m'_{r-1}) \\
    m_{r+1} &= m_r - (m_1 - m'_1) - (m_2 - m'_2) - \ldots - (m_r - m'_r) \\
    m'_{r+1} &= m_{r+1} + \sum_{j=1}^r (m_j - m'_j)
\end{align*}
\]

Now, we simplify each equation separately.

\[
\begin{align*}
    m'_{r+1} &= m'_1 \\
    m'_{r+1} &= m'_2 - (m_1 - m'_1) \quad (1) \\
    &\vdots \\
    m'_{r+1} &= m'_{r} - (m_1 - m'_1) - (m_2 - m'_2) - \ldots - (m_{r-2} - m'_{r-2}) \quad (r-1) \\
    m'_{r+1} &= m'_{r} - (m_1 - m'_1) - (m_2 - m'_2) - \ldots - (m_{r-1} - m'_{r-1}) \quad (r) \\
    m'_{r+1} &= m'_{r+1} - (m_1 - m'_1) - (m_2 - m'_2) - \ldots - (m_r - m'_r) \quad (r+1)
\end{align*}
\]

Next, for \(i = 1, \ldots, r + 1\), we take equation number (i) and subtract equations (1) + (2) + \ldots + (i - 1)
from it. This yields the following set of equations:

\[ m'_1 = m_{r+1} \quad (1) \]
\[ m'_2 = m_1 \quad (2) - (1) \]
\[ \vdots \]
\[ m'_{r-1} = m_{r-2} \quad (r-1) - (1) - (2) - \ldots - (r-2) \]
\[ m'_r = m_{r-1} \quad (r) - (1) - (2) - \ldots - (r-1) \]
\[ m'_{r+1} = m_r \quad (r+1) - (1) - (2) - \ldots - (r) \]

Finally, we notice that because all of the \( m_i \)'s and \( m'_i \)'s run over the set \( \{0, \ldots, N_B - 1\} \), the only possible way for the above equations to be satisfied \( \text{mod } N_B \), is if they are actually satisfied as equations in \( \mathbb{Z} \). This shows that

\[
\left( \prod_{i=1}^{r} \delta_{m_i - m'_i + \sum_{i<j}[m_j - m'_j], 0} \right) \delta_{m_{r+1}, m'_{r+1} - \sum_j[m_j - m'_j]} = \left[ \delta_{m'_1, m_{r+1}} \prod_{i=1}^{r} \delta_{m'_i, m_i} \right], \quad (5.42)
\]

where the Dirac delta's on the RHS are ordinary Dirac delta's. Now, plugging (5.42) into (5.41) proves the theorem.

Having computed the \((r+1)^{\text{th}}\) order Casimir \( \zeta_{r+1} \) for each \( r \in \mathbb{N} \), the next logical step is to try to give physical meaning to the operators computed. This is the topic of the next section.

### 5.2.3 Physical Interpretation of the Casimir Operators

In the previous section we have computed the \((r+1)^{\text{th}}\) order Casimir operator \( \zeta_{r+1} \) for each \( r \in \mathbb{N} \) in the representation given by (5.28). We now address the important question of whether it is possible to assign any physical meaning to the operators \( \zeta_{r+1} \). When working in the formalism of second quantization, as explained in Section 4.2.3, all operators with a clear physical interpretation are normal ordered, meaning that all raising operators are to the left of the lowering operators. Hence, our approach will be to normal order the expression for \( \zeta_{r+1} \) in equation (5.36), and then try to interpret the results.

We will start with the simplest Casimir, the second order operator \( \zeta_2 \), which, according to (5.36), is given by,

\[
\zeta_2 = \frac{1}{2 \pi l_B^2} \sum_{m_1, m'_1, m_2, m'_2} \delta_{m'_1, m_2} \delta_{m'_2, m_1} c_{m'_1}^\dagger c_{m_2} c_{m'_2}^\dagger c_{m_1}.
\]

The ladder operators can be normal ordered as follows,

\[
c_{m_1} c_{m_1'} c_{m_2} c_{m_2'} = c_{m_1}^\dagger (-c_{m_2} c_{m_1'} + \delta_{m_2, m_1'}) c_{m_2'} = -c_{m_1}^\dagger c_{m_2} c_{m_1'} c_{m_2'} + \delta_{m_2, m_1'} c_{m_1}^\dagger c_{m_1'} c_{m_2} c_{m_2'}.
\]

Hence,

\[
\zeta_2 = \frac{1}{2 \pi l_B^2} \left( N_B \sum_m c_m c_m - \sum_{m, m'} c_m c_{m'} c_{m'}^\dagger c_m \right).
\]

The two operators occurring in the above expression have a clear physical meaning. We follow the convention of Definitions 4.2 and 4.3.
5.2. CASIMIR OPERATORS ON THE TORUS

Definition 5.10. For \( k \in \mathbb{N} \), let

\[
n_{m_1, \ldots, m_k} := c_{m_1}^\dagger \cdots c_{m_k}^\dagger c_{m_k} \cdots c_{m_1}.
\]

Recall that the operator \( n_m = c_{m}^\dagger c_m \) counts the number of electrons occupying state \( |\alpha_m\rangle \). That is, its action on a multi-particle basis state \( \phi_{m_1} \wedge \ldots \wedge \phi_{m_k} \) is given by

\[
n_m \phi_{m_1} \wedge \ldots \wedge \phi_{m_k} = \begin{cases} 
\phi_{m_1} \wedge \ldots \wedge \phi_{m_k} & \text{if } m \in \{m_1, \ldots, m_k\} \\
0 & \text{otherwise}
\end{cases}
\]

Thus, the term \( \sum_m n_m \) counts the total number of electrons. Likewise, the operator \( n_{m,m'} = c_{m}^\dagger c_{m'}^\dagger c_{m'} c_{m} \) acts on a basis state \( \phi_{m_1} \wedge \ldots \wedge \phi_{m_k} \) as

\[
n_{m,m'} \phi_{m_1} \wedge \ldots \wedge \phi_{m_k} = \begin{cases} 
\phi_{m_1} \wedge \ldots \wedge \phi_{m_k} & \text{if } m,m' \in \{m_1, \ldots, m_k\} \\
0 & \text{otherwise}
\end{cases}
\]

Hence, the operator \( \sum_{m,m'} n_{m,m'} \) counts the number of pairs with multiplicity, meaning that each pair is counted twice. We conclude that the second order Casimir only counts particles and pairs, it does not care about how the particles are distributed among the available states.

Let us summarize the result in the following corollary.

Corollary 5.7. The second order Casimir \( \zeta_2 \) takes the form

\[
\zeta_2 = \frac{1}{2\pi l_B^2} \left[ N_B \sum_m n_m - \sum_{m,m'} n_{m,m'} \right] \tag{5.43}
\]

when acting on Fock space \( F^* \). Moreover, \( \zeta_2 \) does not distinguish between two different basis states with the same amount of electrons: when restricted to the \( N_{\text{el}} \)-particle subspace \( F^{*}_{N_{\text{el}}} \) of Fock space \( F^* = \bigoplus_{N=0}^{\infty} F^*_N \), \( \zeta_2 \) is equal to the multiplication operator

\[
\zeta_2 \big|_{F^{*}_{N_{\text{el}}}} = \frac{N_{\text{el}}}{2\pi l_B^2} (1 + N_B - N_{\text{el}}). \tag{5.44}
\]

Proof. When acting on \( \phi_{m_1} \wedge \ldots \wedge \phi_{m_{N_{\text{el}}}} \), \( \sum_m n_m \) gives a factor of \( N_{\text{el}} \), and \( \sum_{m,m'} n_{m,m'} \) gives a factor of \( N_{\text{el}}(N_{\text{el}} - 1) \). Hence, whenever \( \zeta_2 \) acts on a basis state \( \phi_{m_1} \wedge \ldots \wedge \phi_{m_{N_{\text{el}}}} \) with \( N_{\text{el}} \) electrons in it, it gives

\[
\zeta_2 \phi_{m_1} \wedge \ldots \wedge \phi_{m_{N_{\text{el}}}} = \frac{1}{2\pi l_B^2} \left[ N_B N_{\text{el}} - N_{\text{el}}(N_{\text{el}} - 1) \right] \phi_{m_1} \wedge \ldots \wedge \phi_{m_{N_{\text{el}}}}
\]

\[
= \frac{N_{\text{el}}}{2\pi l_B^2} (1 + N_B - N_{\text{el}}) \phi_{m_1} \wedge \ldots \wedge \phi_{m_{N_{\text{el}}}}.
\]

Before we proceed to compute the higher order Casimir operators, we will perform a consistency check. Haldane [10] also computed the second order Casimir operator, which he calls \( C_2 \).

\[
C_2 := \frac{1}{2N_B} \sum_{q \neq 0} \rho(q) \rho(-q),
\]

where \( \rho(q) \) is the sum

\[
\rho(q) := \sum_{i=1}^{N_{\text{el}}} \rho_i(q)
\]
of first quantized one-particle operators \( \rho_i(q) := e^{-iqR_i} \), and the index \( i \) is the particle label. Haldane’s definition of the second order Casimir differs from ours in that Haldane excludes \( \rho(0) \) from the sum over the Brillouin zone. Note that this does not affect the centrality of \( C_2 \). Indeed,

\[
\rho(0) = \sum_{i=1}^{N_{el}} \rho_i(0) = N_{el} \text{ is a constant, hence a central operator, and the difference between two central operators is again a central operator, therefore, so is } C_2.
\]

Haldane computes

\[
C_2 = \frac{N_{el}(N_B^2 - N_{el})}{2N_B} + \sum_{m < m'} P_{m,m'},
\]

where \( P_{m,m'} \) is the operator that exchanges the particles at guiding center positions \( m \) and \( m' \).

If we want to compare \( C_2 \) to \( \zeta_2 \), we will have to add \( \rho(0) \) to the sum in \( C_2 \). Let us call the adjusted operator \( C_2' \).

By (5.45),

\[
C_2' := \frac{1}{2N_B} \sum_q \rho(q)\rho(-q) = C_2 + \frac{(\rho(0))^2}{2N_B} = \frac{N_{el}N_B}{2} + \sum_{m < m'} P_{m,m'}.
\]

Next, we want to write the permutation operator \( P_{m,m'} \) in terms of ladder operators. This is straightforward.

\[
P_{m,m'} = c_{m'}^\dagger c_m c_{m'} c_m^\dagger.
\]

Notice that, compared to \( n_{m,m'} = c_{m'}^\dagger c_m c_m^\dagger c_m \) the order of the raising operators \( c_{m'}^\dagger \) and \( c_m^\dagger \) is reversed. That is, \( P_{m,m'} \) annihilates particles at positions \( m \) and \( m' \), and then creates them in reversed order, which is the same thing as exchanging the particles with guiding centre coordinates \( m \) and \( m' \). Using fermionic exchange statistics,

\[
P_{m,m'} = -n_{m,m'}.
\]

Hence, we see that (5.43) restricted to the \( N_{el} \)-particle subspace \( F_{N_{el}}^\ast \) reduces to

\[
\zeta_2 |_{F_{N_{el}}^\ast} = \frac{1}{2\pi l_B^2} \left( N_B \sum_m n_m + \sum_{m < m'} P_{m,m'} \right) = \frac{1}{\pi l_B^2} \left( \frac{N_B N_{el}}{2} + \sum_{m < m'} P_{m,m'} \right).
\]

**Corollary 5.8.** We conclude that both Casimir operators agree

\[
\zeta_2 = \frac{1}{\pi l_B^2} C_2,
\]

up to a numerically irrelevant constant.

Note that the overall prefactor of a Casimir operator does not necessarily carry any physical significance, because for any Casimir \( C, \lambda C \) is also a Casimir for every \( \lambda \in \mathbb{C} \).

Let us move on to the third order Casimir \( \zeta_3 \).

\[
\zeta_3 = \frac{1}{(2\pi l_B^2)^3} \sum_{m_1,m_1',m_3,m_3'} \delta_{m_1',m_3} \delta_{m_1,m_3} \delta_{m_1',m_2} \delta_{m_1',m_3} c_{m_1}^\dagger c_{m_1'}^\dagger c_{m_2} c_{m_2'}^\dagger c_{m_3} c_{m_3'}^\dagger
\]

\[
= \frac{1}{(2\pi l_B^2)^3} \sum_{m_1,m_2,m_3} c_{m_1}^\dagger c_{m_3}^\dagger c_{m_2} c_{m_2} c_{m_1} c_{m_3}.
\]

(5.46)
Normal ordering the ladder operators gives
\[
\begin{equation}
\begin{aligned}
c_{m_1} c_{m_3}^\dagger c_{m_2}^\dagger c_{m_1} c_{m_3} c_{m_2} &= c_{m_1} (\delta_{m_2, m_3} - c_{m_2}^\dagger c_{m_3}) (\delta_{m_1, m_3} - c_{m_1}^\dagger c_{m_3}) c_{m_2} \\
&= \delta_{m_2, m_3} c_{m_1} c_{m_3} c_{m_2} (\delta_{m_1, m_3} c_{m_1}^\dagger c_{m_3} c_{m_2}) \\
&\quad - \delta_{m_1, m_3} c_{m_1}^\dagger c_{m_2} c_{m_3} c_{m_2} (\delta_{m_2, m_3} c_{m_2}^\dagger c_{m_3} c_{m_2}) \\
&= \delta_{m_2, m_3} c_{m_1} c_{m_3} c_{m_2} (\delta_{m_2, m_3} c_{m_2}^\dagger c_{m_3} c_{m_2}) \\
&\quad - \delta_{m_1, m_3} c_{m_1}^\dagger c_{m_2} c_{m_3} c_{m_2} (\delta_{m_2, m_3} c_{m_2}^\dagger c_{m_3} c_{m_2}) \\
&= \delta_{m_2, m_3} c_{m_1} c_{m_3} c_{m_2} \\
&\quad + \delta_{m_1, m_3} (n_{m_1, m_2} - n_{m_1, m_2}) \\
&\quad + n_{m_1, m_2}. 
\end{aligned}
\end{equation}
\]

Finally, performing the sums yields the following corollary.

**Corollary 5.9.** The third order Casimir \( \zeta_3 \) takes the form

\[
\zeta_3 = \frac{1}{(2\pi l_B^2)^2} \left[ \sum_m n_m + (2 - N_B) \sum_{m,m'} n_{m,m'} + \sum_{m,m',m''} n_{m,m',m''} \right] 
\]

when acting on Fock space \( \mathcal{F}^* \). When restricted to the \( N_{el} \)-particle subspace \( \mathcal{F}^*_{N_{el}} \) of Fock space \( \mathcal{F}^* \), \( \zeta_3 \) becomes the multiplication operator

\[
\zeta_3 |_{\mathcal{F}^*_{N_{el}}} = \frac{N_{el}}{(2\pi l_B^2)^2} \left[ 1 + N_B - N_{el}(1 + N_B) + N_{el}^2 \right].
\]

**Proof.** Substituting \( N_{el}, N_{el}(N_{el} - 1) \) and \( N_{el}(N_{el} - 1)(N_{el} - 2) \) for \( \sum_m n_m, \sum_{m,m'} n_{m,m'} \), and \( \sum_{m,m',m''} n_{m,m',m''} \) respectively in the above expression for \( \zeta_3 \) yields

\[
\zeta_3 |_{\mathcal{F}^*_{N_{el}}} = \frac{1}{(2\pi l_B^2)^2} \left[ N_{el} + (2 - N_B) N_{el}(N_{el} - 1) + N_{el}(N_{el} - 1)(N_{el} - 2) \right] 
\]

\[
= \frac{N_{el}}{(2\pi l_B^2)^2} \left[ 1 + N_B - N_{el}(1 + N_B) + N_{el}^2 \right].
\]

\( \square \)

**Remark 5.9.** The higher order Casimir’s \( \zeta_{r+1} \) all have expressions similar to the formulas in (5.44) for \( \zeta_2 \) and (5.48) for \( \zeta_3 \). For each \( r \in \{1, 2, 3, \ldots, N_B - 1\} \), \( \zeta_{r+1} \) is constant on subspaces \( \mathcal{F}_{N_{el}} \) of Fock space, and on each such subspace it acts by multiplication with a polynomial in \( N_{el} \) of degree \( r+1 \). For \( r \geq N_B \), \( \zeta_{r+1} \) acts on each subspace \( \mathcal{F}_{N_{el}} \) of Fock space by multiplication with a polynomial in \( N_{el} \) of degree \( N_B \), because we can have at most \( N_B \) particles in the \( n_{el}^0 \) Landau level.

Besides the above remark, there is no obvious relation between \( r \) and the polynomial \( \zeta_{r+1} |_{\mathcal{F}_{N_{el}}} \).

Let us compute one more Casimir operator to see if some kind of pattern emerges.

The fourth order Casimir is given by

\[
\begin{equation}
\begin{aligned}
\zeta_4 &= \frac{1}{(2\pi l_B^2)^3} \sum_{m_1, \ldots, m_4} \delta_{m_1', m_4} \delta_{m_2', m_1} \delta_{m_3', m_2} \delta_{m_4', m_3} c_{m_1}^\dagger c_{m_1} c_{m_2}^\dagger c_{m_2} c_{m_3}^\dagger c_{m_3} c_{m_4}^\dagger c_{m_4} \\
&= \frac{1}{(2\pi l_B^2)^3} \sum_{m_1, \ldots, m_4} c_{m_1}^\dagger c_{m_4}^\dagger c_{m_2}^\dagger c_{m_2} c_{m_3}^\dagger c_{m_3} c_{m_4}^\dagger c_{m_4}.
\end{aligned}
\end{equation}
\]
As before, we normal order the ladder operators,
\[
e^\dagger_{m_1} c_{m_4}^\dagger c_{m_3}^\dagger c_{m_2}^\dagger c_{m_1}^\dagger c_{m_4} c_{m_3} c_{m_2} c_{m_1}
= c_{m_1}^\dagger (\delta_{m_1,m_4} - c_{m_1}^\dagger c_{m_4}) (\delta_{m_1,m_3} - c_{m_1}^\dagger c_{m_3}) (\delta_{m_1,m_2} - c_{m_1}^\dagger c_{m_2}) c_{m_3}
= \delta_{m_2,m_4}^2 \delta_{m_1,m_3}^2 c_{m_1}^\dagger c_{m_3}
\]
\[
- \delta_{m_1,m_3} \delta_{m_2,m_4} c_{m_1}^\dagger c_{m_3}^\dagger c_{m_4} c_{m_3}
- \delta_{m_2,m_4}^2 c_{m_1}^\dagger c_{m_3}^\dagger c_{m_1}^\dagger c_{m_4} c_{m_3}
- \delta_{m_1,m_3} \delta_{m_2,m_4} c_{m_1}^\dagger c_{m_3}^\dagger c_{m_4} c_{m_3}
+ \delta_{m_2,m_4} c_{m_1}^\dagger c_{m_3}^\dagger c_{m_4} c_{m_2} c_{m_3}
+ \delta_{m_1,m_3} c_{m_1}^\dagger c_{m_2} c_{m_4} c_{m_3} c_{m_3}
+ \delta_{m_2,m_4} c_{m_1}^\dagger c_{m_4} c_{m_3} c_{m_2} c_{m_3}
- c_{m_1}^\dagger c_{m_2} c_{m_4} c_{m_3} c_{m_1} c_{m_4} c_{m_2} c_{m_3}.
\]

The terms (A) and (B) are already normal ordered. (C) need one more adjustment,
\[
(C) = \delta_{m_2,m_4} c_{m_1}^\dagger c_{m_3} (\delta_{m_1,m_4} - c_{m_1}^\dagger c_{m_4}) c_{m_2} c_{m_3}
+ \delta_{m_1,m_3} c_{m_1}^\dagger c_{m_3} (1 - c_{m_1}^\dagger c_{m_4}) c_{m_2} c_{m_3}
+ \delta_{m_2,m_4} c_{m_1}^\dagger c_{m_2} (\delta_{m_3,m_4} - c_{m_1}^\dagger c_{m_4}) c_{m_1} c_{m_3}
= \delta_{m_2,m_4} \delta_{m_1,m_4} c_{m_1}^\dagger c_{m_3} c_{m_2} c_{m_3}
+ \delta_{m_1,m_3} c_{m_1}^\dagger c_{m_3} c_{m_2} c_{m_3}
+ \delta_{m_2,m_4} c_{m_1}^\dagger c_{m_2} c_{m_4} c_{m_3} c_{m_3}
- \delta_{m_1,m_3} c_{m_4} c_{m_3} c_{m_2} c_{m_3}
- \delta_{m_2,m_4} c_{m_1}^\dagger c_{m_2} c_{m_4} c_{m_3} c_{m_3}
- \delta_{m_1,m_3} c_{m_1}^\dagger c_{m_2} c_{m_3} c_{m_4} c_{m_3}.
\]

Lastly, (D) requires a bit more work,
\[
(D) = -c_{m_1}^\dagger c_{m_2}^\dagger (\delta_{m_3,m_4} - c_{m_3}^\dagger c_{m_4}) (\delta_{m_1,m_4} - c_{m_1}^\dagger c_{m_4}) c_{m_2} c_{m_3}
= -\delta_{m_3,m_4} c_{m_1}^\dagger c_{m_1}^\dagger c_{m_3} c_{m_2} c_{m_3}
+ \delta_{m_3,m_4} c_{m_1}^\dagger c_{m_2} c_{m_4} c_{m_3} c_{m_3}
+ \delta_{m_3,m_4} c_{m_1}^\dagger c_{m_2} c_{m_4} c_{m_3} c_{m_3}
- c_{m_1}^\dagger c_{m_2}^\dagger c_{m_3} (1 - c_{m_3}^\dagger c_{m_4}) c_{m_3} c_{m_2} c_{m_3}.
\]

Performing the summations over \(m_1, \ldots, m_4\) gives
\[
(A) = N_B \sum_m n_m
\]
\[
(B) = (N_B - 2) \sum_{m,m'} n_{m,m'}
\]
\[
(C) = (N_B - 2) \sum_{m,m'} n_{m,m'} - 3 \sum_{m,m',m''} n_{m,m',m''}
\]
\[
(D) = -\sum_{m,m'} n_{m,m'} + (N_B - 2) \sum_{m,m',m''} n_{m,m',m''} - \sum_{m,m',m'',m'''} n_{m,m',m'',m'''}.
\]

Finally, adding (A) – (D) yields the following corollary.
Corollary 5.10. The fourth order Casimir \( \zeta_4 \) takes the form

\[
\zeta_4 = \frac{1}{(2\pi l_B^2)^3} \left[ N_B \sum_m n_m + (2N_B - 5) \sum_{m,m'} n_{m,m'} + (N_B - 5) \sum_{m,m',m''} n_{m,m',m''} \right]
\]

when acting on Fock space \( \mathcal{F}^* \). When restricted to the \( N_{el} \)-particle subspace \( \mathcal{F}_{N_{el}}^* \) of Fock space \( \mathcal{F}^* \), \( \zeta_4 \) becomes the multiplication operator

\[
\zeta_4|_{\mathcal{F}_{N_{el}}^*} = \frac{N_{el}}{(2\pi l_B^2)^3} \left[ (1 + N_B) - N_{el}(1 + N_B) + N_{el}^2(1 + N_B) - N_{el}^3 \right].
\]

Proof. Substituting \( N_{el}, N_{el}(N_{el} - 1), N_{el}(N_{el} - 1)(N_{el} - 2) \) and \( N_{el}(N_{el} - 1)(N_{el} - 2)(N_{el} - 3) \) for \( \sum_m n_m, \sum_m n_{m,m'} ; \sum m,m',m'' n_{m,m',m''} \) and \( \sum m,m',m'' n_{m,m',m'',m''} \) respectively, gives

\[
\zeta_4|_{\mathcal{F}_{N_{el}}^*} = \frac{N_{el}}{(2\pi l_B^2)^3} \left[ N_B + (2N_B - 5)(N_{el} - 1) + (N_B - 5)(N_{el} - 1)(N_{el} - 2) - (N_{el} - 1)(N_{el} - 2)(N_{el} - 3) \right]
\]

\[
= \frac{N_{el}}{(2\pi l_B^2)^3} \left[ (1 + N_B) - N_{el}(1 + N_B) + N_{el}^2(1 + N_B) - N_{el}^3 \right].
\]

Looking at the obtained expressions for \( \zeta_2 \), \( \zeta_3 \) and \( \zeta_4 \), we see the following pattern emerging.

Conjecture 5.1. When acting on the \( N_{el} \)-particle sector of Fock space, for \( r \in \{1, 2, \ldots, N_{el} - 1\} \) the \((r + 1)\)-th order Casimir \( \zeta_{r+1} \) takes the following form

\[
\zeta_{r+1}|_{\mathcal{F}_{N_{el}}^*} = \frac{N_{el}}{(2\pi l_B^2)^r} \left[ (1 + N_B) \sum_{i=0}^{r-1} (-N_{el})^i + (-N_{el})^r \right].
\]

Moreover, using \( \sum_{i=0}^{r-1} x^i = \frac{1 - x^r}{1 - x} \), formula (5.50) reduces to

\[
\zeta_{r+1}|_{\mathcal{F}_{N_{el}}^*} = \frac{N_{el}}{(2\pi l_B^2)^r} \left[ (1 + N_B) \frac{1 - (-N_{el})^r}{1 + N_{el}} + (-N_{el})^r \right].
\]

Remark 5.10. We cannot generalize the obtained formulas for \( \zeta_{r+1} \) to orders for which \( r \geq N_{el} \), since for \( r \geq N_{el} \), the highest order terms will drop out because there are only \( N_{el} \) different creation operators, and \( (c_m^\dagger)^2 = 0 \) for any \( m \). It is not obvious how exactly this will affect the obtained formulas for the Casimirs of operators of order \( \mathcal{R} = r + 1 > N_{el} \).

Lastly, inspired by Haldane’s expression (5.45) for the second order Casimir, we mention that there is a way to express the obtained Casimir operators \( \zeta_2 \), \( \zeta_3 \), and \( \zeta_4 \) in terms of permutation operators. In the case of \( \zeta_2 \), there is only one permutation operator that comes to mind: the exchange operator. However, when going to higher order Casimir operators, there are more candidates available, because there are more than 2 elements in the permutation group \( S_N \) for \( N > 2 \). When we rewrite the expressions for the Casimirs in terms of permutation operators, we will not discriminate and use all possible permutation operators.
To start with, rewrite
\[
\sum_{m, m'} n_{m, m'} = \sum_{\sigma \in S_2} \sum_{m > m'} \text{sgn}(\sigma) P_{m, m'}^\sigma \tag{5.52}
\]
\[
\sum_{m, m', m''} n_{m, m', m''} = \sum_{\sigma \in S_2} \sum_{m > m' > m''} \text{sgn}(\sigma) P_{m, m', m''}^\sigma \tag{5.53}
\]
\[
\sum_{m, m', m'', m'''} n_{m, m', m'', m'''} = \sum_{\sigma \in S_2} \sum_{m > m' > m'' > m'''} \text{sgn}(\sigma) P_{m, m', m'', m'''}^\sigma \tag{5.54}
\]
where, for example, \(P_{m, m'}^\sigma\) applies \(\sigma \in S_2\) to the electrons with guiding center positions \(m\) and \(m'\), and likewise for the higher order permutation operators. To see that the above equations hold, observe that for (5.52),
\[
\sum_{m, m'} n_{m, m'} = \sum_{m > m'} 2 c_m^\dagger c_{m'} c_m = \sum_{m > m'} (c_m^\dagger c_m c_{m'} - c_m^\dagger c_{m'} c_m) = \sum_{m > m'} (P_{m, m'}^{(1d)} - P_{m, m'}^{(12)}) = \sum_{\sigma \in S_2} \sum_{m > m'} \text{sgn}(\sigma) P_{m, m'}^\sigma.
\]
Equations (5.53) and (5.54) follow analogously.

Using equations (5.52) – (5.54), we obtain the following expressions for the Casimir operators \(\zeta_2, \zeta_3\) and \(\zeta_4\):
\[
\zeta_2 = \frac{1}{2\pi i B} \left[ N_B N_{el} - \sum_{\sigma \in S_2} \sum_{m > m'} \text{sgn}(\sigma) P_{m, m'}^\sigma \right]
\]
\[
\zeta_3 = \frac{1}{(2\pi i B)^2} \left[ N_{el} + (2 - N_B) \sum_{\sigma \in S_2} \sum_{m > m'} \text{sgn}(\sigma) P_{m, m'}^\sigma + \sum_{\sigma \in S_4} \sum_{m > m' > m''} \text{sgn}(\sigma) P_{m, m', m''}^\sigma \right]
\]
\[
\zeta_4 = \frac{1}{(2\pi i B)^3} \left[ N_B N_{el} + (2N_B - 5) \sum_{\sigma \in S_2} \sum_{m > m'} \text{sgn}(\sigma) P_{m, m'}^\sigma \right.
\]
\[
\left. + (N_B - 5) \sum_{\sigma \in S_3} \sum_{m > m' > m''} \text{sgn}(\sigma) P_{m, m', m''}^\sigma - \sum_{\sigma \in S_4} \sum_{m > m' > m'' > m'''} \text{sgn}(\sigma) P_{m, m', m'', m'''\sigma} \right].
\]
The advantage of the expressions for \(\zeta_2, \zeta_3\) and \(\zeta_4\) as a polynomial in \(N_{el}\) over the expressions above, is that the former expressions generalize to arbitrary order \((\mathcal{R} = r + 1 = N_B)\), whereas the latter do not. However, to the physicist, the latter expressions might capture the imagination more, which is why we have listed both.

**Conclusion**
We have computed the Casimir operators \(\zeta_2, \zeta_3\) and \(\zeta_4\), generalizing Haldane’s result for \(\zeta_2\). Additionally, we were able to extrapolate the expressions found for the lower order Casimirs up to order \(\mathcal{R} = N_B\). The original idea was that, because Haldane’s Casimir contains a permutation operator, perhaps the Casimir operators would be able to give information about the exchange statistics of the particles, even if the particles are not necessarily fermions. For example, for bosons, equation (4.28) holds\(^9\), and therefore the projected density operator algebra remains intact. However, how the rest of the construction of Chapter 5 is affected by a change of statistics is not immediately clear. Besides bosonic and fermionic statistics, there exist even more general *anyonic* statistics, which we will discuss in the next chapter — but we will not be able to compute Casimirs in case of

\(^9\)To prove this, use the bosonic versions of (4.26) – (4.27).
anyons, unfortunately, because there is no straightforward generalization of the formalism of second quantization to anyonic systems (see the introduction of Section 6.1).

The results of this chapter also apply to composite fermions, described in Section 3.2.5, because the projected composite fermion density operators $\bar{\rho}(q)$ — see equation (5) of Goerbig, Lederer and Morais Smith [6] — satisfy the same Lie algebra as the ordinary projected density operators (5.29):

$$[\bar{\rho}(k), \bar{\rho}(l)] = 2i \sin \left( \frac{\pi k \times l}{N_B^*} \right) \bar{\rho}(k + l),$$

(5.55)

where $B^* = B - 2s\phi_0n_{el}$, as explained in Section 3.2.5. Hence, all the expressions for the Casimirs obtained in this chapter also hold for composite fermions, only with $B$ replaced by $B^*$. In particular, (5.51) for composite fermions reads

$$\zeta_{r+1}|_{F_{N_{el}}}^* = \frac{N_{el}}{(2\pi l_B^*)^2r} \left[ (1 + N_B^*)^{1 - (-N_{el})^r} + (-N_{el})^r \right].$$

Having computed the Casimir operators, we can conclude that, despite the fact that we found nice formulas, ultimately, from a physicists perspective, the Casimir operators are perhaps not as interesting as they could have been. Instead of pursuing the line of reasoning followed thus far any further, we will change course and proceed to a topic that is intimately related to the FQHE, but not directly related to the Casimir operators we computed in Chapters 4 and 5.

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$^{10}$The absence of the minus sign in (5.55) does not affect the form of the Casimir or the computation done in this chapter – as the author found out the hard way.
Chapter 6

Charged Anyons in a Magnetic Field

The fundamental particles of nature fall into two categories; depending on their exchange statistics, a fundamental particle can either be a bosons or a fermion. The type of the particles in question determines the symmetry properties of the wavefunction: $N$-particle wavefunctions describing identical bosons (fermions) pick up a plus (minus) sign whenever two particles are interchanged.

In turns out that, in two dimensions, there is a possibility for particles to exist that are neither fermions nor bosons. These particles are called anyons, a term invented by Wilczek [26], because the statistical phase an anyonic $N$-particle wavefunction picks up under the exchange of two identical anyons is — in principle — arbitrary. Since all fundamental particles in nature are either fermions or bosons, it is expected that, if anyons exist at all, they exist in the form of quasi-particles.

The only physical system known to date in which anyons are expected to appear is that of the FQHE, where the anyons are effective quasi-particle excitations of the FQHE fluid in the ground state at a Laughlin filling factor $\nu = \frac{1}{2s+1}$ with $s \in \mathbb{N}$, as described in Section 3.2.4. We will discuss the anyonic nature of these quasi-particle excitations in Section 6.1.1. As described in Section 3.2.4, the quasi-particles carry charge. Therefore, we expect that they also feel the magnetic field that causes the QHE. Motivated by this conjecture, the goal of this chapter is to find an exact description of charged non-interacting anyons in a uniform perpendicular magnetic field.

The approach that we will take is less conventional than the one normally found in literature, where, as will be explained in Section 6.1.1, an anyon is seen as a charged particle-flux tube composite. Instead, we will take the anyon to be our fundamental particle, meaning that we are working with an effective description of the system of quasi-particle excitations, and we will attempt to construct exact anyonic energy eigenfunctions of the Hamiltonian that describes non-interacting charged particles in a uniform perpendicular magnetic field. The difficulty in constructing anyon wavefunctions lies in the fact that even non-interacting anyons feel each other presence, by which we mean that their exchange statistics give rise to a statistical term in the wavefunction that relates anyons at different positions to each other, making non-interacting anyons equally complicated as interacting bosons or fermions.

In Section 6.1, we will introduce anyons in the FQHE, and we will set up the mathematical formalism that describes anyons as fundamental particles. Next, in Section 6.2, we will construct energy eigenfunctions of the Hamiltonian for non-interacting charged anyons in a uniform perpendicular magnetic field.
6.1 Introduction to Anyons

For identical bosons and fermions, the multi-particle wavefunction \( \psi \) picks up a phase \( e^{i\alpha \pi} \), where \( \alpha \) is an even integer for bosons (i.e. \( e^{i\alpha \pi} = 1 \)), and an odd integer for fermions (i.e. \( e^{i\alpha \pi} = -1 \)), under the exchange of two particles. For anyons, we allow \( \alpha \) to be any real number. At first sight, it seems as if anyons are not very different from bosons or fermions. Therefore, one might guess that much of the existing theory for bosons and fermions can readily be applied to anyons. This assumption turns out to false.

For example, one of the first things a physics student might try when constructing a mathematical formalism that describes anyons, is to use the formalism of second quantization, and impose the following commutation relations on the creation and annihilation operators for anyons,

\[
\psi(r)\psi(r') - e^{i\alpha \pi}\psi(r')\psi(r) = 0.
\]

(6.1)

However, closer inspection shows that this naive approach is doomed to fail. For, combining equation (6.1) with the same equation with \( r \) and \( r' \) interchanged:

\[
\psi(r')\psi(r) - e^{i\alpha \pi}\psi(r)\psi(r') = 0,
\]

leads to

\[
(1 - e^{2i\alpha \pi})\psi(r')\psi(r) = 0.
\]

Hence, whenever \( \alpha \notin \mathbb{Z} \), we must have \( \psi(r')\psi(r) = 0 \), which, after taking the Hermitian conjugate, means that we can create at most one anyon! This is undesirable for many reasons, one of which is the fact that anyons statistics only show up when there are at least two anyons present, which we cannot create in this naive approach. The simple argument provided above shows that any second quantized approach to anyons is likely to fail\(^1\).

Now that we have seen how not to describe anyons, we will proceed with the construction of a correct mathematical formalism for describing anyons. There are multiple ways to do this. First, we will discuss the anyon as it is usually described within the context of the QHE: as a charged particle-flux tube composite. Next, we will switch to the mathematically more elegant description that takes anyons as fundamental particles. We will also elaborate on why anyons are only expected to occur in two dimensions.

6.1.1 Anyons in the Quantum Hall Effect

As explained in Section 3.2.4, the quasi-particle excitations of the FQHE fluid at a Laughlin filling factor \( \nu = 1/(2+\epsilon) \) are quasi-holes and quasi-particles, which consist of a magnetic flux tube \( \Phi \) and an electric charge \( e^* = e\nu \) both localized at the same position. A flux tube, refers to an infinitely thin solenoid pierced by a magnetic flux \( \Phi \), such that the magnetic flux outside of the solenoid is zero. The combination of a charged particle bound to a flux tube is commonly referred to as a charged particle flux-tube composite. Its anyonic nature was first noted by Wilczek [25] and [26]. In this section, we will follow Wilczek and describe how a charged particle-flux tube composite gives rise to anyonic statistics.

Suppose that we have two charged particle-flux tube composites of charge \( -e^* \), mass \( m \) and flux \( \Phi \), with corresponding position coordinates \( r_1 \) and \( r_2 \). We will assume that the electrostatic interaction is negligible, and that the only interaction between the two particles is due to the fact

\(^1\)That is, the author is not aware of any description of anyons that uses creation and annihilation operators for anyons. Note that the creation and annihilation operators used in Chern-Simons theory related approaches only create and destroy the charged particle part of the anyon, but they leave the flux-tube part of the anyon untouched. See Dunne [2] for the details.
that the charge of one particle feels the magnetic flux attached to the other. Hence, the Hamiltonian for the two particles is given by

$$H = \frac{1}{2m} (p_1 + e^* a_1) + \frac{1}{2m} (p_2 + e^* a_2),$$

where $a_1$ is the magnetic vector potential that particle 1 feels due to the flux attached to particle 2, and likewise for $a_2$. That is,

$$a_1(r_1, r_2) = \frac{\Phi e_z \times (r_1 - r_2)}{2\pi |r_1 - r_2|^2},$$

$$a_2(r_1, r_2) = \frac{\Phi e_z \times (r_2 - r_1)}{2\pi |r_1 - r_2|^2}.$$

Note that the above vector potentials mimic the properties of a flux tube. Indeed, the vector potential

$$a(r) = \frac{\Phi e_z \times r}{2\pi |r|^2}$$

is irrotational outside the origin, so the corresponding magnetic field $b(r) = \nabla \times a(r) = 0$ for $r \neq 0$, but the total flux penetrating any disc $D$ centered at the origin is

$$\int_D b \cdot e_z \, dr = \oint_{\partial D} A \cdot d\mathbf{r} = \Phi.$$

Additionally, we take the charged particles to be bosonic$^2$. Going to the center of mass $R := \frac{1}{2}(r_1 + r_2)$ and relative $r := r_1 - r_2$ position coordinates, with corresponding momenta $P := M \dot{R} = 2m \dot{r_1} + \dot{r_2} = p_1 + p_2$ and $p := \frac{m}{2} \dot{r} = \frac{p_1 - p_2}{2}$, the Hamiltonian becomes

$$H = \frac{P^2}{4m} + \frac{[p + e^* a(r)]^2}{m},$$

where $a$ is given by equation (6.2). Thus, the two-charged particle-flux tube system is equivalent to a tensor product $\psi_{\text{CM}} \otimes \psi_{\text{rel}}$ of a free center of mass particle with mass $M = 2m$ together with another particle of charge $-e^*$ and mass $\frac{m}{2}$ that feels the vector potential $a$ due to a magnetic flux tube $\Phi$ at the origin.

We are interested in the exchange process of two anyons, which only involves the relative coordinate $r$. Exchanging the anyons amounts to adiabatically — as to not disturb the system and keep it in the same energy state — moving $r$ to $-r$. In terms of polar coordinates, $r = (r, \theta)$, the anyon exchange process is $\theta \mapsto \theta + \pi$. Note that, because the charged particles are bosons, the wavefunction $\psi_{\text{rel}}$ describing the relative motion of the particles satisfies the boundary condition

$$\psi_{\text{rel}}(r, \theta + \pi) = \psi_{\text{rel}}(r, \theta).$$

Remark 6.1. The polar angle $\theta$ is multi-valued on the configuration space. However, as will be explained in the Section 6.1.3, we are actually describing wavefunctions defined on the universal cover of the configuration space, where $\theta \in \mathbb{R}$ is globally defined and single-valued.

The remark above justifies the following gauge transformation:

$$a \mapsto a' = a - \nabla \chi,$$

with

$$\chi(r, \theta) = \frac{\Phi}{2\pi} \theta.$$
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and hence\(^3\)

\[ a' = 0. \]

Therefore, in the primed gauge, also known as the anyon gauge, the Hamiltonian reduces to the free particle Hamiltonian

\[ H' = \frac{P^2}{4m} + \frac{p'^2}{m}. \]

However, the wavefunction \( \psi_{rel}' \) in the primed gauge changes as well. As explained in Section 2.2.3, \( \psi_{rel}' \) gets multiplied by a factor \( U(r) = \exp \left( \frac{-ie^* \chi(r)}{\hbar} \right) \) from equation (2.39). That is,

\[ \psi_{rel}'(r, \theta) = \exp \left( \frac{-ie^* \Phi}{\hbar} \right) \psi_{rel}(r, \theta). \]

We conclude that, because of (6.3), the system comprised of two-charged particle-flux tube composites is equivalent to the system of two free particles with anyonic exchange statistics, meaning that, under particle exchange \( \theta \mapsto \theta + \pi \),

\[ \psi_{rel}'(r, \theta + \pi) = \exp \left( \frac{-ie^* \Phi \pi}{\hbar} \right) \psi_{rel}'(r, \theta). \]

That is, the total wavefunction picks up a phase \( e^{i\alpha \pi} \) under the exchange of two particles, where the statistical parameter \( \alpha \) is given by

\[ \alpha = -\frac{e^* \Phi}{\hbar}. \]

**Remark 6.2.** Note that, for anyons, it matters whether the exchange is clockwise \( \theta \mapsto \theta - \pi \) or anti-clockwise \( \theta \mapsto \theta + \pi \), because the statistical phase that the wavefunction picks up differs for the clockwise \( e^{-i\alpha \pi} \) and anti-clockwise \( e^{i\alpha \pi} \) exchanges.

In the case of quasi-holes, which carry a charge of \( e^* = e\nu \) and a flux \( \Phi = \phi_0 = \frac{h}{e} \), the statistical parameter is given by

\[ \alpha_{qh} = -\nu \]

for quasi-holes. The statistical parameter for quasi-particles in the FQHE has the same value.

To summarize, we have explained how charged particle-flux tube composites show anyonic behavior. In particular, the quasi-hole excitations of the FQHE fluid at fractional filling factor \( \nu \), which consist of a charged particle of charge \( -e^* = -e\nu \) bound to a flux tube with flux \( \phi_0 \), are anyons with statistical parameter \( \alpha_{qh} = -\nu \). Moreover, we now have an idea of how anyonic statistics can arise in nature. However, when working with anyons, it seems cumbersome to think of the anyon as a composite of two separate objects. Instead, we prefer to describe an anyon as a single mathematical entity.

The remainder of Section 6.1 will be devoted to constructing the mathematical framework that we will use to describe anyons.

### 6.1.2 The Configuration Space of Identical Particles

In order to give a rigorous description of anyons, we will have go all the way back to the fundamentals of identical particles in quantum mechanics.

Since the discovery of the Gibbs paradox\(^4\), it has been known that it is necessary to differentiate between multi-particle systems comprised of identical particles and multi-particle systems.

\(^3\)Recall that the \( \frac{\partial}{\partial \theta} \) part of the gradient in polar coordinates is \( \frac{1}{r} \frac{\partial}{\partial \theta} \).

\(^4\)The problem that the entropy of a gas of identical particles does not scale with the number of particles when two identical containers of the same type of gas are joined together. This problem is solved by dividing the multiplicity by \( N! \), which accounts for the fact that the particles are indistinguishable.
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consisting of non-identical particles. The usual way to deal with identical particles, is to impose an (anti)symmetry condition on the multi-particle wavefunction. Other than the fact that (anti)symmetric wavefunctions give rise to experimentally observed distributions in statistical mechanics, the origin of this (anti)symmetrization condition is not immediately clear.

Instead of imposing an (anti)symmetry condition, we will encode the indistinguishability of the identical particles in the configuration space. We assume that each particle can move through all of \( \mathbb{R}^d \), with \( d \) an arbitrary positive integer, but that no two particles can be at exactly the same position, because we don’t know what it means for two particles to occupy the same space.

**Definition 6.1.** The set of points of coincidence \( C \subset (\mathbb{R}^d)^N \) is set \( C := \{(r_1, \ldots, r_N) \in (\mathbb{R}^d)^N \colon r_i = r_j \text{ for some } 1 \leq i < j \leq N \} \) of configurations for which two or more particle coordinates coincide.

As mentioned above, we want \( C \) to not be a part of the configuration space. Taking the indistinguishability into account, the configuration space is defined as follows.

**Definition 6.2.** The configuration space of \( N \) identical particles \( C_{N,d} \) is defined as the space

\[
C_{N,d} := \frac{(\mathbb{R}^d)^N - C}{S_N},
\]

where \( C \subset (\mathbb{R}^d)^N \) is set of elements of coincidence, and the division by \( S_N \) means that we consider any two elements in \( (\mathbb{R}^d)^N - C \) related by a permutation of particle positions to be equal.

For example, in the case of \( N = 2 \) particles, the elements \((a, b) \in (\mathbb{R}^d)^2 \) and \((b, a) \in (\mathbb{R}^d)^2 \) represent the same element in \( C_{2,d} \), which reflects the fact that “two identical particles at positions \( a \) and \( b \)” is the same statement as “two identical particles at positions \( b \) and \( a \)”.

Having defined the configuration space, we can now discuss the fundamental difference between \( d = 2 \) and \( d > 2 \), and why \( d = 2 \) allows for more general exchange statistics than \( d = 3 \). In order to do so, we only have to consider the two particles that we want to exchange. We follow the well-known publication by Leinaas and Myrheim [16].

Suppose we have two particles in \( \mathbb{R}^d \). Using center of mass \( R := \frac{1}{2}(r_1 + r_2) \) and relative \( r := r_1 - r_2 \) position coordinates, the configuration space \( C_{2,d} \) is isomorphic to

\[
C_{2,d} \cong \mathbb{R}^d \otimes \left( \frac{\mathbb{R}^d - 0}{\mathbb{Z}_2} \right),
\]

since the center of mass \( R \) coordinate can be anywhere in \( \mathbb{R}^d \), but the relative position coordinate \( r \) has to be non-zero, and it has to satisfy \( r \sim -r \), because the particles are identical. Now,

\[
\left( \frac{\mathbb{R}^d - 0}{\mathbb{Z}_2} \right) \cong (0, \infty) \times \mathbb{R}^{d-1},
\]

where \( \mathbb{R}^{d-1} \) is the real projective space of dimension \( d - 1 \), which, by definition, is equal to \( \mathbb{R}^d / \sim \), with \( x \sim y \Leftrightarrow x = \lambda y \) for some \( \lambda \in \mathbb{R} \).

Next, we quantize the theory. This means that we construct a complex line bundle \( \mathcal{L} \) over \( C_{2,d} \), such that its square-integrable sections correspond to two-particle wavefunctions. The line bundle is equipped with a connection which has zero curvature. In quantum mechanical language, the process of adiabatically exchanging particles becomes that of parallel transporting localized sections of \( \mathcal{L} \) over a closed loop. Performing a parallel transportation over a closed curve \( \gamma : [0, 1] \to C_{2,d} \) yields a unitary map

\[
P_{\gamma}^x : \mathcal{L}_x \to \mathcal{L}_x,
\]

where \( x = \gamma(0) = \gamma(1) \). Note that \( \mathcal{L}_x \cong \mathbb{C} \), so \( P_{\gamma}^x = e^{i\alpha} \) for some \( \alpha \in \mathbb{R} \), which is the phase that the wavefunction picks up after the two particles have been exchanged via the loop \( \gamma \). Because the
curvature is zero, $P^\gamma_x = P^{\gamma'}_x$ for any $\gamma'$ that can be continuously deformed into $\gamma$. In particular, this means that we can restrict our attention to loops for which the center of mass coordinate $R$ is constant, because any closed loop for which $R$ is not constant can be continuously deformed into one for which $R$ is constant. Hence, we will forget about the center of mass coordinate, and only consider closed loops in the relative coordinate configuration space $(0, \infty) \times \mathbb{R}P^{d-1}$.

Now comes the point where we distinguish between $d = 2$ and $d > 2$. For $d > 2$, the fundamental group of $\mathbb{R}P^{d-1}$ is given by
\[
\pi_1(\mathbb{R}P^{d-1}) \cong \mathbb{Z}_2.
\]
In particular, any closed loop in $(0, \infty) \times \mathbb{R}P^{d-1}$ traversed twice can be continuously deformed back to the trivial loop. Hence, for every $x \in (0, \infty) \times \mathbb{R}P^{d-1}$ and for every $\gamma$ such that $x = \gamma(0) = \gamma(1)$,
\[
(P^\gamma)^2 = 1,
\]
This means that any kind of exchange process in dimension $d > 2$ can result in the wavefunction picking up a phase $e^{i\alpha\pi} = \pm 1$. However, in $d = 2$ dimensions,
\[
\pi_1(\mathbb{R}P^1) \cong \mathbb{Z},
\]
and there is no restriction on $P^\gamma_x = e^{i\alpha\pi}$, which means that the wavefunction can pick up any arbitrary phase under the exchange of two particles.

Now that we have explained why anyons can possibly occur in $d = 2$ dimensions, let us focus on the case $d = 2$, and discuss the anyon configuration space and its universal cover, which we need to properly define anyonic wavefunctions.

### 6.1.3 The Universal Cover of the Configuration Space

From now on, all particles are restricted to move in $d = 2$ dimensions.

**Notation:** because $d = 2$ is fixed, we will simply write $\mathcal{C}_N$ instead of $\mathcal{C}_{N,2}$. Additionally, an arbitrary element of $\mathcal{C}_{N,d}$ will be denoted as a set $\{r_1, \ldots, r_N\}$. We use curly brackets to emphasize the fact that the order of the elements is irrelevant, i.e. $\{b, a\} = \{a, b\}$.

Next, we discuss an important issue that we have swept under the rug in the previous section.

**Remark 6.3.** An anyonic wavefunction is a section of a complex line bundle $\mathcal{L}$ over the configuration space. However, given a fixed quantum mechanical state, there is some ambiguity in which section $\psi \in \Gamma(\mathcal{C}_N, \mathcal{L})$ of the line bundle to assign to this particular state. For example, suppose we have a section $\psi$ that has the same expectation values as the fixed quantum state. Now, we can do a particle exchange, after which $\psi$ picks up a global phase $e^{i\alpha\pi}$, meaning that $\psi(x) \mapsto \psi'(x) = e^{i\alpha\pi}\psi(x)$ for each $x \in \mathcal{C}_N$. The transformed section $\psi'$ describes precisely the same state as $\psi$ does; in particular, all expectation values of $\psi$ and $\psi'$ are the same. Instead of having different descriptions of the same state, we would rather have one mathematical entity that contains, in a well-defined way, both $\psi$ and $\psi'$.

In more mathematical language, the section $\psi \in \Gamma(\mathcal{C}_N, \mathcal{L})$ is *multi-valued*. This means that, strictly speaking, it is not a function at all! In order to remedy this issue, we have to go to the universal cover of the configuration space: a space that itself contains many copies\(^5\) of the configuration space $\mathcal{C}_N$, each copy corresponding to different labellings\(^6\) of the anyons.

---

\(^5\)Modulo a measure zero set.
\(^6\)See next page.
Definition 6.3. The universal cover of $C_N$ is the simply connected covering space of $C_N$, which is unique up to homeomorphism. It is denoted by $\tilde{C}_N$. The corresponding covering map from $\tilde{C}_N$ to $C_N$ will be denoted by $P : \tilde{C}_N \to C_N$.

Without proof, we mention that there is a nice way to visualize the space $\tilde{C}_N$. To each point $\{r_1, \ldots, r_N\} \in C_N$ we associate a collection of $N$ non-intersecting lines $\{\gamma_1, \ldots, \gamma_N\}$ extending to $y \to -\infty$ such that each line begins at precisely one of the points in the set $\{r_1, \ldots, r_N\}$. We consider two elements $\{(r_1, \ldots, r_N), \{\gamma_1, \ldots, \gamma_N\}\}$ and $\{(r_1', \ldots, r_N'), \{\gamma_1', \ldots, \gamma_N'\}\}$ to be equivalent if $\{r_1, \ldots, r_N\} = \{r_1', \ldots, r_N'\}$ as sets, and, $\{\gamma_1, \ldots, \gamma_N\}$ can be continuously deformed into $\{\gamma_1', \ldots, \gamma_N'\}$ without the any of the lines intersecting each other, while keeping the endpoints $\{r_1, \ldots, r_N\}$ fixed throughout the deformation process. Now, $\tilde{C}_N$ is defined as the collection of equivalence classes of sets $\{(r_1, \ldots, r_N), \{\gamma_1, \ldots, \gamma_N\}\}$. We will use square brackets to denote equivalence classes: $[[r_1, \ldots, r_N], \{\gamma_1, \ldots, \gamma_N\}]$.

Having defined the configuration space $C_N$ and its universal cover $\tilde{C}_N$, we can discuss the process of labelling identical anyons. With bosons or fermions, a labelling is an assignment that assigns a position variable to each fermion or boson. For anyons, the situation is a little bit more complicated.

Given a set of non-intersecting lines $\{\gamma_1', \ldots, \gamma_N'\}$ starting at points $\{r_1', \ldots, r_N'\}$ and extending to $y \to -\infty$, we can always find another set of lines $\{\gamma_1, \ldots, \gamma_N\}$ that are homotopy-equivalent to $\{\gamma_1', \ldots, \gamma_N'\}$, such that the lines $\{\gamma_1, \ldots, \gamma_N\}$ after some point $y < c$ are straight and parallel. We then label the lines $\{\gamma_1, \ldots, \gamma_N\}$ with numbers $1, 2, \ldots, N$ from left to right.

**Definition 6.4.** Given a set of points $\{r_1, \ldots, r_N\} \in C_N$, a labelling is a choice of an element in the preimage $P^{-1}(\{r_1, \ldots, r_N\}) \in \tilde{C}_N$. The anyons are then labelled according to the labelling of the lines — as described above — that connect to the anyon positions.

![Figure 6.1: The two dots represent two fixed anyon position coordinates in $C_2$. The figures show three different ways of connecting these two dots with non-intersecting lines going to $y \to -\infty$. We then label the dots with either a 1 or a 2 depending on whether the dot in question is connected to the blue line (1) or the red line (2), where the labelling of the lines is always from left to right in increasing order.](image)

Having defined what it means to label indistinguishable anyons, we can now define the concept of exchanging two indistinguishable anyons.

**Definition 6.5.** For a given configuration $\{r_1, \ldots, r_N\}$, the process of exchanging two indistinguishable anyons is that of physically moving two anyons from one labelling in $P^{-1}(\{r_1, \ldots, r_N\})$ to another labelling in $P^{-1}(\{r_1, \ldots, r_N\})$. In the exchange process, the lines $\{\gamma_1, \ldots, \gamma_N\}$ initially connected to the points in $\{r_1, \ldots, r_N\}$ remain connected to their respective (moving) endpoints, and do not intersect during the exchange process.
Let us try to visualize the process of anyon exchange. In Figure 6.1, focus on the middle picture, where the first anyon is to the left of the second anyon. Now, we rotate both anyons clockwise around their center of mass over an angle of $\pi$, while keeping the red and blue strands attached to the moving endpoints. Doing so bring us to the left picture in Figure 6.1. This completes the exchange process, as we are now back at the starting point, in the sense that we have two anyons at exactly the same positions as we had before; the only difference is the anyons are labelled differently.

We could have also exchanged the anyons the other way around: using an anti-clockwise exchange process. This would have taken us form the middle picture to the right picture in Figure 6.1. As noted in Remark 6.2, clockwise and anti-clockwise exchanges produce different overall phase factors that multiply the wavefunction. This feature distinguishes anyons from bosons or fermions, because for the latter two it does not matter how the particles are exchanged.

In the next section, we will explain how the difference between clockwise and anti-clockwise exchanges will be encoded in the anyon multi-particle wavefunction.

### 6.1.4 Anyon Wavefunctions and the Braid Group

We now turn to the wavefunction $\psi$ describing $N$ indistinguishable anyons. Keeping Remark 6.3 in mind, we want $\psi$ to assign a complex number not only to each configuration, but also to each labelling of a fixed configuration. In other words, $\psi$ is a section of a complex line bundle $\mathcal{L}$ over the universal cover of the configuration space: $\psi \in \Gamma(\tilde{C}_N, \mathcal{L})$. For our present purposes, it is sufficient to consider only globally trivializable line bundles, so we will restrict ourselves to wavefunctions of the form

$$\psi : \tilde{C}_N \to \mathbb{C}.$$ 

Recall that, in contrast to fermions and bosons, there are now two ways to exchange identical anyons: clockwise and anti-clockwise. Since clockwise and anti-clockwise exchanges undo each other, the phases related to clockwise and anti-clockwise exchanges must be each others multiplicative inverse. This hints at the presence of the braid group $B_N$.

Before we continue, let us recall the correspondence between a covering space and the fundamental group of the corresponding base space.

**Remark 6.4.** For a general topological space $X$ covered by a (not necessarily universal) covering space $p : X' \to X$, where $p$ is the covering map, there is a well-defined action of the fundamental group $\pi_1(X)$ on the covering space $X'$, constructed as follows. For any point $x' \in X'$, observe the image $x = p(x') \in X$, and consider a closed loop $\gamma : [0, 1] \to X$ with basepoint $\gamma(0) = \gamma(1) = x$. The loop $\gamma$ can be lifted to a unique loop $\tilde{\gamma} : [0, 1] \to X'$ such that $\tilde{\gamma}(0) = x'$. Now, $\gamma$ acts on $x'$ by mapping $x'$ to $\tilde{\gamma}(1) \in X'$. It can be shown that this action is independent of the homotopy class of $\gamma$. Additionally, the action respects the multiplication of loops by composition, which means that the action of closed loops on $X'$ transends to a group action of the fundamental group $\pi_1(X)$ on $X'$.

Obviously, the universal cover $\tilde{C}_N$ covers the configuration space $C_N$. Moreover, we have the following theorem, for which we omit the proof.

**Theorem 6.1.** The fundamental group of $C_N$ is isomorphic to the braid group on $N$ strands:

$$\pi_1(C_N) \cong B_N.$$ 

Due to Remark 6.4, we see that the braid group $B_N$ naturally acts on $\tilde{C}_N$. Pictorially, the elements $b_{12}$ and $b_{12}^{-1}$ of $B_N$ map the middle picture in Figure 6.1 to the left and right pictures respectively. We conclude that the process of exchanging two anyons can be seen as an action of a particular element of the braid group on $\tilde{C}_N$. 

Notation: we will adopt the shorthand notation \( \xi \) for \( \{r_1, \ldots, r_N\} \in \mathcal{C}_N \), and \( \tilde{\xi} \) for an element \( \{\{r_1, \ldots, r_N\}, \{\gamma_1, \ldots, \gamma_N\}\} \in \tilde{\mathcal{C}}_N \) that projects to \( \xi \). Moreover, for any element \( \tilde{\xi} \in \tilde{\mathcal{C}}_N \) and any \( b \in B_N \), we denote the action of \( b \) on \( \tilde{\xi} \) by a dot, i.e. \( \tilde{\xi} \mapsto b \cdot \tilde{\xi} \).

The requirement that the \( N \)-particle anyonic wavefunction \( \psi : \tilde{\mathcal{C}}_N \to \mathbb{C} \) picks up a phase \( e^{\pm i\alpha \pi} \) when two anyons are interchanged can finally be made precise.

**Requirement.** For any \( b \in B_N \cong \pi_1(\mathbb{C}^N) \) and \( \tilde{\xi} \in \tilde{\mathcal{C}}_N \), we require that
\[
\psi(b \cdot \tilde{\xi}) = \rho(b) \psi(\tilde{\xi}),
\]
where
\[
\rho : B_N \to \mathbb{C}
\]
is the one-dimensional representation of the braid group that maps each left-over-right generator \( b_{j,j+1} \in B_N \) to \( \rho(b_{j,j+1}) = e^{-i\alpha \pi} \in \mathbb{C} \), and each right-over-left generator \( b_{j,j+1}^{-1} \) to \( \rho(b_{j,j+1})^{-1} = e^{i\alpha \pi} \in \mathbb{C} \).

**Remark 6.5.** It is the representation \( \rho : B_N \to \mathbb{C} \) of the braid group that determines the type of particle that we are modelling. Setting \( \alpha = 0 \) will give back the bosonic wavefunctions, as equation (6.4) then reduces to the statement that \( \psi \) takes the same value on each sheet of \( \mathbb{C}_N \) embedded in \( \tilde{\mathcal{C}}_N \), meaning that \( \psi \) is symmetric with respect to the exchange of two particles. Likewise, when setting \( \alpha = 1 \), equation (6.4) reduces to the familiar anti-symmetry condition for fermionic \( N \)-particle wavefunctions. Only when we take \( \alpha \notin \mathbb{Z} \) do we get to see the fact that we are actually dealing with a representation of the braid group, as only in this case we can distinguish between clockwise and anti-clockwise exchanges of particles, since then \( e^{i\alpha \pi} \neq e^{-i\alpha \pi} \).

Now that we have made explicit the condition (6.4) for the wavefunction to satisfy under particle exchange, let us try to construct a function on the universal cover with the required exchange properties.

### 6.1.5 The Statistical Term in the Wavefunction

We will now construct functions on the universal cover \( \tilde{\mathcal{C}}_N \) of the configuration space that behave correctly under particle exchange. Because, later on, we want the obtained wavefunctions to be eigenfunctions of the Hamiltonian, we put the obvious additional restriction on the to be constructed wavefunctions that they are differentiable.

**Remark 6.6.** A possible approach to constructing multi-particle anyon wavefunctions is to try to generalize the construction of multi-particle bosonic (fermionic) wavefunctions by (anti-)symmetrizing the single-particle wavefunctions. Goldin and Majid [8] came up with such a generalization, by making use of what they call the braided tensor product. The problem with their approach, however, is that the multi-particle wavefunctions obtained with the use of the braided tensor product are discontinuous, and can therefore not be eigenfunctions of a differential operator such as the Hamiltonian.

**Notation:** We will use complex coordinates \( z = x - iy \) on \( \mathbb{R}^2 \). An element of \( \mathcal{C}_N \) will now be denoted by complex numbers \( \{z_1, \ldots, z_N\} \).

The complex coordinates introduced above allow for a local\(^7\) identification of \( \mathcal{C}_N \) and \( \tilde{\mathcal{C}}_N \) with \( \mathbb{C}^N \), thereby making \( \mathcal{C}_N \) and \( \tilde{\mathcal{C}}_N \) complex manifolds of complex dimension \( N \). Hence, we know what it means for a function \( \psi : \mathcal{C}_N \to \mathbb{C} \) to be holomorphic.

Because we require that \( \psi \) picks up a phase when two anyons are exchanged, the dependence of \( \psi \) on the paths \( \{\gamma_1, \ldots, \gamma_N\} \) should appear only as a phase factor. We also require that \( \psi \) depends

\(^7\)Dividing by a discrete group (like \( S_N \)) does not change the local properties of a space.
differentiably on its argument. A way to achieve this is by using the paths \( \{ \gamma_1, \ldots, \gamma_N \} \) to define angles \( \theta_{jk} \in \mathbb{R} \) for each pair of particles\(^8\), as shown in Figure 6.2, and then taking these angles as the argument of some periodic function.

Using the angles \( \theta_{jk} \) introduced above, we are now able to construct holomorphic functions on \( \tilde{C}_N \) by considering products of the form \( |z_j - z_k|^\alpha e^{i\alpha \theta_{jk}} \), where \( j, k \in \{1, \ldots, N\} \), which we formally write as

\[
(z_j - z_k)^\alpha := |z_j - z_k|^\alpha e^{i\alpha \theta_{jk}}.
\]

(6.6)

The notation above makes sense because we can locally express the right hand side of (6.6) in terms of a holomorphic branch of the complex logarithm,

\[
(z_j - z_k)^\alpha = \exp \left[ \alpha \left( \log |z_j - z_k| + i \theta_{jk} \right) \right] = \exp \left[ \alpha \log_C(z_j - z_k) \right],
\]

(6.7)

where the choice of the branch of the complex logarithm is determined by the value of \( \theta_{jk} \). This way of writing \((z_j - z_k)^\alpha\) shows that \((z_j - z_k)^\alpha\) is actually holomorphic on \( \tilde{C}_N \).

**Remark 6.7.** We will use the notation \((z_i - z_j)^\alpha\) for \(|z_j - z_k|^\alpha e^{i\alpha \theta_{jk}}\) throughout the rest of this chapter, but it must be kept in mind that \((z_i - z_j)^\alpha\) is a function that not only depends on \(z_j\) and \(z_k\), but also on the angle \(\theta_{jk}\).

**Remark 6.8.** The fact that (6.6) behaves as required under the exchange of the \(j\)th and the \(k\)th particle can be verified directly: e.g. for \((j,k) = (1,2)\), applying either \(b_{12}\) or \(b_{12}^{-1}\) increases \(\theta_{12}\) by \(-\pi\) or \(\pi\) respectively, which results in an extra factor of \(e^{-i\alpha \pi}\) or \(e^{i\alpha \pi}\) in (6.6). Note that the factor \(|z_1 - z_2|^\alpha\) does not give any extra signs because of the absolute value.

Differentiating \((z_i - z_j)^\alpha\) with respect to \(z_i\) yields

\[
\frac{\partial}{\partial z_i} (z_i - z_j)^\alpha = \frac{\partial}{\partial z_i} \exp \left[ \alpha \log_C(z_j - z_k) \right] = \frac{\alpha}{z_i - z_j} \exp \left[ \alpha \log_C(z_j - z_k) \right]
\]

\[
= \frac{\alpha}{(z_i - z_j)} (z_i - z_j)^\alpha.
\]

(6.8)

To summarize, we have constructed the configuration space for identical particles, and we have seen that, due to difference in the topology\(^9\) between the two and the more-dimensional configuration spaces, the phase picked up by the multi-particle wavefunction under the exchange of two particles can be any complex phase in two dimensions, whereas it has to be \(\pm 1\) in three or more dimensions.

\(^8\)More formally, we are actually constructing functions on the abelian cover of \(C_N\).

\(^9\)More precisely, the fundamental group.
6.2. CONSTRUCTION OF THE ANYON EIGENFUNCTIONS

We then focused on the situation of two dimensions, and we argued that, in order to properly describe multi-particle wavefunctions — which are in general of anyonic nature — we have to consider functions defined on the universal cover of the configuration space. We then constructed statistical terms of the form \((z_i - z_j)^\alpha\), which are holomorphic and have the required exchange properties. However, in order to construct a physically meaningful wavefunction, we will have to involve the \(N\)-particle Hamiltonian.

This concludes the mathematical setup for anyons. In the next section, we will proceed with the physically relevant model of non-interacting charged anyons in a uniform perpendicular magnetic field, and we will attempt to construct eigenfunctions of the corresponding Hamiltonian.

6.2 Construction of the Anyon Eigenfunctions

In this section, we will construct \(N\)-particle anyonic wavefunctions describing non-interacting charged anyons in a uniform perpendicular magnetic field. Our approach is based on a general ansatz inspired by our physical knowledge of the one-particle case, the wavefunctions of which we discussed in Section 3.2.1.

Before we start with the construction of the anyon eigenfunctions, there is one important issue that want to touch upon briefly. Anyons arise in the FQHE as effective quasi-particles due to the strong magnetic field that the actual particles — i.e. the electrons — feel. It is not completely agreed upon that the anyons, being quasi-particle excitations of the FQHE fluid, feel the external magnetic field that causes the FQHE themselves, because, in principle, collective excitations can behave completely differently from their constituent particles. However, in the case of the quasi-particle excitations of the FQHE fluid, it is known that the excitations carry charge, which should interact with the external magnetic field. Moreover, the charged particle-flux tube composite description of anyons is based on the interaction between a (localized) magnetic field on one anyon and the charge of another. This suggests that the FQHE anyons should indeed also feel the external magnetic field that causes the FQHE.

To further support our claim that anyons do feel the external magnetic field, here are three papers or books that claim the same: page 58 of Khare [13], page 17 of Stern [22] and the introduction of Johnson and Canright [12]. Additionally, Lerda (page 60 of [17]) claims that the quasi-particles and quasi-holes in the FQHE feel the underlying electrons exactly as a charged particle would feel an external uniform magnetic field.

The author is of the opinion that the arguments presented above give enough motivation to study the system of non-interacting charged anyons in a uniform perpendicular magnetic field. Besides, irrespective of whether anyons feel a magnetic field or not, the Hamiltonian that we will consider is also applicable to free anyons that rotate around the \(z\)-axis with constant angular frequency \(\Omega\), as will be explained in Remark 6.9.

6.2.1 The Wavefunction Ansatz

We now turn to an actual physical system of non-interacting charged anyons in a perpendicular magnetic field \(B = Be_z\). We take our anyons to have mass \(m\), and charge \(-e^*\). In the case of the FQHE at filling factor \(\nu\), we know that \(e^* = e\nu\). However, our description of anyons is in principle not restricted to the FQHE, so we will leave the charge \(-e^*\) as a free parameter.

We want to find exact anyonic eigenfunctions \(\psi\) of the \(N\)-particle Hamiltonian

\[
H_N = \sum_{i=1}^{N} \frac{1}{2m} (\Pi_{x_i}^2 + \Pi_{y_i}^2),
\]

where \(r_i = (x_i, y_i) \in \mathbb{R}^2\) denotes the position of the \(i\)-th anyon, and

\[
\Pi_i = -i\hbar\nabla_i + e^*A_i.
\]
As usual, we will be working in the symmetric gauge (2.41),

\[ A = \frac{B}{2}(-y e_x + x e_y) . \]

**Remark 6.9.** The Hamiltonian \( H_N \) in (6.9) in the symmetric gauge is not only applicable to charged anyons in a magnetic field; it is more general, because it also describes free anyons that are set spinning with angular frequency \( \Omega \) around the z-axis. Indeed, the coordinate transformation from a stationary to a rotating reference frame amounts to replacing the momentum \( p \) by \( p + r \times \Omega \), where \( \Omega := \Omega e_z \), which is mathematically equivalent to the minimal substitution \( p \mapsto p + e A \) in the symmetric gauge.

Going to complex coordinates \( z = x - iy, \quad \bar{z} = x + iy \), \( \partial := \partial_w = 2 l_B^* \partial_z \), \( \bar{\partial} := \partial_{\bar{w}} = 2 l_B^* \partial_{\bar{z}} \).

The \( \theta_{jk} \) variables remain invariant under the rescaling given by (6.11).

In terms of the newly defined variables, the Hamiltonian (6.10) becomes

\[
H_N = \frac{\hbar \omega_C^*}{2} \sum_{i=1}^{N} \left[ \frac{1}{4(l_B^*)^2} \bar{z}_i \bar{z}_i - z_i \partial_{z_i} + z_i \partial_{\bar{z}_i} - 4(l_B^*)^2 \partial_{z_i} \partial_{\bar{z}_i} \right] .
\]

Recall that the magnetic length \( l_B^* \) and the cyclotron frequency \( \omega_C^* \) are defined by \( l_B^* := \sqrt{\hbar/e B} \) and \( \omega_C^* := e^* B/m \).\(^{10} \)

Before we tackle the problem at hand, let us change to dimensionless variables to get rid of the magnetic length in the Hamiltonian.

**Definition 6.6.** A more computational friendly set of variables to work with is given by

\[
w := \frac{z}{2 l_B^*} , \quad \bar{w} := \frac{\bar{z}}{2 l_B^*} , \quad \partial := \partial_w = 2 l_B^* \partial_z , \quad \bar{\partial} := \partial_{\bar{w}} = 2 l_B^* \partial_{\bar{z}} .
\]

The \( \theta_{jk} \) variables remain invariant under the rescaling given by (6.11).

In terms of the newly defined variables, the Hamiltonian (6.10) becomes

\[
H_N = \frac{\hbar \omega_C^*}{2} \sum_{i=1}^{N} \left[ w_i \bar{w}_i - w_i \partial_{z_i} + \bar{w}_i \partial_{\bar{z}_i} - \partial_{z_i} \partial_{\bar{z}_i} \right] ,
\]

The approach that we take is the following. In order to construct eigenfunctions of the Hamiltonian, we will use an smart ansatz based on the following ingredients: (a) the one-particle wavefunctions for charged particles in a magnetic field\(^{11} \), and (b) the statistical terms introduced in Section 6.1.5.

Let us focus on (b) first. We require that \( \psi \) contains a statistical term that accounts for the correct exchange properties. The function that does the job is the **statistical prefactor**, given in terms of the \( w \)-coordinates by

\[
S(w_1, \ldots, w_N, \theta_{12}, \theta_{13}, \ldots, \theta_{N-1,N}) := \prod_{k>j=1}^{N} (w_j - w_k)^\alpha .
\]

This is the only term in the wavefunction that will depend on the angles \( \theta_{jk} \), the remaining terms will depend on the position coordinates alone. Before we proceed, let us check that this prefactor indeed has the correct exchange properties.

\(^{10}\)The star in \( l_B^* \) and \( \omega_C^* \) reminds us of the fact that the charge of the anyons is \(-e^*\). It is not related to star we use for composite fermions, which refers to a weakened magnetic field \( B^* \).

\(^{11}\)Note that the one-particle anyon wavefunctions are the same as the one-particle electron wavefunctions, because there are no exchange statistics in play when there is only one particle present.
Lemma 6.1. For any element $B \in B_N$, the statistical prefactor $S$ satisfies equation (6.4): that is, for any $\xi \in \tilde{C}_N$,

$$S(b \cdot \xi) = \rho(b)S(\xi).$$

Proof. Because $\rho : B_N \to \mathbb{C}$ is a homomorphism, we can restrict ourselves to the case that $b \in B_N$ is a generator $b_{j,j+1}^{-1}$ of the braid group. We will focus on the case of an anti-clockwise exchange, so $\rho(b_{j,j+1}^{-1}) = e^{i\alpha \pi}$. Since the endpoints remain invariant under an exchange process, we only have to focus on the angular part of (6.13), given by,

$$\prod_{k>j=1}^N \exp (i\alpha \theta_{jk}) = \exp \left( i\alpha \sum_{k>j=1}^N \theta_{jk} \right).$$

Applying $b_{j,j+1}^{-1}$ to all the different angles separately gives the following transformations,

$$b_{j,j+1}^{-1}\theta_{j,j+1} = \theta_{j,j+1} + \pi$$

$$b_{j,j+1}^{-1}\theta_{k,j} = \theta_{k,j+1}$$

$$b_{j,j+1}^{-1}\theta_{k,j+1} = \theta_{k,j}$$

$$b_{j,j+1}^{-1}\theta_{j+1,l} = \theta_{j,l}$$

$$b_{j,j+1}^{-1}\theta_{j+1,l} = \theta_{j+1,l}$$

$$b_{j,j+1}^{-1}\theta_{k,l} = \theta_{k,l}$$

where $k < l$ such that $k, l \notin \{j, j+1\}$. Hence, we see that the net effect of $b_{j,j+1}^{-1}$ on the sum $\sum_{k>j=1}^N \theta_{jk}$ is that we get an extra factor of $\pi$, resulting in an extra overall phase factor of $e^{i\alpha \pi}$. For clockwise exchanges $b_{j,j+1}$, we get a $-\pi$ added to the sum instead of a $+\pi$, which results in an extra overall phase factor of $e^{-i\alpha \pi}$.

Next, we take (a) into account. In addition to the statistical prefactor, we will include the ubiquitous exponential factor that occurs in all fermionic and bosonic $N$-particle wavefunctions for charged particles in a uniform perpendicular magnetic field,

$$E(z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N) := \exp \left[ -\sum_{i=1}^N \frac{|z_i|^2}{4(l_B^*)^2} \right],$$

which in terms of the $w$-coordinates takes the form

$$E(w_1, \ldots, w_N, \bar{w}_1, \ldots, \bar{w}_N) := \exp \left[ -\sum_{i=1}^N w_i \bar{w}_i \right],$$

into our ansatz. Including this extra factor will have the added benefit of making all obtained wavefunctions rapidly decreasing at infinity.

Notation: We will abbreviate the vectors $(w_1, \ldots, w_N)$, $(\bar{w}_1, \ldots, \bar{w}_N)$ and $(\theta_{12}, \theta_{13}, \ldots, \theta_{N-1,N})$ with boldface characters $w$, $\bar{w}$ and $\theta$ respectively.

Ansatz. Taking everything together, we arrive at the following ansatz for the $N$-particle anyonic eigenfunction of $H_N$,

$$\psi(w, \bar{w}, \theta) = S(w, \theta) E(w, \bar{w}) F(w, \bar{w}),$$

where $F$ is a yet undetermined function of $w$ and $\bar{w}$.
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**Remark 6.10.** We have constructed $S$ such that it transforms as an anyon wavefunction should under particle exchange. Hence, to ensure that $\psi$ transforms correctly, we require that the remaining term $FE$ is symmetric. By symmetry of $E$, this means that $F$ has to be symmetric with respect to pairwise exchange of variables $(w_i, \bar{w}_i) \leftrightarrow (w_j, \bar{w}_j)$.

In short, based on physical intuition and the statistical properties that we want our wavefunction to satisfy, we have come up with an ansatz of the form (6.14) for a general $N$-particle anyonic energy eigenfunction. To see if our ansatz makes any sense at all, in the next section, we will try to solve the time-independent Schrödinger equation with it.

### 6.2.2 The Time Independent Schrödinger Equation

Finally, we come to the time-independent Schrödinger equation. Motivated by our physical intuition that anyons in a magnetic field will form Landau levels just like ordinary charged particles do, we expect that (at least some of) the $N$-particle energy eigenvalues will be of the form $E_{N,K} = \frac{1}{2} \hbar \omega_C^*(N + 2K)$, where $K \in \mathbb{N}$ is a natural number that signifies the amount of energy packets $\hbar \omega_C^*$ the system has on top of its ground state energy $\frac{1}{2} N \hbar \omega_C^*$. Henceforth, we will refer to $K$ as the excitation energy.

The time-independent Schrödinger equation for $\psi(w, \bar{w}, \theta)$ with energy eigenvalue $E = E_{N,K}$ reads

$$H_N \psi(w, \bar{w}, \theta) = \frac{1}{2} \hbar \omega_C^*(N + 2K) \psi(w, \bar{w}, \theta),$$

(6.15)

where $H_N$ is given by (6.12).

**Lemma 6.2.** Plugging the ansatz (6.14) into Schrödinger equation (6.15) yields the following equation for $F$:

$$\sum_{i=1}^{N} \left( \bar{w}_i - \frac{1}{2} \left( \partial_i + \sum_{j=1, j \neq i}^{N} \frac{\alpha}{w_i - w_j} \right) \right) \bar{\partial}_i F(w, \bar{w}) = K F(w, \bar{w}).$$

(6.16)

**Proof.** We expand the left hand side of (6.15) by applying $H$ in (6.12) to the ansatz (6.14).

$$H_N \psi(w, \bar{w}, \theta) = \frac{\hbar \omega_C^*}{2} \sum_{i=1}^{N} \left[ w_i \bar{w}_i - w_i \partial_i + \bar{w}_i \partial_i - \partial_i \partial_i \right] S(w, \theta) E(w, \bar{w}) F(w, \bar{w})$$

$$= \frac{\hbar \omega_C^*}{2} \sum_{i=1}^{N} \left\{ \begin{array}{l} S(w, \theta) w_i \bar{w}_i E(w, \bar{w}) F(w, \bar{w}) - w_i \partial_i \left[ S(w, \theta) \right] E(w, \bar{w}) F(w, \bar{w}) \\ - S(w, \theta) \bar{w}_i \partial_i \left[ E(w, \bar{w}) \right] F(w, \bar{w}) - S(w, \theta) E(w, \bar{w}) \partial_i \left[ F(w, \bar{w}) \right] \\ + S(w, \theta) \bar{w}_i \partial_i \left[ E(w, \bar{w}) \right] F(w, \bar{w}) + S(w, \theta) E(w, \bar{w}) \partial_i \left[ F(w, \bar{w}) \right] \\ - \partial_i \left[ S(w, \theta) \right] \bar{\partial}_i \left[ E(w, \bar{w}) \right] F(w, \bar{w}) - \partial_i \left[ S(w, \theta) \right] E(w, \bar{w}) \bar{\partial}_i \left[ F(w, \bar{w}) \right] \\ - S(w, \theta) \bar{\partial}_i \left[ E(w, \bar{w}) \right] \bar{\partial}_i \left[ F(w, \bar{w}) \right] - S(w, \theta) E(w, \bar{w}) \partial_i \bar{\partial}_i \left[ F(w, \bar{w}) \right] \\ - S(w, \theta) \partial_i \bar{\partial}_i \left[ E(w, \bar{w}) \right] \bar{\partial}_i \left[ F(w, \bar{w}) \right] - S(w, \theta) E(w, \bar{w}) \partial_i \bar{\partial}_i \left[ F(w, \bar{w}) \right] \end{array} \right\}.$$  

The above expression can be simplified significantly. First of all, by taking the first terms of the first, second, third and fifth line together, we recognize the term $S(w, \theta) F(w, \bar{w}) H_N E(w, \bar{w})$. Because $E(w, \bar{w})$ is the unnormalized wavefunction describing $N$ particles in the lowest Landau level, it has an energy eigenvalue of $\frac{1}{2} N \hbar \omega_C^*$. Hence, this term cancels with $\frac{1}{2} N \hbar \omega_C^* \psi(w, \bar{w}, \theta)$ on the right hand side of (6.15).
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Furthermore, using \( \partial_i E(w, \bar{w}) = -w_i E(w, \bar{w}) \), we see that the second terms on the first and second line cancel with the first terms on the fourth and sixth line. Thus, equation (6.15) becomes

\[
\frac{\hbar \omega^*}{2} C^2 N \sum_{i=1}^N \left\{ S(w, \theta) E(w, \bar{w}) \bar{w}_i \partial_i \left[ F(w, \bar{w}) \right] - \partial_i \left[ S(w, \theta) \right] E(w, \bar{w}) \partial_i \left[ F(w, \bar{w}) \right] \right. \\
- S(w, \theta) \partial_i \left[ E(w, \bar{w}) \right] \partial_i \left[ F(w, \bar{w}) \right] - S(w, \theta) E(w, \bar{w}) \partial_i \partial_i \left[ F(w, \bar{w}) \right] \left. \right\} \\
= \hbar \omega^* KS(w, \theta) E(w, \bar{w}) F(w, \bar{w}). 
\]

(6.17)

Now, observe that \( \partial_i E(w, \bar{w}) = -\bar{w}_i E(w, \bar{w}) \). Also, by equation (6.8),

\[
\partial_i S(w, \theta) = \left( \sum_{j=1, j \neq i}^N - \frac{\alpha}{w_i - w_j} \right) S(w, \theta). 
\]

Hence, (6.17) can be rewritten as

\[
\frac{\hbar \omega^*}{2} S(w, \theta) E(w, \bar{w}) \sum_{i=1}^N \left\{ 2\bar{w}_i - \sum_{j=1, j \neq i}^N \frac{\alpha}{w_i - w_j} - \partial_i \right\} \partial_i F(w, \bar{w}) \\
= \hbar \omega^* KS(w, \theta) E(w, \bar{w}) F(w, \bar{w}), 
\]

which is indeed equivalent to (6.16).

In short, we have been able to rewrite the time-independent Schrödinger equation (6.15) applied to the ansatz (6.14) to equation (6.16) for the unknown function \( F(w, \bar{w}) \). In the next section we will solve equation (6.16) for \( F \).

6.2.3 A Polynomial Ansatz for the Wavefunction

We will now attempt to solve equation (6.16) obtained in the previous section for the unknown function \( F(w, \bar{w}) \). Before we do so, let us mention two things. First, observe that there is a hidden structure to be found in equation (6.16).

Definition 6.7. Define the operators \( \tau_i \) by

\[
\tau_i := -\frac{1}{2} \left( \partial_i + \sum_{j=1, j \neq i}^N - \frac{\alpha}{w_i - w_j} \right). 
\]

(6.18)

Lemma 6.3. The operators \( \tau_i \) commute

Proof. Follows straight from the definition.

Corollary 6.1. Using the newly defined operators \( \tau_i \), equation (6.16) simplifies to:

\[
\sum_{i=1}^N \left[ \bar{w}_i + \tau_i \right] \partial_i F(w, \bar{w}) = K F(w, \bar{w}). 
\]

(6.19)

Remark 6.11. The strength in the above simplification lies not only in the notation. Because the operators \( \tau_i \) commute with each other, and also with \( \bar{w}_i \) and \( \partial_i \), we can regard the \( \tau_i \)'s as numbers when working with equation (6.19) as an equation in \( \bar{w} \).
Second, using our physical intuition, we will restrict the form of $F$. The one-particle wavefunctions for anyons should coincide with the one-particle wavefunctions for electrons, because the difference between anyons and fermions only becomes apparent when there is more than one particle present. Recalling equation (3.14) and the formulas for the ladder operators $a, a^\dagger, b$ and $b^\dagger$ in the symmetric gauge given in Section 3.2.1, the one-particle wavefunctions for electrons in the symmetric gauge are of the form

$$\psi_{\text{electron}}(z, \bar{z}) = P(z, \bar{z}) \exp \left(-\frac{|z|^2}{4\hbar}\right),$$

where $P(z, \bar{z})$ is a polynomial in $z$ and $\bar{z}$ such that:

1. the highest power of $z$ in $P(z, \bar{z})$ corresponds to the guiding centre coordinate of the electron in state $\psi_{\text{electron}}$, and,
2. the highest power of $\bar{z}$ in $P(z, \bar{z})$ corresponds to the energy of the electron in state $\psi_{\text{electron}}$.

Hence, we expect that $F$ in the ansatz (6.14) is also polynomial in $w_1, \ldots, w_N$ and $\bar{w}_1, \ldots, \bar{w}_N$. Moreover, recall that we are trying to solve the time-independent Schrödinger equation (6.15)

$$H_N \psi(w, \bar{w}, \theta) = \frac{1}{2} \hbar \omega_C^* (N + 2K) \psi(w, \bar{w}, \theta),$$

where $K$ is the number of energy packets that separates the anyonic $N$-particle state $\psi$ from the ground state. Now, because of comment (2) above, the highest power of $\bar{w}_i$ in $F$ should correspond to the energy of the $i$th anyon. Therefore, we expect that the sum of the exponents of the variables $\bar{w}_1, \ldots, \bar{w}_N$ in every term of $F$ is at most $K$.

The above arguments naturally lead to the following formula for $F$:

$$F(w, \bar{w}) = \sum_{0 \leq i_1 + \ldots + i_N \leq K} c_{i_1, \ldots, i_N}(w) \bar{w}_1^{i_1} \cdots \bar{w}_N^{i_N}, \quad (6.20)$$

where the $c_{i_1, \ldots, i_N}(w)$ are unknown polynomials in $w_1, \ldots, w_N$.

**Remark 6.12.** As noted by Ouvry [20], taking $F(w, \bar{w}) = F(w)$ to be a holomorphic polynomial yields anyonic $N$-particle eigenfunctions of the Hamiltonian $H_N$ with eigenvalue $\frac{1}{2} \hbar \omega_C^*$, that is, $N$-particle lowest Landau level anyon eigenfunctions. This can easily be checked by applying $H_N$ to $\psi = S E F$ with $F = F(w)$. However, we will see in the next section that the challenge lies in finding wavefunctions for the higher Landau levels.

It turns out that, when we restrict ourselves to a general ansatz of the form $\psi = S E F$, where $F$ is a generic polynomial in $w$ and $\bar{w}$, equation (6.20) is the only formula for $F$ that will yield any solutions to the Schrödinger equation (6.19). Before we show this, here is a definition that will come in handy.

**Definition 6.8.** For each coefficient $c_{i_1, \ldots, i_N}(w)$, we define the order of $c_{i_1, \ldots, i_N}(w)$ to be $i_1 + \ldots + i_N$. Additionally, we introduce the function $\text{Ord}$, defined by

$$\text{Ord}[c_{i_1, \ldots, i_N}(w)] := (i_1 + \ldots + i_N)c_{i_1, \ldots, i_N}(w).$$

**Lemma 6.4.** The only solutions to the Schrödinger equation (6.19) for an arbitrary polynomial function $F$ in the variables $w$ and $\bar{w}$ are those given by formula (6.20), where the only non-zero coefficients $c_{i_1, \ldots, i_N}$ are those for which $i_1 + \ldots + i_N \leq K$.

**Proof.** Assume that $F$ is an arbitrary polynomial in $w$ and $\bar{w}$. Reorder $F$ such that it is a polynomial in $\bar{w}$ with coefficients that depend on $w$ — as in (6.20) — and let $D$ be the degree of $F$ when considered as a polynomial in $\bar{w}$. We will prove that $D = K$. 

First, observe what happens when we apply \((\bar{w}_k + \tau_k)\partial_k\) to a arbitrary term \(c_{i_1, \ldots, i_N}(w)\bar{w}^{i_1}_{1} \cdots \bar{w}^{i_N}_{N}\).

\[
(\bar{w}_k + \tau_k)\partial_k c_{i_1, \ldots, i_N}(w)\bar{w}^{i_1}_{1} \cdots \bar{w}^{i_N}_{N} = i_k c_{i_1, \ldots, i_N}(w)\bar{w}^{i_1}_{1} \cdots \bar{w}^{i_N}_{N} + i_k \tau_k c_{i_1, \ldots, i_N}(w)\bar{w}^{i_1}_{1} \cdots \bar{w}^{i_k-1}_{k} \cdots \bar{w}^{i_N}_{N}.
\]

Hence, the operator \(\sum_{k=1}^{N} [\bar{w}_k + \tau_k] \partial_k\) contains parts \(\bar{w}_k \partial_k\) that preserves the order, and parts \(\tau_k \partial_k\) that lower the order.

Next, we focus on the highest order terms. Because of the above, the only terms on the LHS of the Schrödinger equation (6.19) that contain terms of order \(D\) are the terms of the form \(\bar{w}_k \partial_k c_{i_1, \ldots, i_N}(w)\bar{w}^{i_1}_{1} \cdots \bar{w}^{i_N}_{N}\) with \(i_1 + \ldots + i_N = D\). In particular, for each fixed highest order term \(c_{i_1, \ldots, i_N}(w)\bar{w}^{i_1}_{1} \cdots \bar{w}^{i_N}_{N}\), we have,

\[
\sum_{k=1}^{N} [\bar{w}_k + \tau_k] \partial_k c_{i_1, \ldots, i_N}(w)\bar{w}^{i_1}_{1} \cdots \bar{w}^{i_N}_{N} = \sum_{k=1}^{N} i_k c_{i_1, \ldots, i_N}(w)\bar{w}^{i_1}_{1} \cdots \bar{w}^{i_k-1}_{k} \cdots \bar{w}^{i_N}_{N} + \text{lower order terms}.
\]

Comparing to the RHS of the Schrödinger equation, where each term is multiplied by \(\frac{\hbar}{\omega_L^2} (N + 2K)\) that we can hope to find are those for which \(K\) is a non-negative integer.

Let us summarize where we stand. We wrote the ansatz eigenfunction \(\psi = SEF\), with \(F\) a polynomial in \(w\) and \(\bar{w}\), are concerned, the only possible energy eigenvalues \(\frac{\hbar}{\omega_L^2} N + 2K\) that we can hope to find are those for which \(K\) is a non-negative integer.

**Corollary 6.2.** For as far as eigenfunctions of the form \(\psi = SEF\), with \(F\) a polynomial in \(w\) and \(\bar{w}\), are concerned, the only possible energy eigenvalues \(\frac{\hbar}{\omega_L^2} (N + 2K)\) that we can hope to find are those for which \(K\) is a non-negative integer.

Let us summarize where we stand. We wrote the ansatz eigenfunction \(\psi = SEF\), where \(F\) is given by formula (6.20), with yet to be determined coefficients \(c_{i_1, \ldots, i_N}(w)\).

**Lemma 6.5.** Plugging (6.20) for \(F\) into the Schrödinger equation (6.19) yields the following recurrence relation for the coefficients \(c_{i_1, \ldots, i_N}(w)\) of \(F\),

\[
(K - \text{Ord}) c_{i_1, \ldots, i_N}(w) = \sum_{k=1}^{N} (i_k + 1) \tau_k c_{i_1, \ldots, i_{k-1}, i_k + 1, i_k + 1, \ldots, i_N}(w) \quad \text{for} \quad 0 \leq i_1 + \ldots + i_N \leq K. \tag{6.21}
\]

That is, the coefficient \(c_{i_1, \ldots, i_N}\) on the LHS is equal to a weighted sum over all coefficients obtained by increasing one of the indices of \(c_{i_1, \ldots, i_N}\).

**Remark 6.13.** Before we prove this lemma, let us make the following important remark: the above recurrence relation has plenty of solutions! As a matter of fact, the highest order coefficients, i.e. the \(c_{i_1, \ldots, i_N}\)'s with order \(K\), can be chosen freely, and then (6.21) gives the expressions for the lower order coefficients in terms of the highest order coefficients.

**Proof.** We have

\[
\sum_{k=1}^{N} [\bar{w}_k + \tau_k] \partial_k F(w, \bar{w}) = \sum_{k=1}^{N} [\bar{w}_k + \tau_k] \partial_k \left( \sum_{0 \leq i_1 + \ldots + i_N \leq K} c_{i_1, \ldots, i_N}(w)\bar{w}^{i_1}_{1} \cdots \bar{w}^{i_N}_{N} \right)
\]

\[= \sum_{k=1}^{N} \sum_{0 \leq i_1 + \ldots + i_N \leq K} i_k c_{i_1, \ldots, i_N}(w)\bar{w}^{i_1}_{1} \cdots \bar{w}^{i_N}_{N} \tag{A}\]

\[+ \sum_{k=1}^{N} \sum_{0 \leq i_1 + \ldots + i_N \leq K} i_k \tau_k c_{i_1, \ldots, i_N}(w)\bar{w}^{i_1}_{1} \cdots \bar{w}^{i_k-1}_{k} \cdots \bar{w}^{i_N}_{N} \tag{B}\]
For \((A)\), we have, after interchanging the sums,
\[
(A) = \sum_{0 \leq i_1 + \ldots + i_N \leq K} \text{Ord}[c_{i_1},\ldots,i_N(w)]\bar{w}_1^{i_1} \ldots \bar{w}_N^{i_N}.
\]

Regarding \((B)\), for fixed index \(k\), we have,
\[
\sum_{0 \leq i_1 + \ldots + i_N \leq K} i_k \tau_k c_{i_1,\ldots,i_N}(w)\bar{w}_1^{i_1} \ldots \bar{w}_k^{i_k-1} \ldots \bar{w}_N^{i_N}
\]
\[
= \sum_{i_k=1}^{K} \tilde{\sum}_{0 \leq i_1 + \ldots + i_{k-1} + i_{k+1} + \ldots + i_N \leq K-i_k} i_k \tau_k c_{i_1,\ldots,i_N}(w)\bar{w}_1^{i_1} \ldots \bar{w}_k^{i_k-1} \ldots \bar{w}_N^{i_N}
\]
\[
= \sum_{i_k=0}^{K-1} \tilde{\sum}_{0 \leq i_1 + \ldots + i_{k-1} + i_{k+1} + \ldots + i_N \leq K-1-i_k} (i_k + 1) \tau_k c_{i_1,\ldots,i_k+1,\ldots,i_N}(w)\bar{w}_1^{i_1} \ldots \bar{w}_N^{i_N}
\]

where \(\tilde{\sum}\) means that we are summing over all indices but \(i_k\). Summing over all \(k\) yields
\[
(B) = \sum_{0 \leq i_1 + \ldots + i_N \leq K-1} \left[ \sum_{k=1}^{i_N} (i_k + 1) \tau_k c_{i_1,\ldots,i_k+1,\ldots,i_N}(w) \right] \bar{w}_1^{i_1} \ldots \bar{w}_N^{i_N}.
\]

Taking everything together gives
\[
\sum_{k=1}^{N} \left[ \bar{w}_k + \tau_k \right] \partial_k F(w, \bar{w})
\]
\[
= \sum_{i_1 + \ldots + i_N = K} K c_{i_1,\ldots,i_N}(w)\bar{w}_1^{i_1} \ldots \bar{w}_N^{i_N}
\]
\[
+ \sum_{0 \leq i_1 + \ldots + i_N \leq K-1} \left[ \text{Ord}[c_{i_1,\ldots,i_N}(w)] + \sum_{k=1}^{N} (i_k + 1) \tau_k c_{i_1,\ldots,i_k+1,\ldots,i_N}(w) \right] \bar{w}_1^{i_1} \ldots \bar{w}_N^{i_N}
\]

For the Schrödinger equation (6.19) to hold, the above has to be equal to
\[
KF(w, \bar{w}) = \sum_{0 \leq i_1 + \ldots + i_N \leq K} K c_{i_1,\ldots,i_N}(w)\bar{w}_1^{i_1} \ldots \bar{w}_N^{i_N}.
\]

By comparing the coefficients of \(\bar{w}_1^{i_1} \ldots \bar{w}_N^{i_N}\) on both side of the Schrödinger equation, we see that the highest order coefficients are free to choose, and the lower order coefficients must satisfy (6.21).

To summarize, we plugged the ansatz \(\psi = S E F\) into the Schrödinger equation, and we got a recurrence formula (6.21) for the coefficients of \(F\) seen as a polynomial in \(\bar{w}\). In the next section, we will try to find solutions to the recurrence relation (6.21).

### 6.2.4 Solutions to the Schrödinger Equation

We are looking for anyonic eigenfunctions of the Hamiltonian for \(N\) non-interacting charged anyons in a uniform perpendicular magnetic field, which, in terms of the dimensionless complex coordinates \(w\) and \(\bar{w}\), is given by (6.12), that is
\[
H_N = \frac{\hbar \omega_c^*}{2} \sum_{i=1}^{N} \left[ w_i \bar{w}_i - w_i \partial_i + \bar{w}_i \bar{\partial}_i - \partial_i \bar{\partial}_i \right].
\]
Using the ansatz \( \psi = SEF \), where \( F \) is a polynomial in \( w \) and \( \bar{w} \), we have come to the conclusion that, for this particular ansatz, the eigenvalues of \( H_N \) are given by \( E_{N,K} = \frac{1}{2} \hbar \omega^*_w (N + 2K) \), where \( K \) is a non-negative integer, and the to-be-determined polynomial \( F \) is of the form (6.20), i.e.

\[
F(w, \bar{w}) = \sum_{0 \leq i_1 + \ldots + i_N \leq K} c_{i_1, \ldots, i_N}(w) \bar{w}_1^{i_1} \cdots \bar{w}_N^{i_N}.
\]

As shown in Lemma 6.5, the Schrödinger equation for \( \psi \),

\[
H_N \psi = \frac{1}{2} \hbar \omega^*_w (N + 2K) \psi,
\]

becomes a recurrence relation for the unknown coefficients \( c_{i_1, \ldots, i_N}(w) \) of \( F \),

\[
(K - \text{Ord}) c_{i_1, \ldots, i_N}(w) = \sum_{k=1}^{N} (i_k + 1) \tau_k c_{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_N}(w) \quad \text{for} \quad 0 \leq i_1 + \ldots + i_N \leq K. \tag{6.22}
\]

Additionally, by Remark 6.10, we require that \( F \) is symmetric with respect to pairwise exchange \((w_i, \bar{w}_i) \leftrightarrow (w_j, \bar{w}_j)\) of anyon position coordinates. We will discuss the symmetry conditions on \( F \) in the next section.

In this section, we will provide solutions to the above recurrence relation. To solve (6.22) directly seems quite difficult. Instead, we have opted for a more brute force approach. For different particle numbers \( N \) and excitation energies \( K \), using (6.22), we computed the lower order coefficients in terms of the highest order coefficients, and looked for a pattern. The details of these computations can be found in Appendix B.1. The pattern that we found is the following.

**Solution to the recurrence relation (6.22).**

1. Highest order coefficients \( c_{i_1, \ldots, i_N} \) with \( i_1 + \ldots + i_N = K \) free to choose.
2. Lower order coefficients \( c_{i'_1, \ldots, i'_N} \) with \( 0 \leq i'_1 + \ldots + i'_N < K \) are given by

\[
c_{i'_1, \ldots, i'_N} = \sum_{i_1 + \ldots + i_N = K} \tau_1^{i_1} \cdots \tau_N^{i_N} c_{i_1, \ldots, i_N} \tag{6.23}
\]

In equation (6.23), we use the convention that \( \binom{i}{i'} = 0 \) whenever \( i < i' \).

**Remark 6.14.** The above solution might seem a little intimidating. However, taking a closer look at equation (6.23) reveals that it is actually quite an elegant equation. As a matter of fact, (6.23) is equivalent\(^{12}\) to the following set of equations that express the lower order coefficients \( c_{i'_1, \ldots, i'_N} \) in terms of the highest order coefficients \( c_{i_1, \ldots, i_N} \),

\[
c_{0,0,\ldots,0} = \sum_{i_1 + \ldots + i_N = K} \tau_1^{i_1} \cdots \tau_N^{i_N} c_{i_1, \ldots, i_N}, \tag{6.24}
\]

\[
c_{i'_1,\ldots,i'_N} = \frac{1}{i'_1! \cdots i'_N!} \left[ \left( \frac{\partial}{\partial \tau_1} \right)^{i'_1} \cdots \left( \frac{\partial}{\partial \tau_N} \right)^{i'_N} \right] c_{0,0,\ldots,0} \quad \text{for} \quad 0 < i'_1 + \ldots + i'_N < K, \tag{6.25}
\]

where the derivatives in equation (6.25) are formal derivatives with respect to the operators \( \tau_i \).

Let us go over one example to illustrate the workings of the above equations.

\(^{12}\)When looking for a pattern in the solutions to (6.22), it was actually (6.24) and (6.25) that were found first.
**Example 6.1.** We will keep it simple, and look at the situation of \( N = 2 \) particles with excitation energy \( K = 3 \). We will express all the lower order coefficients in terms of the highest order coefficients \( c_{30}, c_{21}, c_{12} \) and \( c_{03} \), which are free to choose.

From equation (6.24), the lowest order coefficient \( c_{00} \) is given in terms of the highest order coefficients by

\[
c_{00} = \tau_1^3 c_{30} + \tau_1^2 \tau_2 c_{21} + \tau_1 \tau_2^2 c_{12} + \tau_2^3 c_{03}.
\]

(6.26)

The first and second order coefficients are now given by differentiating the expression for \( c_{00} \) (6.26) with respect to the operators \( \tau_i \) and then dividing by the corresponding factorials as shown in equation (6.25). This procedure yields

\[
c_{10} = 3 \tau_1^2 c_{30} + 2 \tau_1 \tau_2 c_{21} + \tau_2^2 c_{12}
\]

\[
c_{01} = \tau_1^2 c_{21} + 2 \tau_1 \tau_2 c_{12} + 3 \tau_2^2 c_{03}
\]

for the first order coefficients, and,

\[
c_{20} = 3 \tau_1 c_{30} + \tau_2 c_{21}
\]

\[
c_{11} = 2 \tau_1 c_{21} + 2 \tau_2 c_{12}
\]

\[
c_{02} = \tau_1 c_{12} + 3 \tau_2 c_{03}
\]

for the second order coefficients. Combining all of the above formulas, the full \((N, K) = (2, 3)\) solution \( \psi_{2,3} \) to the Schrödinger equation is given by

\[
\psi_{2,3} = S E F
\]

\[
= (w_1 - w_2) e^{-w_1 \bar{w}_1 - w_2 \bar{w}_2} \left[ c_{00} + c_{10} \bar{w}_1 + c_{01} \bar{w}_2 + c_{20} \bar{w}_1^2 + c_{11} \bar{w}_1 \bar{w}_2 + c_{02} \bar{w}_2^2
\right.

\[+ c_{30} \bar{w}_1^3 + c_{21} \bar{w}_1^2 \bar{w}_2 + c_{12} \bar{w}_1 \bar{w}_2^2 + c_{03} \bar{w}_2^3\]

\[
= (w_1 - w_2) e^{-w_1 \bar{w}_1 - w_2 \bar{w}_2} \left[ \tau_1^3 c_{30} + \tau_1^2 \tau_2 c_{21} + \tau_1 \tau_2^2 c_{12} + \tau_2^3 c_{03}
\right.

\[+ (3 \tau_1^2 c_{30} + 2 \tau_1 \tau_2 c_{21} + \tau_2^2 c_{12}) \bar{w}_1 + (\tau_1^2 c_{21} + 2 \tau_1 \tau_2 c_{12} + 3 \tau_2^2 c_{03}) \bar{w}_2
\]

\[+ (3 \tau_1 c_{30} + \tau_2 c_{21}) \bar{w}_1^2 + (2 \tau_1 c_{21} + 2 \tau_2) \bar{w}_1 \bar{w}_2 + (\tau_1 c_{12} + 3 \tau_2 c_{03}) \bar{w}_2^2
\]

\[+ c_{30} \bar{w}_1^3 + c_{21} \bar{w}_1^2 \bar{w}_2 + c_{12} \bar{w}_1 \bar{w}_2^2 + c_{03} \bar{w}_2^3\]

\]

where the highest order coefficients are all arbitrary polynomials of \( w_1 \) and \( w_2 \) obeying the symmetry condition — see next section, and the operators \( \tau_i \) are given by (6.18), that is

\[
\tau_i := -\frac{1}{2} \left( \partial_i + \sum_{j=1,j\neq i}^N \frac{\alpha}{w_i - w_j} \right).
\]

(6.27)

It seems that our work is done: we have found a recipe for producing eigenfunctions of the \( N \)-particle Hamiltonian for arbitrary particle number \( N \) and excitation energy \( K \). However, there is one problem that we have swept under the rug. The operators \( \tau_i \) introduce local singularities by multiplying with factors of the form \( \frac{1}{w_i - w_j} \), which are not locally square-integrable. This means that certain wavefunctions obtained through the above procedure are not normalizable. How to deal with this problem is the subject of the next section.
6.2.5 The Normalizability and Symmetry Conditions

There are two more conditions that we have to take into account, the symmetry and the normalization condition. We shall start with the easiest of the two: the symmetry condition.

Recall that we wrote $F$ in equation (6.20) as,

$$F(w, \bar{w}) = \sum_{0 \leq i_1 + \ldots + i_N \leq K} c_{i_1, \ldots, i_N}(w) \bar{w}_{i_1}^{i_1} \cdots \bar{w}_{i_N}^{i_N}.$$  \hspace{1cm} (6.28)

We now focus on the coefficients $c_{i_1, \ldots, i_N}(w)$, which are polynomials in the coordinates $w_1, \ldots, w_N$. Further expanding $c_{i_1, \ldots, i_N}(w)$, which are polynomials in the coordinates $w_1, \ldots, w_N$.

$$c_{i_1, \ldots, i_N}(w) = \sum_{0 \leq j_1 + \ldots + j_N \leq d} \gamma_{j_1, \ldots, j_N}^{i_1, \ldots, i_N} w_{j_1}^{i_1} \cdots w_{j_N}^{i_N}.$$  \hspace{1cm} (6.29)

where $d$ is the maximum degree of all coefficients together, and the $\gamma_{j_1, \ldots, j_N}^{i_1, \ldots, i_N}$'s are complex numbers.

**Definition 6.9.** We define an action of the permutation group $S_N$ on the coefficients $\gamma_{j_1, \ldots, j_N}^{i_1, \ldots, i_N}$ by

$$\sigma(\gamma_{j_1, \ldots, j_N}^{i_1, \ldots, i_N}) := \gamma_{j_{\sigma(1)}, \ldots, j_{\sigma(N)}}^{i_{\sigma(1)}, \ldots, i_{\sigma(N)}}$$  \hspace{1cm} (6.30)

for each $\sigma \in S_N$.

The above definition is chosen such that a permutation $\sigma$ applied to the particle coordinates exactly corresponds to applying $\sigma$ to the coefficients $\gamma_{j_1, \ldots, j_N}^{i_1, \ldots, i_N}$ of the corresponding $N$-particle wavefunction.

With the above definition, the symmetry condition on $F$ can be made explicit.

**Symmetry condition:** we require that $F$ is symmetric under pairwise exchange $(w_i, \bar{w}_i) \leftrightarrow (w_j, \bar{w}_j)$. In terms of the coefficients $\gamma_{j_1, \ldots, j_N}^{i_1, \ldots, i_N}$, the symmetry condition reads

$$\sigma(\gamma_{j_1, \ldots, j_N}^{i_1, \ldots, i_N}) = \gamma_{j_{\sigma(1)}, \ldots, j_{\sigma(N)}}^{i_{\sigma(1)}, \ldots, i_{\sigma(N)}} \text{ for all } \sigma \in S_N \text{ and for all indices } i_1, \ldots, i_N, j_1, \ldots, j_N.$$  \hspace{1cm} (6.31)

**Remark 6.15.** Because expression (6.23) — after inserting equation (6.29) — for the lower order terms $c_{i_1, \ldots, i_N}^{\cdot}$ of order less than $K$ is symmetric with respect to pairwise exchange of the indices, we only have to impose the symmetry condition on the highest order terms $c_{i_1, \ldots, i_N}^{\cdot}$ of order $K$, and the lower order terms will inherit them automatically.

Next, is the more difficult normalizability condition. The problem is that the operators $\tau_i$ create additional local singularities, which are not square-integrable. We can get rid of these singularities by putting extra constraints on the coefficients $\gamma_{j_1, \ldots, j_N}^{i_1, \ldots, i_N}$. The key idea is captured by the following lemma.

**Lemma 6.6.** A homogeneous polynomial $P(x, y) = \sum_{i+j=d} a^{i,j} x^i y^j$ in two variables of degree $d$ is divisible by the polynomial $(x - y)^l$, for a fixed $l \in \mathbb{N}$, $l \leq d$, if and only if

$$\sum_{i+j=d} t^k a^{i,j} = 0 \text{ for all } k \in \{0, 1, \ldots, l-1\}.$$  \hspace{1cm} (6.32)

Note that our unconventional choice of index placement for the coefficients $a^{i,j}$ is because the coefficients $\gamma_{j_1, \ldots, j_N}^{i_1, \ldots, i_N}$ also have indices at the top.
Proof. In order to prove the lemma, we use a basic algebra theorem on the divisibility of polynomials in one variable.\footnote{See for example the (Dutch) lecture notes by G. van der Geer [4], Proposition 4.12, for the case \( l = 2 \). The generalization of the theorem to arbitrary order is straightforward. If \( f \) has an \( l \)th order zero at \( y \), then write \( f(x) = f_l(x)(x - y)^l \) and differentiate with respect to \( x \) up to \( l - 1 \) times at the point \( x = y \). For the converse, by \( f(y) = 0 \), we have \( f(x) = f_1(x)(x - y) \) for some polynomial \( f_1 \). Differentiate with respect to \( x \) at \( x = y \) and use \( f'(y) = 0 \) to obtain \( f_1(y) = 0 \), hence \( f_1(x) = f_2(x)(x - y) \) for some polynomial \( f_2 \), and therefore \( f(x) = f_2(x)(x - y)^2 \). Proceed by induction.} If \( f \) is a polynomial over the field of complex numbers (or any domain for that matter), then the following two statements are equivalent:

\[
f \text{ has a zero of order } l \text{ at } y \in \mathbb{C} \iff f(y) = f'(y) = \ldots = f^{(l-1)}(y) = 0, \tag{6.33}
\]

Next, we apply the statement in (6.33) to the polynomial \( f(x) = P(x, y) \) considered only as a polynomial in \( x \). That is,

\[
f(x) = \sum_{i=0}^{d} b^i x^i,
\]

where \( b^i \) is a function of \( y \),

\[
b^i(y) = a^{i,d-i} y^{d-i}.
\]

Now, the LHS of (6.33) taken to be true for all \( y \in \mathbb{C} \) is precisely the statement that the two-variable polynomial \((x - y)^l\) divides the polynomial \( P(x, y) \). The RHS of (6.33) reads

\[
0 = f(y) = \sum_{i=0}^{d} a^{i,d-i} y^d \\
0 = f'(y) = \sum_{i=0}^{d} ia^{i,d-i} y^{d-1} \\
0 = f''(y) = \sum_{i=0}^{d} i(i-1)a^{i,d-i} y^{d-2} \\
\vdots \\
0 = f^{(l-1)}(y) = \sum_{i=0}^{d} i(i-1) \cdots (i-(l-2))a^{i,d-i} y^{d-(l-1)}.
\]

Note that, in each line, the exponent of \( y \) is independent of \( i \), so we can divide out all the factors of \( y \) when \( y \neq 0 \). We conclude that the RHS of (6.33), taken to be true for all \( y \in \mathbb{C} \), is equivalent to

\[
0 = \sum_{i=0}^{d} a^{i,d-i} \\
0 = \sum_{i=0}^{d} ia^{i,d-i} \\
0 = \sum_{i=0}^{d} i(i-1)a^{i,d-i} \\
\vdots \\
0 = \sum_{i=0}^{d} i(i-1) \cdots (i-(l-2))a^{i,d-i}.
\]
To complete the proof, note that the above set of equations is equivalent to (6.32) by taking linear combinations.

Using the Lemma 6.6, we can find out which constraints we have to put on the $\gamma^{j_1,\ldots,j_N}_{i_1,\ldots,i_N}$’s for the singularities to disappear. We shall illustrate how in the example below. Because Lemma 6.6 only puts a restriction on a fixed degree part, we take our highest order coefficients $c_{i_1,\ldots,i_N}$ to consist of a degree $d$ part only.

**Example 6.2.** We shall investigate the $(N, K) = (2, 1)$ case. In this case, the free to choose highest order coefficients are $c_{10}$ and $c_{01}$, and the lowest order coefficient $c_{00}$ can be expressed in terms of the highest order coefficients through (6.24). Hence, the polynomial $F$ takes the form

$$F(w, \bar{w}) = c_{00} + c_{10}w_1 + c_{01}\bar{w}_2 = \tau_1c_{10} + \tau_2c_{01} + c_{10}\bar{w}_1 + c_{01}\bar{w}_2,$$

Using the definition of the $\tau$-operators (6.27), the above expression for $F$ becomes

$$F(w, \bar{w}) = -\frac{1}{2}(\partial_1 c_{10} + \partial_2 c_{01}) - \frac{1}{2}\frac{\alpha}{w_1 - \bar{w}_2} (c_{10} - c_{01}) + c_{10}\bar{w}_1 + c_{01}\bar{w}_2. \quad (6.34)$$

We see that, in the case of $(N, K) = (2, 1)$, there is only one local singularity. We expand

$$c_{10}(w) = \sum_{j_1 + j_2 = d} \gamma^{j_1,j_2}_{1,0} w_1^{j_1} w_2^{j_2} \quad \text{and} \quad c_{01}(w) = \sum_{j_1 + j_2 = d} \gamma^{j_1,j_2}_{0,1} w_1^{j_1} w_2^{j_2}. \quad (6.36)$$

Now, the condition for the local singularity to disappear, is that we want $c_{10} - c_{01}$ to be divisible by $(w_1 - \bar{w}_2)$. By Lemma 6.6, this equivalent to the condition that

$$\sum_{j_1 + j_2 = d} \left( \gamma^{j_1,j_2}_{1,0} - \gamma^{j_1,j_2}_{0,1} \right) = 0. \quad (6.35)$$

Equation (6.35) is the only equation that has to be satisfied for the $(N, K) = (2, 1)$ wavefunction to be normalizable.

**Remark 6.16.** From Example 6.2, we see that the normalizability condition(s) and the symmetry condition(s) are related. As a matter of fact, in the $(N, K) = (2, 1)$ case, the normalizability follows directly from the symmetry requirements. Also for higher excitation energies, we will find that the symmetry condition reduces the number of normalizability conditions, but for $K > 1$ symmetry alone will no longer be sufficient.

Having explained the mechanism behind the normalizability condition, let us now turn to the general case. The approach we take here is the same approach that we took for solving the recurrence equation for $F$: we list the normalizability conditions for several cases separately, and then look for a pattern. Unfortunately, the normalizability conditions become quite numerous very quickly, so we have only been able to find a general normalizability condition for $N = 2$ particles (and arbitrary excitation energy $K$). See Appendix B.2 for the details. The result is summarized below.

**Normalizability condition for $N = 2$ particles:** for each degree $d$ and $K$ separately, we require that for any $l \in \{1, 2, \ldots, K\}$ and for all $k \in \{0, 1, 2, \ldots, l - 1\}$,

$$\sum_{j_1 + j_2 = d} j_1^K - l \sum_{l_1 + l_2 = K} (-1)^{l_2} (l_2) \gamma^{j_1,j_2}_{l_1,l_2} = 0. \quad (6.36)$$

The above equations do not take the symmetry conditions into account: they have to be imposed separately.
We have developed a method to construct $N$-particle anyonic energy eigenfunctions for non-interacting charged anyons in a uniform perpendicular magnetic field. However, we have only been able to find the normalizability conditions for $N = 2$ particles. This does not mean that the rest of the theory is useless: quite the contrary. However, when constructing anyonic wavefunctions for three or more particles, the normalizability conditions have to be imposed manually, which does require a bit of labor, because, for every choice of $N$ and $K$ there are different — and at first sight unrelated — conditions that have to be met. Note that, as in the $N = 2$ case, the normalizability conditions for $N > 2$ will involve only fixed degree $d$ parts of the polynomials $c_{i_1,...,i_N}(w)$.

**Corollary 6.3.** Because the normalizability conditions have to be imposed for each particle number $N$ and degrees $d$ and $K$ separately, we obtain different energy eigenfunctions

$$\psi_{N,d,K}(w, \bar{w}, \theta) = S_N(w, \theta) E_N(w, \bar{w}) F_{N,d,K}(w, \bar{w})$$

for every choice of $N$, $d$ and $K$, with energy $\frac{1}{2}\hbar \omega_C^*(N + 2K)$. Note that the normalizability conditions relate different highest order coefficients $c_{i_1,...,i_N}$ to each other. In particular, all highest order coefficients of the energy eigenfunction $\psi_{N,d,K}$ have the same degree $d$ in $w$.

**Remark 6.17.** We have not shown that the labels $N$, $d$ and $K$ uniquely determine an energy eigenstate. All that we know is that all the energy eigenstates that we have constructed carry labels $N$, $d$ and $K$. As we will see in Section 6.2.7, for fixed $N$, $d$ and $K$ there can be more than one energy eigenfunction $\psi_{N,d,K}$, which means that the labels $d$ and $K$ actually refer to specific eigenspaces of the $N$-particle Hamiltonian $H_N$.

To summarize, we have found exact eigenfunctions $\psi = SEF$ of the Hamiltonian $H_N$. Within the set of eigenfunctions that we found, certain eigenfunctions turned out to be not normalizable. In order to filter out the normalizable eigenfunctions, we have to impose (6.36). Additionally, we also have to impose the symmetry condition on $F$ mentioned in Remark 6.10, which can be translated to equation (6.31) in terms of the coefficients $\gamma_{A_1,...,A_N}$ of $F$.

This completes our construction of the exact anyonic energy eigenfunctions for non-interacting charged anyons in a uniform perpendicular magnetic field. Of course, it would be a shame to stop here: having constructed all these energy eigenfunctions, we are now able to extract actual physical information from them, and this is exactly what we will do in the next section.

### 6.2.6 Measuring Anyons: Angular Momentum

As of today, there is no experiment that determines the statistical parameter $\alpha$ of the anyons directly. Now that we have a plethora of anyonic energy eigenfunctions that describe non-interacting charged anyons in a uniform perpendicular magnetic field, our hope is that there exist operators with an $\alpha$-dependent spectrum. The only physical operator that we have considered so far, is the Hamiltonian $H_N$. Because we have only found eigenfunctions corresponding to the eigenvalues

$$E_{N,K} = \frac{1}{2}\hbar \omega_C^*(N + 2K),$$

the energy spectrum is independent of $\alpha$ for as far as the eigenfunctions we have constructed are concerned. Actually, this statement is not entirely true, because in the FQHE — which, up to now, is the only physical system wherein anyons are expected to exist — the charge $e^*$ of the quasi-holes is given by $e^* = ev$, where $v$ is the filling factor, which coincidentally is also equal to the statistical angle $\alpha$. Thus, from the energy spectrum $E_{N,K}$ we can read-off the statistical angle $\alpha = \nu$ from

\footnote{Thanks to L. Fritz for pointing this out. Note that, previously, the author used $e$ to denote the anyon charge, which obscured the fact that in the case of the FQHE, the anyon charge is related to the filling factor, and hence to the statistical angle.}
the spacing between the Landau levels, which is determined by the cyclotron frequency

\[ \omega^* = \frac{e^* B}{m}. \]

However, note that we could have also measured the filling factor directly to determine the statistical angle, and for that all the theory developed in this chapter is not even necessary. Instead, what we mean by an \( \alpha \)-dependent spectrum, is empirical evidence that the quasi-particle excitations in the FQHE show anyonic behavior in the first place, and for that purpose, we can use the theory developed in this chapter, because the wavefunctions we have found apply to any kind of non-interacting charged anyons in a uniform perpendicular magnetic field, irrespective of what the underlying phenomenon is that causes the anyons to appear.

Anyonic statistics show up when two anyons are exchanged, which we do by adiabatically transporting the anyons around each other. This suggests that the angular momentum operator might be dependent on the statistical parameter \( \alpha \).

**Definition 6.10.** Let

\[ L_i := -i\hbar (x_i \partial_i - y_i \partial_i) \]

be the operator associated to the \( z \)-component of the angular momentum of the \( i \)th anyon around the origin. Furthermore, we denote the (\( z \) component of) the total angular momentum operator relative to the origin by

\[ L_N := \sum_{i=1}^{N} L_i. \]

In terms of the dimensionless complex coordinates \( w \) and \( \bar{w} \), the total angular momentum operator reads

\[ L_N = \hbar \sum_{i=1}^{N} \left(-w_i \partial_i + \bar{w}_i \bar{\partial}_i\right). \]

As can be checked by hand, \( L_N \) commutes with the \( N \)-particle Hamiltonian \( H_N \), so we can hope to simultaneously diagonalize both operators. We will apply \( L_N \) to an arbitrary energy eigenfunction of the form \( \psi_N = S_N E_N F_N \). Since \( L_N \) is a differential operator, we can apply \( L_N \) to each of the three factors separately, and then add up the results.

First, the statistical term \( S_N(w, \theta) = \prod_{j>i=1}^{N} (w_i - w_j)^{\alpha} \). For a fixed factor \( (w_i - w_j)^{\alpha} \), we have

\[ L_N(w_i - w_j)^{\alpha} = \hbar (-w_i \partial_i - w_j \partial_j) (w_i - w_j)^{\alpha} = -\hbar \alpha (w_i - w_j)^{\alpha}. \quad (6.37) \]

Hence, using equation (6.37) multiple times, we get

\[ L_N S_N(w) = L_N \left[ \prod_{j>i=1}^{N} (w_i - w_j)^{\alpha} \right] = -\frac{\alpha \hbar}{2} N(N-1) \prod_{j>i=1}^{N} (w_i - w_j)^{\alpha} \]

\[ = -\frac{\alpha \hbar}{2} N(N-1)S_N(w), \]

where \( \frac{1}{2}N(N-1) \) is the number of pairs that we can make out of \( N \) anyons. As expected, the parameter \( \alpha \) shows up in the spectrum of \( L_N \).

Next is the exponential term. But first, a useful lemma.

**Lemma 6.7.** When acting with \( L_N \) on polynomials in \((w, \bar{w})\) of fixed degree \( (d, K) \), by which we mean that in each term of the polynomial the orders of the coordinates \( w_1, \ldots, w_N \) add up to \( d \), and the orders of the coordinates \( \bar{w}_1, \ldots, \bar{w}_N \) add up to \( K \), we have the following operator identity

\[ L_N = \hbar (-d + K) \]

where the operator on the RHS is a multiplication operator.
Proof. When acting on a term \( w_1^{i_1} \cdots w_N^{i_N} \bar{w}_1^{i_1} \cdots \bar{w}_N^{i_N} \),
\[
w_k \partial_k \left( w_1^{i_1} \cdots w_N^{i_N} \bar{w}_1^{i_1} \cdots \bar{w}_N^{i_N} \right) = j_k w_1^{i_1} \cdots w_N^{i_N} \bar{w}_1^{i_1} \cdots \bar{w}_N^{i_N}
\]
\[
\bar{w}_k \partial_k \left( w_1^{i_1} \cdots w_N^{i_N} \bar{w}_1^{i_1} \cdots \bar{w}_N^{i_N} \right) = i_k w_1^{i_1} \cdots w_N^{i_N} \bar{w}_1^{i_1} \cdots \bar{w}_N^{i_N}.
\]
Now use that \( i_1 + \ldots + i_N = K \) and \( j_1 + \ldots + j_N = d \).

Thus, by Lemma 6.7,
\[
L_N E_N (\mathbf{w}, \mathbf{w}) = L_N \exp \left( -\sum_{i=1}^N w_i \bar{w}_i \right) = 0,
\]
since \( \sum_{i=1}^N w_i \bar{w}_i \) has \( d = K = 1 \). Finally, we get to the polynomial part \( F_N (\mathbf{w}, \mathbf{w}) \). Recall that, by equation (6.28), \( F_N \) is a polynomial of the form
\[
F_N (\mathbf{w}, \mathbf{w}) = \sum_{0 \leq i_1 + \ldots + i_N \leq K} c_{i_1, \ldots, i_N} (\mathbf{w}) \bar{w}_1^{i_1} \cdots \bar{w}_N^{i_N},
\]
where all the lower order coefficients \( c_{i_1', \ldots, i_N'} \) can be expressed in terms of the highest order coefficients \( c_{i_1, \ldots, i_N} \) of order \( K \) through equation (6.23),
\[
c_{i_1', \ldots, i_N'} = \sum_{i_1 + \ldots + i_N = K} \left( \begin{array}{c} i_1 \\ \vdots \\ i_N \end{array} \right) \left( \begin{array}{c} i_1' \\ \vdots \\ i_N' \end{array} \right) \tau_1^{i_1 - i'_1} \cdots \tau_N^{i_N - i'_N} c_{i_1, \ldots, i_N}.
\]
Furthermore, to satisfy the normalizability conditions (6.36) we restrict ourselves to the case where all highest order coefficients have the same degree \( d \) in \( \mathbf{w} \),
\[
c_{i_1, \ldots, i_N} (\mathbf{w}) = \sum_{j_1 + \ldots + j_N = d} \gamma_{j_1, \ldots, j_N} w_1^{j_1} \cdots w_N^{j_N}.
\]
This means that, for as far as the highest order coefficients \( c_{i_1, \ldots, i_N} (\mathbf{w}) \) are concerned, by Lemma 6.7,
\[
L_N c_{i_1, \ldots, i_N} (\mathbf{w}) \bar{w}_1^{i_1} \cdots \bar{w}_N^{i_N} = h(-d + K) c_{i_1, \ldots, i_N} (\mathbf{w}) \bar{w}_1^{i_1} \cdots \bar{w}_N^{i_N}.
\]
But what about the lower order coefficients?

By equation (6.39), each time we go down one step in the order of the coefficients \( c_{i_1', \ldots, i_N'} \), we get an additional factor of one of the \( \tau_i \)'s in each term in the sum. Now, the operator \( \tau_i \) lowers the order of \( \mathbf{w} \) by one, because each term in \( \tau_i \), see (6.18),
\[
\tau_i := -\frac{1}{2} \left( \partial_i + \sum_{j=1, j \neq i}^N \frac{\alpha}{w_i - w_j} \right),
\]
differentiates or divides by the coordinates \( w_1, \ldots, w_N \). This means that, each time we go down one order in \( \mathbf{w} \) by decreasing \( i_1 + \ldots + i_N \) by one, we simultaneously decrease the order \( j_1 + \ldots + j_N \) of \( \mathbf{w} \) by one. Hence, the difference \( i_1 + \ldots + i_N - (j_1 + \ldots + j_N) \) remains constant. We conclude that
\[
L_N F_{N, d, K} (\mathbf{w}, \mathbf{w}) = h(-d + K) F_{N, d, K} (\mathbf{w}, \mathbf{w}),
\]
where \( F_{N, d, K} \) is a polynomial corresponding to particle number \( N \), excitation energy \( K \) and highest order coefficients of degree \( d \) in \( \mathbf{w} \).

Let us summarize the obtained results in the following corollary.
Corollary 6.4. If we label the obtained energy eigenfunctions with particle number $N$, excitation energy $K$ and $w$ degree $d$, by which we mean that all highest order coefficients

$$c_{i_1, ..., i_N}(w) = \sum_{j_1 + \ldots + j_N = d} w_1^{j_1} \ldots w_N^{j_N},$$

have degree $j_1 + \ldots + j_N = d$ in $w$, then the energy eigenfunctions $\psi_{N,d,K} = S_N E_N F_{N,d,K}$ are simultaneously eigenfunctions of the total angular momentum operator $L_N$ with eigenvalue

$$L_N \psi_{N,d,K}(w, \bar{w}) = \hbar \left[ -\frac{\alpha}{2} N(N - 1) - d + K \right] \psi_{N,d,K}(w, \bar{w}).$$

Corollary 6.5. From the above corollary, we see that the spectrum of the total angular momentum operator depends on the statistical parameter $\alpha$. This means that the statistical parameter $\alpha$ for non-interacting charged anyons in a uniform perpendicular magnetic field can theoretically be measured directly by looking at the spectrum of the (z component of the) total angular momentum operator $L_N$.

Having concluded that the total angular momentum operator could possibly lead to a direct method for detecting anyons by measuring their statistical phase, in the next and final section, we will finish our discussion on anyon wavefunctions by constructing some explicit formulas for two-particle anyonic wavefunctions using the method constructed in this chapter.

6.2.7 Explicit Formulas for Two-Particle Anyon Eigenfunctions

In this section, we want to give the reader an idea of what the anyon eigenfunctions look like. We will restrict ourselves to case of $N = 2$ particle eigenfunctions. The particle number $N = 2$ will be suppressed in the notation.

Recall that the complete eigenfunction $\psi_{d,K} = \psi_{N=2,d,K}$ is of the form

$$\psi_{d,K}(w_1, \bar{w}_1, w_2, \bar{w}_2) = (w_1 - w_2)^\alpha e^{-w_1 \bar{w}_1 - w_2 \bar{w}_2} F_{d,K}(w_1, \bar{w}_1, w_2, \bar{w}_2),$$

where $F$ is a polynomial of degree $d$ in $w$ and $K$ in $\bar{w}$, i.e. by (6.38):

$$F_{d,K}(w_1, \bar{w}_1, w_2, \bar{w}_2) = \sum_{0 \leq i_1 + i_2 \leq K} c_{i_1, i_2}(w) \bar{w}^{i_1}_1 w^{i_2}_2.$$  

We shall consider different values of $K$ and $d$. In trying to find a balance between being complete but not too exhaustive, we will to discuss the values $(0, 0) - (3, 0)$, $(0, 1) - (3, 1)$ and $(0, 2) - (2, 2)$ for $(d, K)$.

Remark 6.18. We will leave the ubiquitous prefactor $(w_1 - w_2)^\alpha e^{-w_1 \bar{w}_1 - w_2 \bar{w}_2}$, which contributes $\frac{1}{2}N\hbar\omega_C^* = \hbar \omega_C^*$ to the energy\textsuperscript{15} and $-\frac{\alpha}{2} \hbar N(N - 1) = -\alpha \hbar$ to the angular momentum out of the picture, and only focus on the polynomial part $F_{d,K}$ of the solution. Do keep in mind that, when considering eigenvalues, we will include $\hbar \omega_C^*$ and $-\alpha \hbar$ in the energy and angular momentum eigenvalues respectively.

For $K = 0$, $F_{d,0}$ is just a symmetric polynomial in $w_1$ and $w_2$,

$$F_{d,0}(w, \bar{w}) = c_{00}(w).$$

Below is a table of all linearly independent symmetric polynomials up to degree $d = 3$. The column labelled by $L_{d,0}$ gives the angular momentum eigenvalue $\hbar(-\alpha - d)$ of the total wavefunction $\psi_{d,0}$.

\textsuperscript{15}In the sense that $K$ signifies the energy that the system has on top of the ground state energy $\hbar \omega_C^*$. 
The upper indices label the different basis vectors of the corresponding \((d, K)\)-eigenspace, which, for the cases of \(d = 2\) and \(d = 3\), is two-dimensional.

**K=1, \(E = 2h\omega_c^*\).** For \(K = 1\), \(F_{d,K}\) is equal to

\[
F_{d,1}(\mathbf{w}, \bar{\mathbf{w}}) = c_{00}(\mathbf{w}) + c_{10}(\mathbf{w})\bar{w}_1 + c_{01}(\mathbf{w})\bar{w}_2, \tag{6.40}
\]

where the lowest order coefficient \(c_{00}\) is given by

\[
c_{00}(\mathbf{w}) = \tau_1 c_{10}(\mathbf{w}) + \tau_2 c_{01}(\mathbf{w})
\]

\[
= -\frac{1}{2} \left[ \partial_1 c_{10}(\mathbf{w}) + \partial_2 c_{01}(\mathbf{w}) + \frac{\alpha}{w_1 - w_2} \left( c_{10}(\mathbf{w}) - c_{01}(\mathbf{w}) \right) \right] \tag{6.41}
\]

in terms of the highest order coefficients. The highest order coefficients have to satisfy the symmetry condition (6.31)

\[
\gamma^{j_1,j_2}_{1,0} = \gamma^{j_2,j_1}_{0,1}
\]

and the normalizability condition, the latter of which, by Remark 6.16, follows from the former.

**Remark 6.19.** We shall use upper case characters \(A, B, C, \ldots\) to denote the coefficients \(\gamma_{i_1,i_2}^{j_1,j_2}\) because the characters are easier on the eyes.

\(d = 0\).

For \(d = 0\), \(c_{10}\) and \(c_{01}\) are constants, which, by the symmetry condition, are related by

\[
c_{10} = c_{01} = A \tag{6.42}
\]

for some constant \(A \in \mathbb{C}\). Hence, the \((d, K) = (0,1)\) eigenfunctions form a one-dimensional subspace.

Setting \(A = 1\) and plugging (6.42) into (6.41) and then into (6.40) yields the polynomial \(F_{0,1}\) in this subspace, given by

\[
F_{0,1}(\mathbf{w}, \bar{\mathbf{w}}) = \bar{w}_1 + \bar{w}_2.
\]

\(d = 1\).

For \(d = 1\), using the symmetry condition, the coefficients \(c_{10}\) and \(c_{01}\) are given by

\[
c_{10}(\mathbf{w}) = Aw_1 + Bw_2 \tag{6.43}
\]

\[
c_{01}(\mathbf{w}) = Bw_1 + Aw_2 \tag{6.44}
\]

for arbitrary constants \(A\) and \(B\). Hence, the \((d, K) = (1,1)\) subspace of eigenfunctions is two-dimensional. Setting \(A = 1, B = 0\) and \(A = 0, B = 1\) respectively and inserting (6.43) and (6.44) into (6.41) and then into (6.40) yields the polynomials \(F_{1,1}^{A,B}\), given by

\[
F_{1,1}^{1,0}(\mathbf{w}, \bar{\mathbf{w}}) = -1 - \frac{\alpha}{2} + w_1 \bar{w}_1 + w_2 \bar{w}_2,
\]

\[
F_{1,1}^{0,1}(\mathbf{w}, \bar{\mathbf{w}}) = \frac{\alpha}{2} + w_2 \bar{w}_1 + w_1 \bar{w}_2.
\]
The two polynomials above form a basis for the \((d, K) = (1, 1)\) subspace of eigenfunctions.

In what follows, we shall always use the symmetry condition to obtain the expressions for \(c_{10}\) and \(c_{01}\), which we will then insert into (6.41) and then into (6.40) to obtain the solution.

\(d = 2\).

For \(d = 2\), the coefficients \(c_{10}\) and \(c_{01}\) are given by

\[c_{10}(\vec{w}) = A\vec{w}_1^2 + B\vec{w}_1\vec{w}_2 + C\vec{w}_2^2,\]
\[c_{01}(\vec{w}) = C\vec{w}_1^2 + B\vec{w}_1\vec{w}_2 + A\vec{w}_2^2\]

for arbitrary constants \(A, B\) and \(C\). Setting \(A = 1, B = C = 0; A = C = 0, B = 1;\) and \(A = B = 0, C = 1\) respectively, yields the following basis for the three-dimensional \((d, K) = (2, 1)\) subspace of eigenfunctions

\[F^{1,0,0}_{2,1}(\vec{w}, \vec{w}) = -\frac{2 + \alpha}{2}(\vec{w}_1 + \vec{w}_2) + \vec{w}_1^2\vec{w}_1 + \vec{w}_2^2\vec{w}_2,\]
\[F^{0,1,0}_{2,1}(\vec{w}, \vec{w}) = -\frac{1}{2}(\vec{w}_1 + \vec{w}_2) + \vec{w}_1\vec{w}_2(\vec{w}_1 + \vec{w}_2),\]
\[F^{0,0,1}_{2,1}(\vec{w}, \vec{w}) = \frac{\alpha}{2}(\vec{w}_1 + \vec{w}_2) + \vec{w}_2\vec{w}_1 + \vec{w}_1^2\vec{w}_2.\]

\(d = 3\).

For \(d = 3\), the coefficients \(c_{10}\) and \(c_{01}\) are given by

\[c_{10}(\vec{w}) = A\vec{w}_1^3 + B\vec{w}_1^2\vec{w}_2 + C\vec{w}_1\vec{w}_2^2 + D\vec{w}_2^3,\]
\[c_{01}(\vec{w}) = D\vec{w}_1^3 + C\vec{w}_1^2\vec{w}_2 + B\vec{w}_1\vec{w}_2^2 + A\vec{w}_2^3\]

for arbitrary constants \(A, B, C\) and \(D\). Setting \(A = 1, B = C = D = 0; A = C = D = 0, B = 1;\) \(A = B = D = 0, C = 1;\) and \(A = B = C = 0, D = 1\) respectively, yields the following basis for the four-dimensional \((d, K) = (3, 1)\) subspace of eigenfunctions

\[F^{1,0,0,0}_{3,1}(\vec{w}, \vec{w}) = -\frac{3}{2}(\vec{w}_1^2 + \vec{w}_2^2) - \frac{\alpha}{2}(\vec{w}_1^2 + \vec{w}_1\vec{w}_2 + \vec{w}_2^2) + \vec{w}_1^3\vec{w}_1 + \vec{w}_2^3\vec{w}_2,\]
\[F^{0,1,0,0}_{3,1}(\vec{w}, \vec{w}) = -\frac{1}{2}(\alpha + 4)\vec{w}_1\vec{w}_2 + \vec{w}_1\vec{w}_2(\vec{w}_1\vec{w}_1 + \vec{w}_2\vec{w}_2),\]
\[F^{0,0,1,0}_{3,1}(\vec{w}, \vec{w}) = -\frac{1}{2}(\vec{w}_1^2 + \vec{w}_2^3) + \frac{\alpha}{2}\vec{w}_1\vec{w}_2 + \vec{w}_1\vec{w}_2(\vec{w}_2\vec{w}_1 + \vec{w}_1\vec{w}_2),\]
\[F^{0,0,0,1}_{3,1}(\vec{w}, \vec{w}) = \frac{\alpha}{2}(\vec{w}_1^2 + \vec{w}_1\vec{w}_2 + \vec{w}_2^2) + \vec{w}_2^3\vec{w}_1 + \vec{w}_1^3\vec{w}_2.\]

The obtained polynomials for \(K = 1\) together with the total angular momentum eigenvalues \(\hbar(-\alpha - d + 1)\) of the corresponding wavefunctions \(\psi_{d,1}\) are shown in the table below.
6.2. CONSTRUCTION OF THE ANYON EIGENFUNCTIONS

<table>
<thead>
<tr>
<th>$d$</th>
<th>$F$</th>
<th>$L_{d,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$F_{0,1}(\bar{w}, \bar{w}) = \bar{w}_1 + \bar{w}_2$</td>
<td>$h(-\alpha + 1)$</td>
</tr>
<tr>
<td>1</td>
<td>$F_{1,1}^{1,0}(\bar{w}, \bar{w}) = -1 - \alpha + w_1 \bar{w}_1 + w_2 \bar{w}_2,$</td>
<td>$-h\alpha$</td>
</tr>
<tr>
<td>1</td>
<td>$F_{1,1}^{0,1}(\bar{w}, \bar{w}) = \frac{\alpha}{2} + w_2 \bar{w}_1 + w_1 \bar{w}_2$</td>
<td>$-h\alpha$</td>
</tr>
<tr>
<td>2</td>
<td>$F_{2,1}^{1,0,0}(\bar{w}, \bar{w}) = -\frac{2 - \alpha}{2} (w_1 + w_2) + w_1^2 \bar{w}_1 + w_2^2 \bar{w}_2$</td>
<td>$h(\alpha - 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$F_{2,1}^{2,1,0}(\bar{w}, \bar{w}) = -\frac{1}{2} (w_1 + w_2) + w_1 w_2 (\bar{w}_1 + \bar{w}_2)$</td>
<td>$h(\alpha - 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$F_{2,1}^{0,1,1}(\bar{w}, \bar{w}) = \frac{\alpha}{2} (w_1 + w_2) + w_1^2 \bar{w}_1 + w_2^2 \bar{w}_2$</td>
<td>$h(\alpha - 1)$</td>
</tr>
<tr>
<td>3</td>
<td>$F_{3,1}^{1,0,0,0}(\bar{w}, \bar{w}) = -\frac{3}{2} (w_1^2 + w_2^2) - \frac{\alpha}{2} (w_1^2 + w_1 w_2 + w_2^2) + w_1^3 \bar{w}_1 + w_2^3 \bar{w}_2$</td>
<td>$h(\alpha - 1)$</td>
</tr>
<tr>
<td>3</td>
<td>$F_{3,1}^{0,1,0,0}(\bar{w}, \bar{w}) = -\frac{1}{2} (\alpha + 4) w_1 w_2 + w_1 w_2 (w_1 \bar{w}_1 + w_2 \bar{w}_2)$</td>
<td>$h(\alpha - 1)$</td>
</tr>
<tr>
<td>3</td>
<td>$F_{3,1}^{0,1,1,0}(\bar{w}, \bar{w}) = -\frac{1}{2} (w_1^2 + w_2^2) + \frac{\alpha}{2} w_1 w_2 + w_1 w_2 (w_2 \bar{w}_1 + w_1 \bar{w}_2)$</td>
<td>$h(\alpha - 1)$</td>
</tr>
<tr>
<td>3</td>
<td>$F_{3,1}^{0,0,1,1}(\bar{w}, \bar{w}) = \frac{\alpha}{2} (w_1^2 + w_1 w_2 + w_2^2) + w_1^2 \bar{w}_1 + w_2^2 \bar{w}_2$</td>
<td>$h(\alpha - 1)$</td>
</tr>
</tbody>
</table>

**K=2, $E = 3h\omega_C$.**

For $K = 2$, $F_{d,K}$ is equal to

$$F_{d,2}(\bar{w}, \bar{w}) = c_{00}(\bar{w}) + c_{10}(\bar{w}) \bar{w}_1 + c_{01}(\bar{w}) \bar{w}_2 + c_{20}(\bar{w}) \bar{w}_1^2 + c_{11}(\bar{w}) \bar{w}_1 \bar{w}_2 + c_{02}(\bar{w}) \bar{w}_2^2,$$  

(6.45)

where the lower order coefficients $c_{00}, c_{10}$ and $c_{01}$ are given by

$$c_{00}(\bar{w}) = \tau_1^2 c_{20}(\bar{w}) + \tau_1 \tau_2 c_{11}(\bar{w}) + \tau_2^2 c_{02}(\bar{w})$$

$$= \frac{1}{4} \left\{ \partial_1^2 c_{20}(\bar{w}) + \partial_1 \partial_2 c_{11}(\bar{w}) + \partial_2^2 c_{02}(\bar{w}) \\
+ \frac{\alpha}{w_1 - w_2} \left[ \partial_1 \left( 2c_{20}(\bar{w}) - c_{11}(\bar{w}) \right) + \partial_2 \left( c_{11}(\bar{w}) - 2c_{02}(\bar{w}) \right) \right] \\
+ \frac{\alpha(\alpha - 1)}{(w_1 - w_2)^2} \left[ c_{20}(\bar{w}) - c_{11}(\bar{w}) + c_{02}(\bar{w}) \right] \right\};$$  

(6.46)

$$c_{10}(\bar{w}) = 2\tau_1 c_{20}(\bar{w}) + \tau_2 c_{11}(\bar{w})$$

$$= -\frac{1}{2} \left\{ 2\partial_1 c_{20}(\bar{w}) + \partial_2 c_{11}(\bar{w}) + \frac{\alpha}{w_1 - w_2} \left( 2c_{20}(\bar{w}) - c_{11}(\bar{w}) \right) \right\};$$  

(6.47)

$$c_{01}(\bar{w}) = \tau_1 c_{11}(\bar{w}) + 2\tau_2 c_{02}(\bar{w})$$

$$= -\frac{1}{2} \left\{ \partial_1 c_{11}(\bar{w}) + 2\partial_2 c_{02}(\bar{w}) + \frac{\alpha}{w_1 - w_2} \left( c_{11}(\bar{w}) - 2c_{02}(\bar{w}) \right) \right\}. $$  

(6.48)

In the case of $K = 2$, the symmetry condition

$$\gamma_{j_1,j_2} = \gamma_{j_2,j_1}$$
does not imply the normalizability conditions\textsuperscript{16}, which read
\[
\sum_{j_1+j_2=d} j_1 \left( \gamma_{j_1,j_2} - \gamma_{j_1, j_2} + \gamma_{j_1, j_2} \right) = 0,
\]
\[
\sum_{j_1+j_2=d} \gamma_{j_1,j_2} - \gamma_{j_1, j_2} + \gamma_{j_1, j_2} = 0.
\]

See equations (B.8) - (B.10). Note that, by the symmetry condition, (B.9) and (B.10) are equivalent, so we can neglect (B.10).

We proceed as before: for each separate value of \(d\) we will list the general form of the highest order coefficients \(c_{20}, c_{11}\) and \(c_{02}\) satisfying the symmetry and normalizability conditions, and then we plug these into equations (6.45) - (6.48) to obtain a basis for the corresponding \((d, K)\) eigenspaces.

\(d = 0\)

For \(d = 0\), the coefficients \(c_{20}, c_{11}\) and \(c_{02}\) are given by
\[
c_{20} = c_{02} = \frac{1}{2} c_{11} = A
\]
for arbitrary \(A\). Taking \(A = 1\) yields
\[
F_{0,2}(\mathbf{w}) = (\bar{w}_1 + \bar{w}_2)^2.
\]

It might seem surprising that \((\bar{w}_1 - \bar{w}_2)^2\) is not also a \((d, K) = (0, 2)\) eigenfunction of the Hamiltonian. However, as can be checked by hand, \((\bar{w}_1 - \bar{w}_2)^2\) does not solve the Schrödinger equation (6.19) for the polynomial \(F\).

\(d = 1\)

For \(d = 1\), the coefficients \(c_{20}, c_{11}\) and \(c_{02}\) are given by
\[
c_{20}(\mathbf{w}) = Aw_1 + Bw_2
\]
\[
c_{11}(\mathbf{w}) = (A + B)(w_1 + w_2)
\]
\[
c_{02}(\mathbf{w}) = Bw_1 + Aw_2
\]
for arbitrary constants \(A\) and \(B\). Taking \(A = 1, B = 0\) and \(A = 0, B = 1\) respectively yields the following basis for the two-dimensional \((d, K) = (1, 2)\) subspace of eigenfunctions
\[
F_{1,2}^{0,0}(\mathbf{w}, \bar{\mathbf{w}}) = -\frac{1}{2} (3 + \alpha)(\bar{w}_1 + \bar{w}_2) + w_1 \bar{w}_1^2 + (w_1 + w_2)\bar{w}_1 \bar{w}_2 + w_2 \bar{w}_2^2,
\]
\[
F_{1,2}^{0,1}(\mathbf{w}, \bar{\mathbf{w}}) = -\frac{1}{2} (1 - \alpha)(\bar{w}_1 + \bar{w}_2) + w_2 \bar{w}_2^2 + (w_1 + w_2)\bar{w}_1 \bar{w}_2 + w_1 \bar{w}_1^2.
\]

\(d = 2\)

For \(d = 2\), the coefficients \(c_{20}, c_{11}\) and \(c_{02}\) are given by
\[
c_{20}(\mathbf{w}) = Aw_1^2 + Bw_1 w_2 + Cw_2^2
\]
\[
c_{11}(\mathbf{w}) = Dw_1^2 + 2(A + B + C - D)w_1 w_2 + Dw_2^2
\]
\[
c_{02}(\mathbf{w}) = Cw_1^2 + Bw_1 w_2 + Aw_2^2
\]
for arbitrary constants \(A, B, C,\) and \(D\). Setting \(A = 1, B = C = D = 0; A = C = D = 0, B = 1;\)
\(A = B = D = 0, C = 1;\) and \(A = B = C = 0, D = 1\) respectively, yields the following basis for the

\textsuperscript{16}But it does reduce the number of normalizability conditions to two.
four-dimensional \((d, K) = (2, 2)\) subspace of eigenfunctions

\[
F_{2,2}^{1,0,0,0}(\mathbf{w}, \mathbf{w}) = \frac{1}{4}(6 + 5\alpha + \alpha^2) - (3 + \alpha)(w_1 \bar{w}_1 + w_2 \bar{w}_2) + (w_1 \bar{w}_1 + w_2 \bar{w}_2)^2,
\]

\[
F_{2,2}^{0,1,0,0}(\mathbf{w}, \mathbf{w}) = \frac{1}{2} - (w_1 + w_2)(\bar{w}_1 + \bar{w}_2) + w_1 w_2(\bar{w}_1 + \bar{w}_2)^2,
\]

\[
F_{2,2}^{0,0,1,0}(\mathbf{w}, \mathbf{w}) = \frac{1}{4}(2 + \alpha + \alpha^2) - w_1 \bar{w}_1 - w_2 \bar{w}_2 + \alpha(w_2 \bar{w}_1 + w_1 \bar{w}_2) + (w_2 \bar{w}_1 + w_1 \bar{w}_2)^2,
\]

\[
F_{2,2}^{0,0,0,1}(\mathbf{w}, \mathbf{w}) = -\frac{1}{4}(2 + 5\alpha + \alpha^2) + \frac{1}{2}(2 + \alpha)(w_1 - w_2)(\bar{w}_1 + \bar{w}_2) + (w_1 - w_2)^2\bar{w}_1 \bar{w}_2.
\]

The table below shows all the \(K = 2\) eigenfunctions constructed above, together with the angular momentum eigenvalues \(\hbar(-\alpha - d + 2)\) of the corresponding wavefunctions \(\psi_{d,2}\).

<table>
<thead>
<tr>
<th>(d)</th>
<th>(F)</th>
<th>(L_{d,2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(F_{0,2}(\mathbf{w}, \mathbf{w}) = (\bar{w}_1 + \bar{w}_2)^2)</td>
<td>(\hbar(-\alpha + 2))</td>
</tr>
<tr>
<td>1</td>
<td>(F_{1,2}^{1,0,0,0}(\mathbf{w}, \mathbf{w}) = -\frac{1}{2}(3 + \alpha)(\bar{w}_1 + \bar{w}_2) + w_1 \bar{w}_1^2 + (w_1 + w_2)(\bar{w}_1 \bar{w}_2 + w_2 \bar{w}_2)^2)</td>
<td>(\hbar(-\alpha + 1))</td>
</tr>
<tr>
<td>1</td>
<td>(F_{1,2}^{1,0,0,0}(\mathbf{w}, \mathbf{w}) = -\frac{1}{2}(1 - \alpha)(\bar{w}_1 + \bar{w}_2) + w_2 \bar{w}_1^2 + (w_1 + w_2)(\bar{w}_1 \bar{w}_2 + w_1 \bar{w}_2)^2)</td>
<td>(\hbar(-\alpha + 1))</td>
</tr>
<tr>
<td>2</td>
<td>(F_{2,2}^{1,0,0,0}(\mathbf{w}, \mathbf{w}) = \frac{1}{4}(6 + 5\alpha + \alpha^2) - (3 + \alpha)(w_1 \bar{w}_1 + w_2 \bar{w}_2) + (w_1 \bar{w}_1 + w_2 \bar{w}_2)^2)</td>
<td>(-\hbar\alpha)</td>
</tr>
<tr>
<td>2</td>
<td>(F_{2,2}^{1,0,0,0}(\mathbf{w}, \mathbf{w}) = -\frac{1}{2} - (w_1 + w_2)(\bar{w}_1 + \bar{w}_2) + w_1 w_2(\bar{w}_1 + \bar{w}_2)^2)</td>
<td>(-\hbar\alpha)</td>
</tr>
<tr>
<td>2</td>
<td>(F_{2,2}^{0,0,1,0}(\mathbf{w}, \mathbf{w}) = \frac{1}{4}(2 + \alpha + \alpha^2) - w_1 \bar{w}_1 - w_2 \bar{w}_2 + \alpha(w_2 \bar{w}_1 + w_1 \bar{w}_2) + (w_2 \bar{w}_1 + w_1 \bar{w}_2)^2)</td>
<td>(-\hbar\alpha)</td>
</tr>
<tr>
<td>2</td>
<td>(F_{2,2}^{0,0,0,1}(\mathbf{w}, \mathbf{w}) = -\frac{1}{4}(2 + 5\alpha + \alpha^2) + \frac{1}{2}(2 + \alpha)(w_1 - w_2)(\bar{w}_1 + \bar{w}_2) + (w_1 - w_2)^2\bar{w}_1 \bar{w}_2)</td>
<td>(-\hbar\alpha)</td>
</tr>
</tbody>
</table>

This concludes our discussion on \(N\)-particle anyon eigenfunctions. Below is the conclusion of this part of the thesis — i.e. the part on anyonic wavefunctions — and an overview of the obtained results. In the final chapter, we shall compare our results to those obtained by other physicists, and present an outlook for possible further research.

**Conclusion**

Using a smart ansatz we have found energy eigenstates \(\psi_{N,d,K}\) for energies \(\frac{1}{2}\hbar \omega_{C}^2(N + 2K)\) for each particle number \(N\), excitation energy \(K\) and \(w\) degree \(d\). We realized that not all constructed eigenfunctions are normalizable, so we had to impose extra normalizability conditions, which we were able to formulate explicitly for the case of \(N = 2\) particles.

Moreover, we showed that the energy eigenstates we found are simultaneously angular momentum eigenstates with eigenvalue

\[
L\psi_{N,d,K} = \hbar \left(-\frac{\alpha}{2} N(N - 1) - d + K\right) \psi_{N,d,K}.
\]

Comparing to the one-particle angular momentum operator, which, in terms of the one-particle ladder operators \(a, a^\dagger, b\) and \(b^\dagger\) — see Appendix A.1 — can be computed to be equal to

\[
L = \hbar(a^\dagger a - b^\dagger b) = \hbar(n - m),
\]

where \(n\) labels the energy and \(m\) the guiding center of a single charged particle in a uniform perpendicular magnetic field, it seems plausible that the degree \(d\) is related to some kind of position\(^\text{17}\) of the \(N\)-anyon state \(\psi_{N,d,K}\).

\(^\text{17}\)Perhaps the center of mass position?
Additionally, through the angular momentum operator we have obtained a theoretical way to measure the statistical angle $\alpha$ directly for any of the $N$-particle anyon states that reside in the linear span of the $\psi_{N,d,K}$ energy eigenstates.

As can be seen from the tables in Section 6.2.7, the parameters $N$, $d$ and $K$ do not uniquely specify a state, so the labelling $\psi_{N,d,K}$ that we use for any state with $N$ particles, excitation energy $K$ and $w$ degree $d$ refers to a subspace of states, rather than a single state. A possible direction for further research would be to find additional operators that commute with the Hamiltonian, such that their eigenvalues do constitute a complete set, in the sense that specifying the eigenvalues completely classifies a state. However, before going into this question, a more important question should be addressed first: we do not know whether we have found all energy eigenstates. This question will be addressed in the next chapter.
Chapter 7

Conclusion and Outlook

In this thesis we have been discussing two related topics: Casimir operators in the fractional quantum Hall effect (FQHE), and multi-particle wavefunctions for non-interacting charged anyons in a uniform perpendicular magnetic field.

The first topic we studied to gain insight in the FQHE fluid in the low temperature limit, where all dynamics occur within a single Landau level, and are therefore described by the algebra of projected density operators. In order to obtain a better understanding of this algebra, we sought to find and compute Casimir operators, which are specific operators that commute with all the elements of the algebra.

In a first attempt, we computed the second order Casimir operator for the FQHE on the infinite plane. The obtained expression is rife with infinities, which led us to consider Casimir operators for the FQHE on the torus instead. The torus turned out to be computationally friendlier than its planar counterpart, enabling us to compute a Casimir operator \( \zeta_{r+1} \) for each \( r \in \{1, 2, \ldots, N_B - 1\} \):

\[
\zeta_{r+1} \mid_{F^{N_{el}}_{N_{el}}} = \frac{N_{el}}{(2\pi l_B^2)^r} \left[ \left(1 + N_B\right) \frac{1 - (-N_{el})^r}{1 + N_{el}} + (-N_{el})^r \right].
\]

The obtained expressions for the Casimir operators are in agreement with Dixmier’s lemma\(^1\), which says that any central element of an associative algebra over \( \mathbb{C} \) is represented by a scalar when the algebra acts irreducibly on a vector space with a countable basis. Indeed, in the FQHE, the algebra of projected density operators acts on Fock space, which is a direct sum of countably infinite-dimensional fixed-particle-number subspaces, on each of which the algebra acts irreducibly. Equation (7.1) gives the scalar that represents the \((r + 1)\)th order Casimir \( \zeta_{r+1} \) when acting on the irreducible subspace \( F^*_{N_{el}} \subset F^* \) of constant particle number \( N_{el} \).

Many quantum numbers in physics, such as total spin and total angular momentum, arise as a scalar that represents a Casimir operator acting on an irreducible vector space. It was our hope that, similar to total angular momentum and total spin, the scalars on the RHS of equation (7.1) would also be physically significant. However, thus far, the physical meaning of the scalars representing the Casimir operators remains unclear.

We then proceeded on a tangent, and focused on the elementary quasi-particle excitations of the FQHE fluid, which are of anyonic nature. Because these quasi-particle excitations are fractionally charged, we expected that they feel the external magnetic field that causes the FQHE to occur. This led us to an effective description of non-interacting charged anyons in a uniform perpendicular magnetic field. By effective, we mean a description that takes the anyon as a fundamental particle, instead of a collective excitation. It is at this particular point that our approach deviates from

\(^1\)An infinite dimensional generalization of Schur’s lemma for associative algebras over an uncountable and algebraically closed field.
the standard approach to anyons: instead of thinking of anyons as interacting charged boson-flux tube composites, we treat anyons as non-interacting charged particles with actual anyonic exchange statistics\(^2\).

Using a general ansatz inspired by the single particle wavefunctions for a charged particle in a uniform perpendicular magnetic field, we constructed a plethora of exact energy eigenfunctions, which we simultaneously diagonalized with respect to the (\(z\) component of) the total angular momentum operator relative to the origin. In contrast to the energy eigenvalues, the angular momentum eigenvalues that we found turned out to be explicitly dependent on the statistical angle \(\alpha\). This realization brought us to the conclusion that the angular momentum operator could possibly be used to design a method for measuring the anyonic nature of the quasi-particle excitations directly.

The most relevant question that we have not yet discussed, is whether we have found all energy eigenfunctions of the system of \(N\) non-interacting charged anyons in a uniform perpendicular magnetic field. The answer is given by several sources, such as Khare [13], which discusses mostly the two-anyon system in a perpendicular magnetic field, and, Johnson and Canright [12], which produces multi-particle anyon wavefunctions using the single particle ladder operators; but, the most extensive discussion on \(N\) anyons in a perpendicular magnetic field is given by Lerda [17], which we now discuss in some detail.

Lerda uses an approach that is comparable to ours. However, the ansatz he considers is more general, in the sense that his statistical prefactor is also allowed to contain an anti-holomorphic part. This leads to what Lerda calls type I and type II wavefunctions, corresponding to statistical prefactors of the form

\[
S_I(w, \bar{w}, \theta) = \prod_{j > i=1}^N (w_i - w_j)^\alpha
\]

and

\[
S_{II}(\bar{w}, \theta) = \prod_{j > i=1}^N (\bar{w}_i - \bar{w}_j)^{2-\alpha}
\]

respectively. The statistical prefactor used in this thesis is exactly \(S_I\). As a consequence, all wavefunctions with statistical prefactor \(S_{II}\) do not appear in this thesis. We can therefore immediately conclude that we have not found all energy eigenfunctions. Also, on a sidenote, the statistical phase \(\alpha\) that is absent in the energy eigenvalues of type I states, does appear in the energy eigenvalues of type II states, which implies that the statistical angle should in principle be measurable from the energy spectrum alone.

Does this mean that our approach completely falls short with respect to Lerda’s? Let us compare Lerda’s results for type I wavefunctions to ours. Lerda constructs explicit formulas for anyonic wavefunctions in Section 6.3 of his book [17]. The eigenfunctions he obtains are of the form

\[
\psi_I(w, \bar{w}, \theta) = S_I(w, \theta) E(w, \bar{w}) P^{d_s}(w) L_{n+1}^{d_s+N-1+\frac{N(N-1)}{2}} \left( -2 \sum_{i=1}^N w_i \bar{w}_i \right),
\]

where \(E(w, \bar{w}) = \exp \left( -\sum_{i=1}^N w_i \bar{w}_i \right)\) is the ubiquitous exponential occurring in all wavefunctions constructed in this thesis, \(P^{d_s}(w)\) is an arbitrary symmetric polynomial of degree \(d_s\), and \(L_n^d\) is the \(n\)th associated Laguerre polynomial. Comparing to our energy eigenfunctions, we see that \(n\) is equal to the excitation energy \(K\), and the sum \(n + d_s\) is equal to the degree \(d\), because the Laguerre polynomials also contain factors of \(w\). It turns out that, all solutions obtained by Lerda that fall within the range of the \(N = 2\) particle wavefunctions that we computed explicitly in Section 6.2.7, can be written as linear combinations of the solutions that we have found. Specifically, the following table

\[^2\text{In other words, we are using Wilczek’s singular gauge transformation to gauge away the vector potentials due to the flux tubes — see Wilczek [25] and [26] for details.}\]
shows how, for different choices of $n$ and $d_+$, $P^{d_+}(\mathbf{w})L_n^{d_++1+\alpha}(-2w_1\bar{w}_1 - 2w_2\bar{w}_2)$ can be written as a linear combination of $F_{d,K}(\mathbf{w}) = F_{n+d_+,n}(\mathbf{w})$'s. Note that, for $n = 0$, Lerda’s solution consists of symmetric polynomials in $\mathbf{w}$, which agrees with our $K = 0$ solutions.

**Comparison of Lerda’s wavefunctions to the wavefunctions constructed in Section 6.2.7**

<table>
<thead>
<tr>
<th>$n$</th>
<th>$d_+$</th>
<th>$P^{d_+}F_n^{d_++1+\alpha}$</th>
<th>lin. comb. of $F_{n+d_+,n}$’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$2 + \alpha - 2(w_1\bar{w}_1 + w_2\bar{w}_2)$</td>
<td>$-2F_{1,1}^{1,0}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$(w_1 + w_2)[3 + \alpha - 2(w_1\bar{w}_1 + w_2\bar{w}_2)]$</td>
<td>$-2(F_{2,1}^{1,0,0} + F_{2,1}^{0,1,0})$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$(w_1 + w_2)^2[4 + \alpha - 2(w_1\bar{w}_1 + w_2\bar{w}_2)]$</td>
<td>$-2(F_{3,1}^{1,0,0,0} + 2F_{3,1}^{0,1,0,0} + F_{3,1}^{0,0,1,0})$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$(w_1 - w_2)^2[4 + \alpha - 2(w_1\bar{w}_1 + w_2\bar{w}_2)]$</td>
<td>$-2(F_{3,1}^{1,0,0,0} - 2F_{3,1}^{0,1,0,0} + F_{3,1}^{0,0,1,0})$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$\frac{(\alpha+2)(\alpha+3)}{2} - 2(\alpha + 3)(w_1\bar{w}_1 + w_2\bar{w}_2)$</td>
<td>$F_{2,2}^{1,0,0,0}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+2(w_1\bar{w}_1 + w_2\bar{w}_2)^2$</td>
<td></td>
</tr>
</tbody>
</table>

Hence, we predict that all the wavefunctions found by Lerda are also obtained by following our construction. Moreover, our approach is more thorough, because it leads to additional wavefunctions that Lerda does not find. We conclude that the method for constructing anyonic eigenfunctions construction. Moreover, our approach is more thorough, because it leads to additional wavefunctions that Lerda does not find. We conclude that the method for constructing anyonic eigenfunctions discussed in this thesis is of added value compared to the existing methods for generating anyonic eigenfunctions.

One might be tempted to conjecture that, if our approach were to be applied also to Lerda’s type II ansatz, all energy eigenfunctions could be found. However, as Lerda explains, even his ansatz is not general enough, as there exist three-particle statistical prefactors not of the form $S_{II}$ or $S_{II}$. Instead, is seems that there are more promising approaches than those based on trying different types of statistical prefactors. For example, one could try to look for the complete algebra of operators that is responsible for the degeneracy of the energy spectrum, which could possibly be used to generate solutions to the Schrödinger equation.

In the long term, there is plenty of research to be done in the field of charged anyons in a uniform perpendicular magnetic field. For example, we could include anyon interactions in the theory, or, we could try to repeat the setup of Chapter 6 for non-abelian anyons, in which case the $N$-particle anyon wavefunctions must be considered as functions on the full universal cover of the anyon configuration space, instead of the abelian cover. On a more hypothetical level, by comparing the Laughlin ground state wavefunction to the $d = 0$ anyon ground state wavefunction for quasi-holes, where $\alpha$ is equal to the filling factor $\nu$,

$$
\psi_{\text{Anyon}}(\mathbf{w}, \bar{\mathbf{w}}, \theta) = \left[ \prod_{j>i=1}^{N} (w_i - w_j)^{\nu} \right] \exp \left( - \sum_{i=1}^{N} w_i \bar{w}_i \right)
$$

$$
\psi_{\text{Laughlin}}(\mathbf{w}, \bar{\mathbf{w}}) = \left[ \prod_{j>i=1}^{N} (w_i - w_j)^{\frac{1}{2}} \right] \exp \left( - \sum_{i=1}^{N} w_i \bar{w}_i \right)
$$

it seems as if there could be some kind of duality between the two, where $\nu \leftrightarrow \frac{1}{2}$. But, before embarking on any of the above topics, it is noteworthy to emphasize that even the case of non-interacting charged anyons in a uniform perpendicular magnetic field is far from completely solved. Looking at the wavefunctions constructed in this thesis, it seems that there is a lot of hidden structure ready to be found, which should provide more than enough challenge for future anyon-specialists to sink their teeth into.

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3 Analogous to the guiding centre operators for a charged particle in a uniform perpendicular magnetic field.
Appendix A

The Quantum Hall Effects: Addendum

A.1 Overview of the Single Particle Operators

Definitions
Presented here is an overview of the one-particle operators for a particle of charge $-e$ and mass $m$ confined to two dimensions in a uniform perpendicular magnetic field. All operators are defined in terms of the fundamental position and momentum operators acting on the one-particle Hilbert space $L^2(\mathbb{R}^2, \mathbb{C})$. In the position representation, they are given by

$$\hat{x} = x, \quad \hat{p}_x = -i\hbar \frac{\partial}{\partial x},$$
$$\hat{y} = y, \quad \hat{p}_y = -i\hbar \frac{\partial}{\partial y},$$

where $\hat{x} = x$ means that the operator $\hat{x}$ acts on $\psi(x)$ by multiplying with the variable $x$, and likewise for $y$. The magnetic vector potential $A(x,y)$ is an analytic function in the variables $x$ and $y$ – it is actually polynomial in the Landau and Symmetric gauge, so $\hat{A} := A(\hat{x}, \hat{y})$ makes sense as an operator. We will drop the hats distinguishing the operator from the corresponding variable from now on.

Using the operators $r = (x, y)$, $p = (p_x, p_y)$ and $A(r)$ defined above we define the mechanical momentum operator $\Pi$ as

$$\Pi := p + eA.$$

Classically, an electron of mass $m$ and charge $-e$ confined to the $x$-$y$ plane subject to a perpendicular magnetic field $B = Be_z$ moves in (anti-clockwise – when viewed from the $+z$ direction) circles with angular frequency

$$\omega_C := \frac{eB}{m},$$

called the *cyclootron frequency*. Since the radius $\eta$ and the velocity $v$ of the electron's orbit are related by $v = \frac{\Pi}{m} = \omega_C \eta$, we have

$$\eta = \frac{\Pi}{eB} \times e_z.$$

We use the above equation to define the operator $\eta$, which, in the classical picture, corresponds to the radial vector pointing from the centre of the cyclotroon orbit $R$ to the position of the electron $r$, i.e. $r = R + \eta$. Either $\eta$ or $\Pi$ can be used to measure the electrons kinetic energy.
Having defined the operator $\eta$, we can make sense of the operator $R$ corresponding to the classical guiding centre of the cyclotron orbit,

$$R := r - \eta.$$

### Commutation relations

Using the basic commutation relations

$$[x, p] = i\hbar = [y, p_y],$$

with all remaining commutators of $r$ and $p$ vanishing, the other commutators will be computed below.

As computed in sections 3.1.1 and 3.1.2, we have

$$[X, Y] = ilB^2 \quad \text{and} \quad [\Pi_x, \Pi_y] = -i\frac{\hbar^2}{l_B^2}.$$  \hspace{1cm} (A.1)

Now, for $r$ and $\Pi$, we have, for example, $[x, \Pi_x] = [x, p_x + eA_x] = [x, p_x] = i\hbar$. In general,

$$[x, \Pi_x] = i\hbar, \quad [x, \Pi_y] = 0, \quad [y, \Pi_x] = 0, \quad [y, \Pi_y] = i\hbar.$$  \hspace{1cm} (A.3)

Likewise, for $R$ and $\Pi$, we have, for example,

$$[X, \Pi_x] = i\hbar, \quad [X, \Pi_y] = 0, \quad [Y, \Pi_x] = 0, \quad [Y, \Pi_y] = 0.$$  \hspace{1cm} (A.4)

which shows that indeed $X$ and $Y$ commute with the free particle Hamiltonian

$$H = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2).$$

Finally, for $r$ and $R$, we have, for example, $[x, Y] = [x, y - \eta_y] = +\frac{1}{eB}[x, \Pi_x] = i\frac{\hbar}{eB} = il_B^2$. In general,

$$[x, X] = 0, \quad [x, Y] = il_B^2, \quad [y, X] = -il_B^2, \quad [y, Y] = 0.$$  \hspace{1cm} (A.2)

### Ladder operators

From equation (A.2), we see that $(X, Y)$ and $(\Pi_x, \Pi_y)$ are canonically conjugate observables. This suggests introducing corresponding ladder operators $a$, $a^\dagger$, $b$ and $b^\dagger$,

$$a := \frac{l_B}{\sqrt{2l_B}}(\Pi_x - i\Pi_y), \quad a^\dagger := \frac{l_B}{\sqrt{2l_B}}(\Pi_x + i\Pi_y)$$

$$\Pi_x = \frac{\hbar}{\sqrt{2l_B}}(a^\dagger + a), \quad \Pi_y = \frac{\hbar}{\sqrt{2l_B}}(a^\dagger - a),$$

and

$$b := \frac{1}{\sqrt{2l_B}}(X + iY), \quad b^\dagger := \frac{1}{\sqrt{2l_B}}(X - iY)$$

$$X = \frac{l_B}{\sqrt{2}}(b^\dagger + b), \quad Y = \frac{i\hbar}{\sqrt{2}}(b^\dagger - b).$$  \hspace{1cm} (A.5)

We can also express $\eta$ in terms of $a$ and $a^\dagger$,

$$a = \frac{i}{\sqrt{2l_B}}(-\eta_x + i\eta_y), \quad a^\dagger = \frac{i}{\sqrt{2l_B}}(\eta_x + i\eta_y),$$

$$\eta_x = \frac{i\hbar}{\sqrt{2}}(-a^\dagger + a), \quad \eta_y = \frac{-i\hbar}{\sqrt{2}}(a^\dagger + a).$$  \hspace{1cm} (A.6)
A.2 Fourier Transform of the Coulomb Potential in 2D

The Coulomb interaction potential $U(r)$ gives the potential energy two electrons have when separated by a distance $r = |r|$. It is defined by the following formula,

$$U(r) := \frac{e^2}{4\pi\epsilon_0} \frac{1}{r}.$$ 

In the usual case, where $U$ is seen as a function on $\mathbb{R}^3$, $U$ is not in any $L^p(\mathbb{R}^3)$ space with $1 \leq p \leq 2$, so the Coulomb potential can technically not be Fourier transformed. The way physicists circumvent this problem is by Fourier transforming the Yukawa potential, which is the Coulomb potential multiplied by a damping term $e^{-ur}$, and then taking the limit $u \to 0$. In two dimensions, however, no such trickery is needed as the Coulomb potential can be Fourier transformed directly.

For a fixed wavevector $q$, we will adopt polar coordinates such that $q = qe_x$.

$$U(q) = \int_{\mathbb{R}^2} d\mathbf{r} \frac{e^2}{4\pi\epsilon_0} \frac{1}{r} e^{-iq \cdot \mathbf{r}} = \frac{e^2}{4\pi\epsilon_0} \int_0^{\infty} dr \int_0^{2\pi} d\theta \frac{1}{r} e^{-iqr\cos\theta}$$

$$= \frac{e^2}{4\pi\epsilon_0} \int_0^{\infty} dr 2\pi J_0(qr) = \frac{e^2}{2\epsilon_0 \ |q|} \int_0^{\infty} ds J_0(s)$$

$$= \frac{e^2}{2\epsilon_0 \ |q|},$$

where we have used that the zeroth order Bessel function of the first kind $J_0$ is normalized to 1.
Appendix B

Anyon Wavefunctions: Addendum

B.1 Solution to the Recurrence Relation

In this part of the appendix, we will elaborate on the solution (6.23) to the recurrence relation (6.21), given by

\[(K - \text{Ord}) c_{i_1, \ldots, i_N}(w) = \sum_{k=1}^{N} (i_k + 1) \tau_k c_{i_1, \ldots, i_k-1, i_k+1, \ldots, i_N}(w) \text{ for } 0 \leq i_1 + \ldots + i_N \leq K. \]  \hfill (B.1)

As noted in Remark 6.13, the highest order coefficients \(c_{i_1, \ldots, i_N}\) with \(i_1 + \ldots + i_N = K\) are free to choose, and, using (B.1), the lower order coefficients can be expressed in terms of the highest order coefficients. Indeed, if we rewrite (B.1) as

\[c_{i_1, \ldots, i_N} = \sum_{k=1}^{N} \frac{i_k + 1}{K - (i_1 + \ldots + i_N)} \tau_k c_{i_1, \ldots, i_k-1, i_k+1, \ldots, i_N}(w) \text{ for } 0 \leq i_1 + \ldots + i_N \leq K - 1, \]  \hfill (B.2)

then it becomes more obvious that we can express a fixed coefficient of order \(R\) in terms of all the neighboring coefficients of order \(R + 1\). Hence, starting from the highest order coefficients, and then working our way down using (B.2) repeatedly, we can express every coefficient in terms of the highest order coefficients.

In the next couple of pages, using Mathematica, we have expressed all the lower order coefficients in terms of the highest order coefficients for the cases of \(N = 2, K = 1, 2, 3, 4, 5, 6, 7; N = 3, K = 1, 2, 3, 4, 5;\) and \(N = 4, K = 1, 2, 3\). As can be verified directly for any of the expressions computed in the enclosed mathematica file, all cases are in agreement with formula (6.23), wherein the lower order coefficient \(c_{i_1, \ldots, i_N}'\) is expressed in terms of the highest order coefficients \(c_{i_1, \ldots, i_N}\) as

\[c_{i_1, \ldots, i_N}' = \sum_{i_i + \ldots + i_N = K} i_1^{i_1} \cdots i_N^{i_N} \tau_1^{i_1 - i_1'} \cdots \tau_N^{i_N - i_N'} c_{i_1, \ldots, i_N}, \]  \hfill (B.3)

where we use the convention that \(\binom{i}{j} = 0\) whenever \(i < j\). The author is of the opinion that this is convincing evidence that (B.3) is indeed the solution to the recurrence relation (B.1).

The mathematica files for the cases \(N = 2, K = 1, 2, 3, 4, 5, 6, 7; N = 3, K = 1, 2, 3, 4, 5;\) and \(N = 4, K = 1, 2, 3\) can be found below.

\textbf{Notation:} instead of writing \(\tau_1, \tau_2, \tau_3\) and \(\tau_4\), we have opted for \(A, B, C,\) and \(D\) respectively, because it adds to the readability of the Mathematica file, and also because mathematica orders the latter characters more naturally.
(* Recurrence relation for two particles *)
\[ c[K, i, j] := (i+1)/(K-i-j) \cdot c[i+1, j] + (j+1)/(K-i-j) \cdot c[i, j+1] \]

(* K=1 *)

\[ c[0, 0] = \text{Simplify}[c[1, 0, 0]] \]
\[ Bc[0, 1] + Ac[1, 0] \]

(* K=2 *)

\[ c[1, 0] = \text{Simplify}[c[2, 1, 0]] \]
\[ c[0, 1] = \text{Simplify}[c[2, 0, 1]] \]
\[ Bc[1, 1] + 2Ac[2, 0] \]
\[ 2Bc[0, 2] + Ac[1, 1] \]
\[ c[0, 0] = \text{Simplify}[c[2, 0, 0]] \]
\[ B^2 c[0, 2] + ABC[1, 1] + A^2 c[2, 0] \]

(* K=3 *)

\[ c[2, 0] = \text{Simplify}[c[3, 2, 0]] \]
\[ c[1, 1] = \text{Simplify}[c[3, 1, 1]] \]
\[ c[0, 2] = \text{Simplify}[c[3, 0, 2]] \]
\[ Bc[2, 1] + 3Ac[3, 0] \]
\[ 2(Bc[1, 2] + Ac[2, 1]) \]
\[ 3Bc[0, 3] + Ac[1, 2] \]
\[ c[1, 0] = \text{Simplify}[c[3, 1, 0]] \]
\[ c[0, 1] = \text{Simplify}[c[3, 0, 1]] \]
\[ B^2 c[1, 2] + 2ABC[2, 1] + 3A^2 c[3, 0] \]
\[ 3B^2 c[0, 3] + 2ABC[1, 2] + A^2 c[2, 1] \]
\[ c[0, 0] = \text{Simplify}[c[3, 0, 0]] \]
\[ B^3 c[0, 3] + AB^2 c[1, 2] + A^2 Bc[2, 1] + A^3 c[3, 0] \]

(* K=4 *)

\[ c[3, 0] = c[4, 3, 0] \]
\[ c[2, 1] = c[4, 2, 1] \]
\[ c[1, 2] = c[4, 1, 2] \]
\[ c[0, 3] = c[4, 0, 3] \]
\[ Bc[3, 1] + 4Ac[4, 0] \]
\[ 2Bc[2, 2] + 3Ac[3, 1] \]
\[ 3Bc[1, 3] + 2Ac[2, 2] \]
\[ 4Bc[0, 4] + Ac[1, 3] \]
B.1. SOLUTION TO THE RECURRENCE RELATION

\[ c[2, 0] = \text{Simplify}[c[4, 2, 0]] \]
\[ c[1, 1] = \text{Simplify}[c[4, 1, 1]] \]
\[ c[0, 2] = \text{Simplify}[c[4, 0, 2]] \]
\[ B^2 c[2, 2] + 3 A B c[3, 1] + 6 A^2 c[4, 0] \]
\[ 3 B^2 c[1, 3] + 4 A B c[2, 2] + 3 A^2 c[3, 1] \]
\[ 6 B^2 c[0, 4] + 3 A B c[1, 3] + A^2 c[2, 2] \]
\[ c[0, 1] = \text{Simplify}[c[4, 0, 1]] \]
\[ c[1, 0] = \text{Simplify}[c[4, 1, 0]] \]
\[ c[0, 0] = \text{Simplify}[c[4, 0, 0]] \]

(* K=5 *)

\[ c[4, 0] = c[5, 4, 0] \]
\[ c[3, 1] = c[5, 3, 1] \]
\[ c[2, 2] = c[5, 2, 2] \]
\[ c[1, 3] = c[5, 1, 3] \]
\[ c[0, 4] = c[5, 0, 4] \]
\[ B c[4, 1] + 5 A c[5, 0] \]
\[ 2 B c[3, 2] + 4 A c[4, 1] \]
\[ 3 B c[2, 3] + 3 A c[3, 2] \]
\[ 4 B c[1, 4] + 2 A c[2, 3] \]
\[ 5 B c[0, 5] + A c[1, 4] \]
\[ c[3, 0] = \text{Simplify}[c[5, 3, 0]] \]
\[ c[2, 1] = \text{Simplify}[c[5, 2, 1]] \]
\[ c[1, 2] = \text{Simplify}[c[5, 1, 2]] \]
\[ c[0, 3] = \text{Simplify}[c[5, 0, 3]] \]
\[ B^2 c[3, 2] + 4 A B c[4, 1] + 10 A^2 c[5, 0] \]
\[ 6 B^2 c[1, 4] + 6 A B c[2, 3] + 3 A^2 c[3, 2] \]
\[ 10 B^2 c[0, 5] + 4 A B c[1, 4] + A^2 c[2, 3] \]
\[ c[2, 0] = \text{Simplify}[c[5, 2, 0]] \]
\[ c[1, 1] = \text{Simplify}[c[5, 1, 1]] \]
\[ c[0, 2] = \text{Simplify}[c[5, 0, 2]] \]
\[ c[1, 0] = \text{Simplify}[c[5, 1, 0]] \]
\[ c[0, 1] = \text{Simplify}[c[5, 0, 1]] \]
B.1. SOLUTION TO THE RECURRENCE RELATION

\[ c[0, 0] = \text{Simplify}[c[5, 0, 0]] \]

(* K=6 *)

\[ c[5, 0] = c[6, 5, 0] \]
\[ c[4, 1] = c[6, 4, 1] \]
\[ c[3, 2] = c[6, 3, 2] \]
\[ c[2, 3] = c[6, 2, 3] \]
\[ c[1, 4] = c[6, 1, 4] \]
\[ c[0, 5] = c[6, 0, 5] \]
\[ B c[5, 1] + 6 A c[6, 0] \]
\[ 2 B c[4, 2] + 5 A c[5, 1] \]
\[ 3 B c[3, 3] + 4 A c[4, 2] \]
\[ 4 B c[2, 4] + 3 A c[3, 3] \]
\[ 5 B c[1, 5] + 2 A c[2, 4] \]
\[ 6 B c[0, 6] + A c[1, 5] \]
\[ c[4, 0] = \text{Simplify}[c[6, 4, 0]] \]
\[ c[3, 1] = \text{Simplify}[c[6, 3, 1]] \]
\[ c[2, 2] = \text{Simplify}[c[6, 2, 2]] \]
\[ c[1, 3] = \text{Simplify}[c[6, 1, 3]] \]
\[ c[0, 4] = \text{Simplify}[c[6, 0, 4]] \]
\[ B^5 c[4, 2] + 5 A B c[5, 1] + 15 A^2 c[6, 0] \]
\[ 15 B^2 c[0, 6] + 5 A B c[1, 5] + A^2 c[2, 4] \]
\[ c[3, 0] = \text{Simplify}[c[6, 3, 0]] \]
\[ c[2, 1] = \text{Simplify}[c[6, 2, 1]] \]
\[ c[1, 2] = \text{Simplify}[c[6, 1, 2]] \]
\[ c[0, 3] = \text{Simplify}[c[6, 0, 3]] \]
\[ B^5 c[3, 3] + 4 A B^4 c[4, 2] + 10 A^2 B c[5, 1] + 20 A^3 c[6, 0] \]
\[ c[2, 0] = \text{Simplify}[c[6, 2, 0]] \]
\[ c[1, 1] = \text{Simplify}[c[6, 1, 1]] \]
\[ c[0, 2] = \text{Simplify}[c[6, 0, 2]] \]
B.1. SOLUTION TO THE RECURRENCE RELATION

\[
c[1, 0] = \text{Simplify}[c[6, 1, 0]]
c[0, 1] = \text{Simplify}[c[6, 0, 1]]
\]


\[
c[0, 0] = \text{Simplify}[c[6, 0, 0]]
\]


(* K=7 *)

\[
c[6, 0] = c[7, 6, 0]
c[5, 1] = c[7, 5, 1]
c[4, 2] = c[7, 4, 2]
c[3, 3] = c[7, 3, 3]
c[2, 4] = c[7, 2, 4]
c[1, 5] = c[7, 1, 5]
c[0, 6] = c[7, 0, 6]
\]

B c[6, 1] + 7 A c[7, 0]

2 B c[5, 2] + 6 A c[6, 1]

3 B c[4, 3] + 5 A c[5, 2]

4 B c[3, 4] + 4 A c[4, 3]

5 B c[2, 5] + 3 A c[3, 4]

6 B c[1, 6] + 2 A c[2, 5]

7 B c[0, 7] + A c[1, 6]

\[
c[5, 0] = \text{Simplify}[c[7, 5, 0]]
c[4, 1] = \text{Simplify}[c[7, 4, 1]]
c[3, 2] = \text{Simplify}[c[7, 3, 2]]
c[2, 3] = \text{Simplify}[c[7, 2, 3]]
c[1, 4] = \text{Simplify}[c[7, 1, 4]]
c[0, 5] = \text{Simplify}[c[7, 0, 5]]
\]

B c[5, 2] + 6 A B c[6, 1] + 21 A^2 c[7, 0]

3 B c[4, 3] + 10 A B c[5, 2] + 15 A^2 c[6, 1]

2 (3 B c[3, 4] + 6 A B c[4, 3] + 5 A^2 c[5, 2])

2 (5 B c[2, 5] + 6 A B c[3, 4] + 3 A^2 c[4, 3])


21 B c[0, 7] + 6 A B c[1, 6] + A^2 c[2, 5]
B.1. SOLUTION TO THE RECURRENCE RELATION

\[
\begin{align*}
\text{c}[4, 0] &= \text{Simplify}[\text{c}[7, 4, 0]] \\
\text{c}[3, 1] &= \text{Simplify}[\text{c}[7, 3, 1]] \\
\text{c}[2, 2] &= \text{Simplify}[\text{c}[7, 2, 2]] \\
\text{c}[1, 3] &= \text{Simplify}[\text{c}[7, 1, 3]] \\
\text{c}[0, 4] &= \text{Simplify}[\text{c}[7, 0, 4]] \\
4 (B^3 c[3, 4] + 3 A B^2 c[4, 3] + 5 A^2 B c[5, 2] + 5 A^3 c[6, 1]) \\
\text{c}[3, 0] &= \text{Simplify}[\text{c}[7, 3, 0]] \\
\text{c}[2, 1] &= \text{Simplify}[\text{c}[7, 2, 1]] \\
\text{c}[1, 2] &= \text{Simplify}[\text{c}[7, 1, 2]] \\
\text{c}[0, 3] &= \text{Simplify}[\text{c}[7, 0, 3]] \\
\text{c}[2, 0] &= \text{Simplify}[\text{c}[7, 2, 0]] \\
\text{c}[1, 1] &= \text{Simplify}[\text{c}[7, 1, 1]] \\
\text{c}[0, 2] &= \text{Simplify}[\text{c}[7, 0, 2]] \\
\text{c}[1, 0] &= \text{Simplify}[\text{c}[7, 1, 0]] \\
\text{c}[0, 1] &= \text{Simplify}[\text{c}[7, 0, 1]] \\
\text{c}[0, 0] &= \text{Simplify}[\text{c}[7, 0, 0]] \\
B.1. SOLUTION TO THE RECURSION RELATION

(* Recurrence relation for three particles *)
\[ c[i, j, k] := (i+1)/(K-i-j-k) A c[i+1, j, k] + (j+1)/(K-i-j-k) B c[i, j+1, k] + (k+1)/(K-i-j-k) C c[i, j, k+1] \]

(* K=1 *)
\[ c[0, 0, 0] = \text{Simplify}[c[1, 0, 0, 0]] \]
\[ C c[0, 0, 1] + B c[0, 1, 0] + A c[1, 0, 0] \]

(* K=2 *)
\[ c[1, 0, 0] = c[2, 1, 0, 0] \]
\[ c[0, 1, 0] = c[2, 0, 1, 0] \]
\[ c[0, 0, 1] = c[2, 0, 0, 1] \]
\[ C c[1, 0, 1] + B c[1, 0, 1] + 2 A c[2, 0, 0] \]
\[ C c[0, 1, 1] + 2 B c[0, 2, 0] + A c[1, 1, 0] \]
\[ 2 C c[0, 0, 2] + B c[0, 1, 1] + A c[1, 0, 1] \]
\[ c[0, 0, 0] = \text{Simplify}[c[2, 0, 0, 0]] \]
\[ C^2 c[0, 0, 2] + B C c[0, 0, 1] + B^2 c[0, 2, 0] + A C c[1, 0, 1] + A B c[1, 1, 0] + A^2 c[2, 0, 0] \]

(* K=3 *)
\[ c[2, 0, 0] = c[3, 2, 0, 0] \]
\[ c[1, 1, 0] = c[3, 1, 1, 0] \]
\[ c[1, 0, 1] = c[3, 1, 0, 1] \]
\[ c[0, 2, 0] = c[3, 0, 2, 0] \]
\[ c[0, 1, 1] = c[3, 0, 1, 1] \]
\[ c[0, 0, 2] = c[3, 0, 0, 2] \]
\[ C c[1, 0, 1] + B c[2, 1, 0] + 3 A c[3, 0, 0] \]
\[ C c[1, 1, 1] + 2 B c[1, 2, 0] + 2 A c[2, 1, 0] \]
\[ 2 C c[1, 0, 2] + B c[1, 1, 1] + 2 A c[1, 2, 0] \]
\[ C c[0, 2, 1] + 3 B c[0, 3, 0] + A c[1, 2, 0] \]
\[ 2 C c[0, 1, 2] + 2 B c[0, 2, 1] + A c[1, 1, 1] \]
\[ 3 C c[0, 0, 3] + B c[0, 1, 2] + A c[1, 0, 2] \]
\[ c[1, 0, 0] = \text{Simplify}[c[3, 1, 0, 0]] \]
\[ c[0, 1, 0] = \text{Simplify}[c[3, 0, 1, 0]] \]
\[ c[0, 0, 1] = \text{Simplify}[c[3, 0, 0, 1]] \]
\[ C^2 c[1, 0, 2] + B C c[1, 1, 1] + B^2 c[1, 2, 0] + 2 A C c[2, 0, 1] + 2 A B c[2, 1, 0] + 3 A^2 c[3, 0, 0] \]
\[ C^2 c[0, 1, 2] + 2 B C c[0, 2, 1] + 3 B^2 c[0, 3, 0] + A C c[1, 1, 1] + 2 A B c[1, 2, 0] + A^2 c[2, 1, 0] \]
\[ 3 C^2 c[0, 0, 3] + B^2 c[0, 2, 1] + 2 C (B c[0, 1, 2] + A c[1, 0, 2]) + A B c[1, 1, 1] + A^2 c[2, 0, 1] \]
B.1. SOLUTION TO THE RECURSION RELATION

\( c[0, 0, 0] = \text{Simplify}[c[3, 0, 0, 0]] \)

\[ C^3 c[0, 0, 0, 3] + B^3 c[0, 3, 0] + C^2 (B c[0, 1, 2] + A c[1, 0, 2]) + A B^2 c[1, 2, 0] + C (B^2 c[0, 2, 1] + A B c[1, 1, 1] + A^2 c[2, 0, 1]) + A^3 B c[2, 1, 0] + A^3 c[3, 0, 0] \]

\((* K=4 *)\)

\( c[3, 0, 0] = c[4, 3, 0, 0] \)
\( c[2, 1, 0] = c[4, 2, 1, 0] \)
\( c[2, 0, 1] = c[4, 2, 0, 1] \)
\( c[1, 2, 0] = c[4, 1, 2, 0] \)
\( c[1, 1, 1] = c[4, 1, 1, 1] \)
\( c[1, 0, 2] = c[4, 1, 0, 2] \)
\( c[0, 3, 0] = c[4, 0, 3, 0] \)
\( c[0, 2, 1] = c[4, 0, 2, 1] \)
\( c[0, 1, 2] = c[4, 0, 1, 2] \)
\( c[0, 0, 3] = c[4, 0, 0, 3] \)

\( C c[3, 0, 1] + B c[3, 1, 0] + 4 A c[4, 0, 0] \)
\( C c[2, 1, 1] + 2 B c[2, 2, 0] + 3 A c[3, 1, 0] \)
\( 2 C c[2, 0, 2] + B c[2, 1, 1] + 3 A c[3, 0, 1] \)
\( C c[1, 2, 1] + 3 B c[1, 3, 0] + 2 A c[2, 2, 0] \)
\( 2 C c[1, 1, 2] + 2 B c[1, 2, 1] + 2 A c[2, 1, 1] \)
\( 3 C c[1, 0, 3] + B c[1, 1, 2] + 2 A c[2, 0, 2] \)
\( C c[0, 3, 1] + 4 B c[0, 4, 0] + A c[1, 3, 0] \)
\( 2 C c[0, 2, 2] + 3 B c[0, 3, 1] + A c[1, 2, 1] \)
\( 3 C c[0, 1, 3] + 2 B c[0, 2, 2] + A c[1, 1, 2] \)
\( 4 C c[0, 0, 4] + B c[0, 1, 3] + A c[1, 0, 3] \)

\( c[2, 0, 0] = \text{Simplify}[c[4, 2, 0, 0]] \)
\( c[1, 1, 0] = \text{Simplify}[c[4, 1, 1, 0]] \)
\( c[1, 0, 1] = \text{Simplify}[c[4, 1, 0, 1]] \)
\( c[0, 2, 0] = \text{Simplify}[c[4, 0, 2, 0]] \)
\( c[0, 1, 1] = \text{Simplify}[c[4, 0, 1, 1]] \)
\( c[0, 0, 2] = \text{Simplify}[c[4, 0, 0, 2]] \)

\( C^3 c[2, 0, 2, 1] + B C c[2, 1, 2] + B^2 c[2, 2, 0] + 3 A B c[3, 0, 1] + 3 A B c[3, 1, 0] + 6 A^2 c[4, 0, 0] \)
\( C^2 c[1, 1, 2] + 3 B^2 c[1, 3, 0] + 2 C (B c[1, 2, 1] + A c[2, 1, 1]) + 4 A B c[2, 2, 0] + 3 A^2 c[3, 1, 0] \)
\( 3 C^2 c[1, 0, 3] + 2 B C c[1, 1, 2] + B^2 c[1, 2, 1] + 4 A B c[2, 0, 2] + 2 A B c[2, 1, 1] + 3 A^2 c[3, 0, 1] \)
\( C^3 c[0, 2, 2] + 3 B C c[0, 3, 1] + 6 B^2 c[0, 4, 0] + A c[1, 2, 1] + 3 A B c[1, 3, 0] + A^2 c[2, 2, 0] \)
\( 3 C^2 c[0, 1, 3] + 4 B C c[0, 2, 2] + 3 B^2 c[0, 3, 1] + 2 A B c[1, 2, 1] + 2 A B c[1, 1, 2] + A^2 c[2, 1, 1] \)
\( 6 C^2 c[0, 0, 4] + B^2 c[0, 2, 2] + 3 C (B c[0, 1, 3] + A c[1, 0, 3]) + A B c[1, 1, 2] + A^2 c[2, 0, 2] \)
B.1. SOLUTION TO THE RECURSION RELATION

\[ c[1, 0, 0] = \text{Simplify}[c[4, 1, 0, 0]] \]
\[ c[0, 1, 0] = \text{Simplify}[c[4, 0, 1, 0]] \]
\[ c[0, 0, 1] = \text{Simplify}[c[4, 0, 0, 1]] \]
\[
\begin{align*}
&c^3 c[1, 0, 3] + B^3 c[1, 3, 0] + C^2 (B c[1, 1, 2] + 2 A c[2, 0, 2]) + 2 A B^2 c[2, 2, 0] + \\
&C (B^2 c[1, 2, 1] + 2 A B c[2, 1, 1] + 3 A^3 c[3, 0, 1]) + 3 A^2 B c[3, 1, 0] + 4 A^3 c[4, 0, 0] \\
&C^3 c[0, 1, 3] + 4 B^3 c[0, 4, 0] + C^2 (2 B c[0, 2, 2] + A c[1, 1, 2]) + 3 A B^2 c[1, 3, 0] + \\
&C (3 B^2 c[0, 3, 1] + 2 A B c[1, 2, 1] + A^2 c[2, 1, 1]) + 2 A^2 B c[2, 2, 0] + A^3 c[3, 1, 0] \\
&4 C^3 c[0, 0, 4] + B^3 c[0, 3, 1] + 3 C^2 (B c[0, 1, 3] + A c[1, 0, 3]) + A B^2 c[1, 2, 1] + \\
&2 C (B^2 c[0, 2, 2] + A B c[1, 1, 2] + A^2 c[2, 0, 2]) + A^2 B c[2, 1, 1] + A^3 c[3, 0, 1] \\
&c[0, 0, 0] = \text{Simplify}[c[4, 0, 0, 0]] \\
&c^4 c[0, 0, 4] + B^4 c[0, 4, 0] + C^3 (B c[0, 1, 3] + A c[1, 0, 3]) + \\
&A B^3 c[1, 3, 0] + C^2 (B^2 c[0, 2, 2] + A B c[1, 1, 2] + A^2 c[2, 0, 2]) + A^2 B^2 c[2, 2, 0] + \\
&C (B^3 c[0, 3, 1] + A B^2 c[1, 2, 1] + A^2 B c[2, 1, 1] + A^3 c[3, 0, 1]) + A^3 B c[3, 1, 0] + A^4 c[4, 0, 0]
\end{align*}
\]
B.1. SOLUTION TO THE RECURSION RELATION

(* K=5 *)

c[4, 0, 0] = c[5, 4, 0, 0]
c[3, 1, 0] = c[5, 3, 1, 0]
c[3, 0, 1] = c[5, 3, 0, 1]
c[2, 2, 0] = c[5, 2, 2, 0]
c[2, 1, 1] = c[5, 2, 1, 1]
c[2, 0, 2] = c[5, 2, 0, 2]
c[1, 3, 0] = c[5, 1, 3, 0]
c[1, 2, 1] = c[5, 1, 2, 1]
c[1, 1, 2] = c[5, 1, 1, 2]
c[1, 0, 3] = c[5, 1, 0, 3]
c[0, 4, 0] = c[5, 0, 4, 0]
c[0, 3, 1] = c[5, 0, 3, 1]
c[0, 2, 2] = c[5, 0, 2, 2]
c[0, 1, 3] = c[5, 0, 1, 3]
c[0, 0, 4] = c[5, 0, 0, 4]

5 c[0, 0, 5] + B c[0, 1, 4] + A c[1, 0, 4]
B.1. Solution to the Recurrence Relation

\[ c[3, 0, 0] = \text{Simplify}[c[5, 3, 0, 0]] \]
\[ c[2, 1, 0] = \text{Simplify}[c[5, 2, 1, 0]] \]
\[ c[2, 0, 1] = \text{Simplify}[c[5, 2, 0, 1]] \]
\[ c[1, 2, 0] = \text{Simplify}[c[5, 1, 2, 0]] \]
\[ c[1, 1, 1] = \text{Simplify}[c[5, 1, 1, 1]] \]
\[ c[1, 0, 2] = \text{Simplify}[c[5, 1, 0, 2]] \]
\[ c[0, 3, 0] = \text{Simplify}[c[5, 0, 3, 0]] \]
\[ c[0, 2, 1] = \text{Simplify}[c[5, 0, 2, 1]] \]
\[ c[0, 1, 2] = \text{Simplify}[c[5, 0, 1, 2]] \]
\[ c[0, 0, 3] = \text{Simplify}[c[5, 0, 0, 3]] \]

\[ c^2 c[3, 0, 2] + B^2 c[3, 2, 0] + 4 A c[3, 0, 1] + 4 A B c[1, 0, 1] \]
\[ c^2 c[2, 1, 2] + 3 B^2 c[2, 3, 0] + 3 A c[3, 1, 1] + 6 A B c[3, 2, 0] + 6 A^2 c[4, 0, 1] \]
\[ 3 c^2 c[2, 0, 3] + 2 B C c[2, 1, 2] + B^2 c[2, 2, 1] + 6 A c[3, 0, 2] + 3 A B c[3, 1, 1] + 6 A^2 c[4, 0, 1] \]
\[ c^3 c[1, 2, 2] + 3 B^2 c[1, 3, 1] + 6 B^2 c[1, 4, 0] + 6 Ac[2, 2, 1] + 6 A B c[2, 3, 0] + 3 A^2 c[3, 2, 0] \]
\[ 3 c^2 c[1, 1, 3] + 4 B^2 c[1, 4, 0] + 2 A c[3, 2, 0] + 3 A B c[2, 2, 1] + 3 A B c[3, 1, 1] + 4 A c[1, 0, 1] + 4 A^2 c[2, 2, 1] \]
\[ 6 c^2 c[1, 0, 4] + 3 B^2 c[1, 5, 0] + 3 A c[2, 3, 0] + 2 A B c[2, 2, 1] + 4 A^2 c[3, 2, 0] + 4 A^3 c[5, 0, 0] \]
\[ 4 c^3 c[1, 1, 4] + 4 B^2 c[1, 3, 0] + 3 c^2 (B c[1, 3, 1] + 2 A c[2, 0, 3]) + 2 A B c[2, 2, 1] + 2 A B c[3, 2, 1] + 3 A^2 c[3, 0, 2] + 4 A^2 c[4, 0, 1] \]
\[ c^4 c[1, 0, 5] + 6 B^2 c[1, 4, 0] + 3 c^3 (B c[1, 4, 0] + 2 A c[3, 2, 0]) + 6 A B c[2, 1, 0] + 4 A^2 c[2, 2, 1] + 3 A^2 c[2, 1, 0] + 4 A^3 c[3, 2, 0] \]
\[ 4 c^4 c[1, 0, 4] + 6 B^2 c[1, 3, 0] + 3 c^3 (B c[1, 3, 1] + 2 A c[2, 0, 3]) + 3 A B c[2, 2, 1] + 3 A B c[3, 2, 0] + 4 A^2 c[3, 1, 1] + 4 A^3 c[4, 0, 1] \]

\[ 10 c^5 c[0, 0, 5] + 6 B^2 c[0, 3, 2] + 6 c^4 (B c[0, 3, 3] + 2 A c[1, 0, 4]) + 4 A B c[1, 2, 0] + 2 A B c[2, 1, 0] + A^2 c[2, 2, 1] + 3 A^2 c[2, 1, 0] + 4 A^3 c[3, 2, 0] \]
B.1. SOLUTION TO THE RECURRENT RELATION

\[ c[1, 0, 0] = \text{Simplify} [c[5, 1, 0, 0]] \]
\[ c[0, 1, 0] = \text{Simplify} [c[5, 0, 1, 0]] \]
\[ c[0, 0, 1] = \text{Simplify} [c[5, 0, 0, 1]] \]

\[ c_0^3 c[1, 0, 4] + B^4 c[1, 4, 0] + C^3 (B c[1, 1, 3] + 2 A c[2, 0, 3]) + 2 A^2 B^3 c[2, 3, 0] +
C^2 (B^2 c[1, 2, 2] + 2 A B c[2, 1, 2] + 3 A^3 c[3, 0, 2]) + 3 A^2 B^2 c[3, 2, 0] +
C (B^3 c[1, 3, 1] + 2 A B^2 c[2, 2, 1] + 3 A^3 B c[3, 1, 1] + 4 A^3 c[4, 0, 1]) + 4 A^3 B c[4, 1, 0] + 5 A^4 c[5, 0, 0] \]

\[ c_0^5 c[0, 1, 4] + 5 B^4 c[0, 5, 0] + C^5 (2 B c[0, 2, 3] + A c[1, 1, 3]) +
4 A B^3 c[1, 4, 0] + C^5 (3 B^2 c[0, 3, 2] + 2 A B c[1, 2, 2] + A^3 c[2, 1, 2]) + 3 A^3 B^2 c[2, 3, 0] +
C (4 B^3 c[0, 4, 1] + 3 A B^2 c[1, 3, 1] + 2 A^2 B c[2, 2, 1] + A^3 c[3, 1, 1]) + 2 A^3 B c[3, 2, 0] + A^4 c[4, 1, 0] \]

\[ c_0^5 c[0, 0, 5] + B^4 c[0, 4, 1] + 4 C^4 (B c[0, 1, 4] + A c[1, 0, 4]) +
A B^3 c[1, 3, 1] + 3 C^4 (B^2 c[0, 2, 3] + A B c[1, 1, 3] + A^2 c[2, 0, 3]) + A^2 B^2 c[2, 2, 1] +
2 C (B^2 c[0, 3, 2] + A B c[1, 2, 2] + A^2 B c[2, 1, 2] + A^3 c[3, 0, 2]) + A^3 B c[3, 1, 1] + A^4 c[4, 0, 1] \]

\[ c_0^5 c[0, 0, 0] = \text{Simplify} [c[5, 0, 0, 0]] \]

\[ c_0^5 c[0, 0, 5] + B^4 c[0, 5, 0] + C^4 (B c[0, 1, 4] + A c[1, 0, 4]) +
A B^3 c[1, 4, 0] + C^4 (B^2 c[0, 2, 3] + A B c[1, 1, 3] + A^2 c[2, 0, 3]) + A^2 B^2 c[2, 3, 0] +
C^2 (B^2 c[0, 3, 2] + A B c[1, 2, 2] + A^2 B c[2, 1, 2] + A^3 c[3, 0, 2]) + A^2 B^2 c[3, 2, 0] +
A^4 B c[4, 1, 0] + A^5 c[5, 0, 0] \]
B.1. SOLUTION TO THE RECURRENCE RELATION

(*) Recurrence relation for four particles *)
c_{[K, i, j, k, l]} :=
\begin{align*}
(i + 1) / (K - i - j - k - l) & A c_{[i + 1, j, k, l]} + (j + 1) / (K - i - j - k - l) B c_{[i, j + 1, k, l]} + \\
(k + 1) / (K - i - j - k - l) & C c_{[i, j, k + 1, l]} + (l + 1) / (K - i - j - k - l) D c_{[i, j, k, l + 1]}
\end{align*}

(*) K=1 *)
c_{[0, 0, 0, 0]} = c_{[1, 0, 0, 0]}
D c_{[0, 0, 0, 1]} + C c_{[0, 0, 1, 0]} + B c_{[0, 1, 0, 0]} + A c_{[1, 0, 0, 0]}

(*) K=2 *)
c_{[1, 0, 0, 0]} = c_{[2, 1, 0, 0]}
c_{[0, 1, 0, 0]} = c_{[2, 0, 1, 0]}
c_{[0, 0, 1, 0]} = c_{[2, 0, 0, 1]}
c_{[0, 0, 0, 1]} = c_{[2, 0, 0, 1]}
D c_{[1, 0, 0, 1]} + C c_{[1, 0, 1, 0]} + B c_{[1, 1, 0, 0]} + 2 A c_{[2, 0, 0, 0]}
D c_{[0, 1, 0, 1]} + C c_{[0, 1, 1, 0]} + 2 B c_{[0, 2, 0, 0]} + A c_{[1, 1, 0, 0]}
D c_{[0, 0, 1, 1]} + 2 C c_{[0, 0, 2, 0]} + B c_{[0, 1, 1, 0]} + A c_{[1, 0, 1, 0]}
2 D c_{[0, 0, 0, 2]} + C c_{[0, 0, 0, 1]} + B c_{[0, 0, 1, 1]} + A c_{[1, 0, 0, 1]}
c_{[0, 0, 0, 0]} = \text{Simplify}[c_{[2, 0, 0, 0]}]
D^2 c_{[0, 0, 0, 2]} + C^2 c_{[0, 0, 2, 0]} + B C c_{[0, 1, 1, 0]} + \\
B^2 c_{[0, 2, 0, 0]} + D (C c_{[0, 0, 0, 1]} + B c_{[0, 1, 0, 1]} + A c_{[1, 0, 0, 1]} + \\
A C c_{[1, 0, 1, 0]} + A B c_{[1, 0, 0, 0]} + A^2 c_{[2, 0, 0, 0]} +}
B.1. SOLUTION TO THE RECURRENCE RELATION

(* K=3 *)

c[2, 0, 0, 0] = c[3, 2, 0, 0, 0]
c[1, 1, 0, 0] = c[3, 1, 1, 0, 0]
c[1, 0, 1, 0] = c[3, 1, 0, 1, 0]
c[1, 0, 0, 1] = c[3, 1, 0, 0, 1]
c[0, 2, 0, 0] = c[3, 0, 2, 0, 0]
c[0, 1, 1, 0] = c[3, 0, 1, 1, 0]
c[0, 1, 0, 1] = c[3, 0, 1, 0, 1]
c[0, 0, 2, 0] = c[3, 0, 0, 2, 0]
c[0, 0, 1, 1] = c[3, 0, 0, 0, 1]
c[0, 0, 0, 2] = c[3, 0, 0, 0, 2]

dc[2, 0, 0, 1] + cc[2, 0, 1, 0] + bc[2, 1, 0, 0] + 3 ac[3, 0, 0, 0]
dc[1, 1, 0, 1] + cc[1, 1, 1, 0] + 2 bc[1, 1, 0, 0] + 2 ac[2, 1, 0, 1]
dc[1, 0, 1, 1] + 2 cc[1, 0, 1, 0] + bc[1, 1, 0, 1] + 2 ac[2, 0, 1, 0]
dc[0, 2, 0, 1] + cc[0, 2, 1, 0] + 3 bc[0, 3, 0, 0] + ac[1, 2, 0, 0]
dc[0, 1, 2, 0] + 2 cc[0, 2, 1, 0] + 2 bc[0, 2, 1, 0] + ac[1, 1, 1, 0]
dc[0, 1, 0, 2] + 3 cc[0, 1, 1, 0] + 2 bc[0, 1, 0, 1] + ac[1, 1, 1, 0]
dc[0, 2, 1, 0] + 3 cc[0, 0, 1, 0] + bc[0, 1, 2, 0] + ac[1, 0, 2, 0]
dc[0, 2, 2, 0] + 3 cc[0, 0, 0, 1] + ab[1, 1, 0, 1] + ac[1, 0, 0, 1]
dc[0, 3, 0, 0] + bc[0, 0, 2, 1] + ab[0, 1, 1, 0] + ac[1, 0, 0, 1]

c[1, 0, 0, 0] = simplify[c[3, 1, 0, 0, 0]]
c[0, 1, 0, 0] = simplify[c[3, 0, 1, 0, 0]]
c[0, 0, 1, 0] = simplify[c[3, 0, 0, 1, 0]]
c[0, 0, 0, 1] = simplify[c[3, 0, 0, 0, 1]]

(= K=3 =)

c[2, 0, 0, 0] = c[3, 2, 0, 0, 0]
c[1, 1, 0, 0] = c[3, 1, 1, 0, 0]
c[1, 0, 1, 0] = c[3, 1, 0, 1, 0]
c[1, 0, 0, 1] = c[3, 1, 0, 0, 1]
c[0, 2, 0, 0] = c[3, 0, 2, 0, 0]
c[0, 1, 1, 0] = c[3, 0, 1, 1, 0]
c[0, 1, 0, 1] = c[3, 0, 1, 0, 1]
c[0, 0, 2, 0] = c[3, 0, 0, 2, 0]
c[0, 0, 1, 1] = c[3, 0, 0, 0, 1]
c[0, 0, 0, 2] = c[3, 0, 0, 0, 2]
B.2 Normalizability Condition for N=2 Particles

In this section of the appendix, we will elaborate on the normalizability condition given in equation (6.36), which reads: For every excitation energy $K$ and degree $d$ separately, we require that for all $l \in \{1, 2, \ldots, K\}$ and for all $k \in \{0, 1, 2, \ldots, l-1\}$,

$$
\sum_{j_1+j_2=d} j_1^{K-l} \sum_{i_1+i_2=K} (-1)^i_1 (i_1)^{k} \gamma_{i_1,i_2} = 0. \tag{B.4}
$$

The normalizability condition guarantees that we get rid of the local singularities introduced by the differential operators (6.27),

$$
\tau_i := -\frac{1}{2} \left( \partial_i + \sum_{j=1,j\neq i}^N \frac{\alpha}{w_i - w_j} \right). \tag{B.5}
$$

We will not take the symmetry condition (6.31) into account here: it has to be imposed in addition to the normalizability condition.

The approach that we take is the following: we will explicitly list the normalizability conditions for $N=2$, $K=1, 2, 3, 4$, and then we will look for a pattern that generalizes to arbitrary $K$.

The normalizability conditions are obtained by substituting (B.5) into the expression for $F$,

$$
F(w, \bar{w}) = \sum_{0 \leq i_1 + \ldots + i_N \leq K} c_{i_1, \ldots, i_N}(w) \bar{w}_1^{i_1} \ldots \bar{w}_N^{i_N},
$$

where, from (6.23), the lower order coefficients $c_{i_1', \ldots, i_N'}$ are given by

$$
c_{i_1', \ldots, i_N'} = \sum_{i_1 + \ldots + i_N = K} \binom{i_1}{j_1'} \ldots \binom{i_N}{j_N'} \tau_1^{i_1-i_1'} \ldots \tau_N^{i_N-i_N'} c_{i_1, \ldots, i_N}
$$

in terms of the highest order coefficients $c_{i_1, \ldots, i_N}$ with $i_1 + \ldots + i_N = K$, and then sorting all terms together that get multiplied by the same singular prefactor $(w_1 - w_2)^{-k}$ for different values of $k \in \mathbb{N}$. Doing so yields quite a large number of equations that have to be satisfied. We will then get rid of the redundant equations, and from what remains we will try to find a pattern that can be generalized to arbitrary $K$.

**K=1**

Recall the easiest case, that of $K = 1$, which we have already discussed in Example 6.2. As follows from equation (6.34), the only condition for the wavefunction to be normalizable is,

$$
w_1 - w_2 \mid c_{10} - c_{01}, \tag{B.6}
$$

where the long bar means “divides”. We write $c_{i_1,i_2}(w_1, w_2) = \sum_{j_1+j_2=d} \gamma_{i_1,i_2}^{j_1,j_2} w_1^{j_1} w_2^{j_2}$ for some fixed but arbitrary degree $d$. Now, by Lemma 6.6, (B.6) is equivalent to

$$
\sum_{j_1+j_2=d} \gamma_{1,0}^{j_1,j_2} - \gamma_{0,1}^{j_1,j_2} = 0
$$

which indeed agrees with (B.4) for $K = 1$, and therefore $l = 1$ and $k = 0$.

**K=2**

For $K = 2$, $F$ is given by

$$
F(w, \bar{w}) = c_{00} + c_{10} \bar{w}_1 + c_{01} w_2 + c_{20} \bar{w}_1^2 + c_{11} \bar{w}_1 w_2 + c_{02} \bar{w}_2^2,
$$
Likewise, for \( (B.5) \), and collect all terms that are accompanied by the same singular term. This yields,
\[
\begin{align*}
c_{00} &= \frac{\alpha}{4} (\alpha - 1) \frac{1}{(w_1 - w_2)^2} \left( c_{20} - c_{11} + c_{02} \right) + \frac{\alpha}{4} \frac{1}{w_1 - w_2} \left( 2 \partial_1 c_{20} - \partial_1 c_{11} + \partial_2 c_{11} - 2 \partial_2 c_{02} \right) \\
+ \text{regular terms}, \\
c_{10} &= -\frac{\alpha}{2} \frac{1}{w_1 - w_2} \left( 2c_{20} - c_{11} \right) + \text{regular terms}, \\
c_{01} &= -\frac{\alpha}{2} \frac{1}{w_1 - w_2} \left( c_{11} - 2c_{02} \right) + \text{regular terms}.
\end{align*}
\]

Hence, the normalizability conditions for \( K = 2 \) are:
\[
\begin{align*}
(w_1 - w_2)^2 &\quad | \quad c_{20} - c_{11} + c_{02} \quad (A) \\
(w_1 - w_2) &\quad | \quad \partial_1 (2c_{20} - c_{11}) + \partial_2 (c_{11} - 2c_{02}) \quad (B)
\end{align*}
\]
for \( c_{00} \), and
\[
\begin{align*}
(w_1 - w_2) &\quad | \quad 2c_{20} - c_{11} \quad (C) \\
(w_1 - w_2) &\quad | \quad c_{11} - 2c_{02} \quad (D)
\end{align*}
\]
for \( c_{10} \) and \( c_{01} \).

As before, we write \( c_{i_1, i_2}(w_1, w_2) = \sum_{j_1 + j_2 = d} \gamma_{i_1, i_2}^{j_1, j_2} w_1^{j_1} w_2^{j_2} \) for some fixed but arbitrary degree \( d \), and use Lemma 6.6 to turn equations \((A) - (D)\) into equations on the coefficients \( \gamma_{i_1, i_2}^{j_1, j_2} \). Thus, for \((A)\), we have two equations,
\[
\begin{align*}
\sum_{j_1 + j_2 = d} \gamma_{1,1}^{j_1, j_2} - \gamma_{1,1}^{0,2} - \gamma_{0,2}^{j_1, j_2} = 0, \quad (A1) \\
\sum_{j_1 + j_2 = d} j_1 \left( \gamma_{1,1}^{j_1, j_2} - \gamma_{1,1}^{0,2} + \gamma_{0,2}^{j_1, j_2} \right) = 0, \quad (A2)
\end{align*}
\]
Likewise, for \((C)\) and \((D)\) we have
\[
\begin{align*}
\sum_{j_1 + j_2 = d} 2\gamma_{1,0}^{j_1, j_2} - \gamma_{1,1}^{j_1, j_2} = 0 \quad (C) \\
\sum_{j_1 + j_2 = d} \gamma_{1,1}^{j_1, j_2} - 2\gamma_{0,2}^{j_1, j_2} = 0 \quad (D)
\end{align*}
\]
Notice that there is already one redundancy here, as \((C) - 2(A1) = (D)\) so we can safely drop \((D)\) form our list of conditions. Continuing in the same fashion, for \((B)\) we have
\[
\begin{align*}
\partial_1 [2c_{20}(w) - c_{11}(w)] + \partial_2 [c_{11}(w) - 2c_{02}(w)] &= \sum_{j_1 + j_2 = d} j_1 \left( \gamma_{1,0}^{j_1, j_2} - \gamma_{1,1}^{j_1, j_2} \right) w_1^{j_1-1} w_2^{j_2} + \sum_{j_1 + j_2 = d} j_2 \left( \gamma_{1,1}^{j_1, j_2} - 2\gamma_{0,2}^{j_1, j_2} \right) w_1^{j_1} w_2^{j_2-1} \\
&= \sum_{j_1 + j_2 = d} j_1 \left( \gamma_{1,0}^{j_1, j_2} - \gamma_{1,1}^{j_1, j_2} \right) w_1^{j_1-1} w_2^{j_2} + \sum_{j_1 + j_2 = d} (d - j_1) \left( \gamma_{1,1}^{j_1, j_2} - 2\gamma_{0,2}^{j_1, j_2} \right) w_1^{j_1} w_2^{j_2-1}
\end{align*}
\]
The polynomial on the RHS is a polynomial of constand degree $d-1$, so by Lemma 6.6, it is divisible by $(w_1 - w_2)$ if and only if the sum of all the coefficients is zero, that is, if and only if

$$0 = \sum_{j_1 + j_2 = d} j_1 \left( 2\gamma_{1,0}^{j_1,j_2} - \gamma_{1,1}^{j_1,j_2} \right) + \sum_{j_1 + j_2 = d} (d - j_1) \left( \gamma_{1,1}^{j_1,j_2} - 2\gamma_{0,2}^{j_1,j_2} \right)$$

$$= 2 \sum_{j_1 + j_2 = d} j_1 \left( \gamma_{2,0}^{j_1,j_2} - \gamma_{1,1}^{j_1,j_2} + \gamma_{0,2}^{j_1,j_2} \right) + d \sum_{j_1 + j_2 = d} \left( \gamma_{1,1}^{j_1,j_2} - 2\gamma_{0,2}^{j_1,j_2} \right) \tag{B.7}$$

However, the above equation is implied by (A2) and (D), so (B) can be dropped as well.

What remains the following set of equations:

$$\sum_{j_1 + j_2 = d} j_1 \left( \gamma_{2,0}^{j_1,j_2} - \gamma_{1,1}^{j_1,j_2} + \gamma_{0,2}^{j_1,j_2} \right) = 0, \tag{B.8}$$

$$\sum_{j_1 + j_2 = d} \gamma_{2,0}^{j_1,j_2} - \gamma_{1,1}^{j_1,j_2} + \gamma_{0,2}^{j_1,j_2} = 0, \tag{B.9}$$

$$\sum_{j_1 + j_2 = d} 2\gamma_{2,0}^{j_1,j_2} - \gamma_{1,1}^{j_1,j_2} = 0. \tag{B.10}$$

which agrees with (B.4) for $K = 2$, where $l = 1, k = 0$ corresponds to (B.8), $l = 2, k = 0$ corresponds to (B.9), and $l = 2, k = 1$ corresponds to (B.10).

For $K = 3$ and $K = 4$, the reasoning is analogous to the arguments used above. We will therefore not go into all the details, but we shall list the set of initial equation that have to be satisfied as well as the reduced set of equations after all redundancies have been taken care of.

**K=3**

The normalizability conditions for $K = 3$ are:

\[
\begin{align*}
(w_1 - w_2)^3 &\quad | \quad c_{30} - c_{21} + c_{12} - c_{03} \\
(w_1 - w_2)^2 &\quad | \quad \partial_1(3c_{30} - 2c_{21} + c_{12}) + \partial_2(c_{21} - 2c_{12} + 3c_{03}) \\
(w_1 - w_2) &\quad | \quad \partial_1^2(3c_{30} - c_{21}) + \partial_1\partial_2(2c_{21} - 2c_{12}) + \partial_2^2(c_{12} - 3c_{03})
\end{align*}
\]

for $c_{00}$,

\[
\begin{align*}
(w_1 - w_2)^2 &\quad | \quad 3c_{30} - 2c_{21} + c_{12} \\
(w_1 - w_2) &\quad | \quad \partial_1(3c_{30} - c_{21}) + \partial_2(c_{21} - c_{12}) \\
(w_1 - w_2)^2 &\quad | \quad c_{21} - 2c_{12} + 3c_{03} \\
(w_1 - w_2) &\quad | \quad \partial_1(c_{21} - c_{12}) + \partial_2(c_{12} - 3c_{03})
\end{align*}
\]

for $c_{10}$ and $c_{01}$, and

\[
\begin{align*}
(w_1 - w_2) &\quad | \quad 3c_{30} - c_{21} \tag{B.11} \\
(w_1 - w_2) &\quad | \quad c_{21} - c_{12} \tag{B.12} \\
(w_1 - w_2) &\quad | \quad c_{12} - 3c_{03} \tag{B.13}
\end{align*}
\]

for $c_{20}, c_{11}$ and $c_{02}$ respectively.

As in the case of $K = 2$, there is a lot of redundancy to be found in the above equations. For example, (B.13) and $(w_1 - w_2)^2(c_{21} - 2c_{12} + 3c_{03})$ imply $(w_1 - w_2)[\partial_1(c_{21} - c_{12}) + \partial_2(c_{12} - 3c_{03})]$ using a similar argument as in (B.7).
After all the redundant equations have been removed, the following equations remain

\[ \sum_{j_1+j_2=d} j_1^2 \left( \gamma_{3,0}^{j_1,j_2} - \gamma_{2,1}^{j_1,j_2} + \gamma_{1,2}^{j_1,j_2} - \gamma_{0,3}^{j_1,j_2} \right) = 0 \]
\[ \sum_{j_1+j_2=d} j_1 \left( \gamma_{3,0}^{j_1,j_2} - \gamma_{2,1}^{j_1,j_2} + \gamma_{1,2}^{j_1,j_2} - \gamma_{0,3}^{j_1,j_2} \right) = 0 \]
\[ \sum_{j_1+j_2=d} j_1 \left( 3 \gamma_{3,0}^{j_1,j_2} - 2 \gamma_{2,1}^{j_1,j_2} + \gamma_{1,2}^{j_1,j_2} \right) = 0 \]
\[ \sum_{j_1+j_2=d} \left( \gamma_{3,0}^{j_1,j_2} - \gamma_{2,1}^{j_1,j_2} + \gamma_{1,2}^{j_1,j_2} - \gamma_{0,3}^{j_1,j_2} \right) = 0 \]
\[ \sum_{j_1+j_2=d} \left( 3 \gamma_{3,0}^{j_1,j_2} - 2 \gamma_{2,1}^{j_1,j_2} + \gamma_{1,2}^{j_1,j_2} \right) = 0 \]
\[ \sum_{j_1+j_2=d} \left( 9 \gamma_{3,0}^{j_1,j_2} - 4 \gamma_{2,1}^{j_1,j_2} + \gamma_{1,2}^{j_1,j_2} \right) = 0 \]

which is in agreement with (B.4). Note that last three equations are equivalent to (B.11) - (B.13).

**K=4**

The normalizability conditions for K = 4 are:

\[(w_1 - w_2)^4 \quad \begin{array}{l} c_{40} - c_{31} + c_{22} - c_{13} + c_{04} \end{array} \]
\[(w_1 - w_2)^3 \quad \begin{array}{l} \partial_1 (4c_{40} - 3c_{31} + 2c_{22} - c_{13}) + \partial_2 (c_{31} - 2c_{22} + 3c_{13} - 4c_{04}) \end{array} \]
\[(w_1 - w_2)^2 \quad \begin{array}{l} \partial_1^2 (6c_{40} - 3c_{31} + c_{22}) + \partial_1 \partial_2 (3c_{31} - 4c_{22} + 3c_{13}) + \partial_2^2 (c_{22} - 3c_{13} + 6c_{04}) \end{array} \]
\[(w_1 - w_2) \quad \begin{array}{l} \partial_1^3 (4c_{40} - c_{31}) + \partial_1^2 \partial_2 (3c_{31} - 2c_{22}) + \partial_1 \partial_2^2 (2c_{22} - 3c_{13}) + \partial_2^3 (c_{13} - 4c_{04}) \end{array} \]

for \( c_{00} \),

\[(w_1 - w_2)^3 \quad \begin{array}{l} 4c_{40} - 3c_{31} + 2c_{22} - c_{13} \end{array} \]
\[(w_1 - w_2)^2 \quad \begin{array}{l} \partial_1 (12c_{40} - 6c_{31} + 2c_{22}) + \partial_2 (3c_{31} - 4c_{22} + 3c_{13}) \end{array} \]
\[(w_1 - w_2) \quad \begin{array}{l} \partial_1^2 (12c_{40} - 3c_{31}) + \partial_1 \partial_2 (6c_{31} - 4c_{22}) + \partial_2^2 (2c_{22} - 3c_{13}) \end{array} \]
\[(w_1 - w_2)^3 \quad \begin{array}{l} c_{31} - 2c_{22} + 3c_{13} - 4c_{04} \end{array} \]
\[(w_1 - w_2)^2 \quad \begin{array}{l} \partial_1 (3c_{31} - 4c_{22} + 3c_{13}) + \partial_2 (2c_{22} - 6c_{13} + 12c_{04}) \end{array} \]
\[(w_1 - w_2) \quad \begin{array}{l} \partial_1^2 (3c_{31} - 2c_{22}) + \partial_1 \partial_2 (4c_{22} - 6c_{13}) + \partial_2^2 (3c_{13} - 12c_{04}) \end{array} \]

for \( c_{10} \) and \( c_{01} \),

\[(w_1 - w_2)^2 \quad \begin{array}{l} 6c_{40} - 3c_{31} + c_{22} \end{array} \]
\[(w_1 - w_2) \quad \begin{array}{l} \partial_1 (12c_{40} - 3c_{31}) + \partial_2 (3c_{13} - 2c_{22}) \end{array} \]
\[(w_1 - w_2)^2 \quad \begin{array}{l} 3c_{31} - 4c_{22} + 3c_{13} \end{array} \]
\[(w_1 - w_2) \quad \begin{array}{l} \partial_1 (6c_{31} - 4c_{22}) + \partial_2 (4c_{22} - 6c_{13}) \end{array} \]
\[(w_1 - w_2)^2 \quad \begin{array}{l} c_{22} - 3c_{13} + 6c_{04} \end{array} \]
\[(w_1 - w_2) \quad \begin{array}{l} \partial_1 (2c_{22} - 3c_{13}) + \partial_2 (3c_{13} - 12c_{04}) \end{array} \]
for \(c_{20}, c_{11}\) and \(c_{02}\), and,

\[
\begin{align*}
(w_1 - w_2) & | \ 4c_{40} - c_{31} \\
(w_1 - w_2) & | \ 3c_{31} - 2c_{22} \\
(w_1 - w_2) & | \ 2c_{22} - 3c_{13} \\
(w_1 - w_2) & | \ c_{13} - 4c_{04}
\end{align*}
\]

for \(c_{30}, c_{21}, c_{12}\) and \(c_{03}\) respectively.

Removing all redundant equations results in,

\[
\sum_{j_1 + j_2 = d} j_1^3 \left( \gamma_{4,0}^{j_1, j_2} - \gamma_{3,1}^{j_1, j_2} + \gamma_{2,2}^{j_1, j_2} - \gamma_{1,3}^{j_1, j_2} + \gamma_{0,4}^{j_1, j_2} \right) = 0
\]

\[
\sum_{j_1 + j_2 = d} j_1^2 \left( \gamma_{4,0}^{j_1, j_2} - \gamma_{3,1}^{j_1, j_2} + \gamma_{2,2}^{j_1, j_2} - \gamma_{1,3}^{j_1, j_2} + \gamma_{0,4}^{j_1, j_2} \right) = 0
\]

\[
\sum_{j_1 + j_2 = d} j_1 \left( 4\gamma_{4,0}^{j_1, j_2} - 3\gamma_{3,1}^{j_1, j_2} + 2\gamma_{2,2}^{j_1, j_2} - \gamma_{1,3}^{j_1, j_2} + \gamma_{0,4}^{j_1, j_2} \right) = 0
\]

\[
\sum_{j_1 + j_2 = d} j_1 \left( 4\gamma_{4,0}^{j_1, j_2} - 3\gamma_{3,1}^{j_1, j_2} + 2\gamma_{2,2}^{j_1, j_2} - \gamma_{1,3}^{j_1, j_2} + \gamma_{0,4}^{j_1, j_2} \right) = 0
\]

\[
\sum_{j_1 + j_2 = d} j_1 \left( 4\gamma_{4,0}^{j_1, j_2} - 3\gamma_{3,1}^{j_1, j_2} + 2\gamma_{2,2}^{j_1, j_2} - \gamma_{1,3}^{j_1, j_2} + \gamma_{0,4}^{j_1, j_2} \right) = 0
\]

which is in agreement with (B.4).

Since (B.4) holds up to \(K = 4\), and there is no reason to assume that the apparent structure in the normalizability conditions will change for higher excitation energies, we will stop here with checking the validity of (B.4). Besides, for larger \(K\), the work gets too tedious to do by hand.\(^1\)

\(^1\)It would have been possible to write a program that checks whether the normalizability conditions also hold for higher excitation energies or not, if there were more time available, which there is not.
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