Four-point functions of $\mathcal{N} = 4$ SYM$_4$ in the AdS/CFT correspondence

Pieter Naaijkens

Master’s Thesis in Theoretical Physics
Supervisor: Dr. Gleb Arutyunov
Institute for Theoretical Physics
Utrecht University

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Abstract

The AdS/CFT correspondence, proposed by Maldacena, conjectures a duality between $\mathcal{N} = 4$ SYM with a $SU(N)$ gauge group and Type IIB string theory on $\text{AdS}_5 \times S^5$. In particular, in the large $N$ limit, correlation functions in the strongly coupled gauge theory side can be calculated by considering the supergravity approximation on the string theory side. In this thesis I will use this correspondence to compute a specific four-point correlator in the supergravity approximation. I will consider a four-point correlator of two weight-2 and two weight-3 $^{1\over 2}$-BPS operators. These operators correspond to supergravity scalar fields of mass $m^2 = -4$ and $m^2 = -3$ respectively.

First it is shown that the effective Lagrangian reduces to a simple expression. This is then used to compute the on-shell value, as prescribed by the AdS/CFT correspondence. To calculate this, a method to calculate vector and tensor exchange diagrams is extended to include couplings to currents which aren’t conserved on-shell. Using this on-shell action, the four-point function is calculated.

From the gauge theory side, one can show that the amplitude can be described by a single function of the conformal cross-ratios. This does not follow from superconformal symmetry only, but also encodes dynamical information. Surprisingly, the correlator calculated from the supergravity side obeys this restriction, although there is no apparent reason why this should be the case. Hence, this can be seen as evidence supporting the AdS/CFT correspondence. A similar thing was observed in the calculation of correlator of four weight-3 and that of four weight-4 $^{1\over 2}$-BPS operators. Just like in these cases, the four-point function calculated here splits into a “free” and an “interacting” part.
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Chapter 1

Introduction

The large $N$ behaviour of $SU(N)$ gauge theories is interesting, because it is believed that this might lead to a better understanding of the strong coupling behaviour of QCD. Interest in this large $N$ behaviour was renewed by Maldacena’s proposal that large $N$ conformally invariant gauge theories can be described by some string theory on Anti-de Sitter (AdS) space times some compact manifold [1]. There have been many tests of the correspondence so far. For example, recently it was shown numerically [2] that there is a smooth transition from the weak coupling limit, where classical gauge theory can be used, to the strong coupling limit, where a string theory description is adequate. Another interesting is the suggestion that the AdS/CFT correspondence can be applied to heavy ion collision experiments done at the Relativistic Heavy Ion Collider (RHIC) [3]. This could provide a relation between string theory and experiments.

The idea that large $N$ gauge theory is dual to some string theory is actually already around for a few decades. ’t Hooft suggested that the low energy QCD calculations may be simplified by taking the number of colors $N$ (which is 3 in QCD) large. At low energies QCD is strongly coupled (because QCD is asymptotically free), and calculations become very difficult. The idea is to solve the theory for $N = \infty$, and do an expansion in $1/N$ [4]. More precisely, one takes the limit where the ’t Hooft coupling $\lambda = g^2_{YM} N$ is fixed but large. A diagrammatic expansion can then be done in the field theory. This expansion suggests that the gauge theory can be related to a string theory. See Section 1.2 of [5] for a review of the argument. The AdS/CFT correspondence is an explicit example of this idea.

There is a simple heuristic argument why $\mathcal{N} = 4$ SYM$_4$ is dual to a string theory on $\text{AdS}_5 \times S^5$. The supersymmetric Yang-Mills theory in four dimensions, with gauge group $SU(N)$, has the conformal symmetry group $SO(4,2)$. In addition to the conformal symmetry it also has a global $SU(4)_R$ R-symmetry, which rotates the supercharges into each other. It is natural to assume that the corresponding string theory must have the same symmetries. The only five-dimensional space with $SO(4,2)$ symmetry is $\text{AdS}_5$, five-dimensional Anti-de Sitter space. This is the maximally symmetric solution of the Einstein equations with a negative cos-
mological constant.

Since the gauge theory includes supersymmetry, the string side should as well. Superstrings live in 10 dimensions, so we have to add five more. The most natural choice is $S^5$, since we can identify the symmetry group $SO(6) \sim SU(4)$ with the R-symmetry. Hence, $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory is the same as superstring theory on $AdS_5 \times S^5$. Later a more precise relation will be outlined.

It is interesting to compare results obtained from the gauge theory with those obtained from supergravity calculations. This could give more insight in the precise mechanisms of the AdS/CFT correspondence. This thesis deals with a specific case: the four-point function of fields dual to two weight-2 and two weight-3 $\frac{1}{2}$-BPS operators in $SYM_4$,

$$\langle O_2^1(x_1)O_2^2(x_2)O_3^3(x_3)O_3^4(x_4) \rangle,$$

will be calculated in the supergravity approximation. The results will be compared with field-theoretical considerations. This is a new result: up to now only correlators of four $\frac{1}{2}$-BPS operators of equal weight have been considered (apart from some trivial extremal and sub-extremal correlators). Surprisingly, the same features are found in this case.

In particular, in $\mathcal{N} = 4$ $SYM_4$, one can use the insertion procedure to deduce a very specific form of the quantum part of the four-point function. In the case under consideration here, it turns out that it can be described by a single function of the conformally invariant cross-ratios. From a supergravity point of view, there is no apparent reason why this should be the case. However, the supergravity induced four-point function does show this behaviour. This can be seen as strong evidence for the AdS/CFT correspondence.

To arrive at this remarkable result, one has to do the following basic steps:

- Find the effective Lagrangian for the scalar fields $s_2$ and $s_3$, which are dual to weight-2 and weight-3 $\frac{1}{2}$-BPS operators respectively, of the supergravity theory on $AdS_5 \times S^5$.

- Calculate the on-shell value of the action. To find this, one has to calculate exchange diagrams of scalar, vector and tensor fields.

- Specify boundary conditions for the supergravity fields, and find the correlator by varying the on-shell action with respect to this boundary conditions.

- Write the resulting correlator in terms of conformally invariant cross-ratios $s$ and $t$. The predictions from the field theory can then be checked.

There are a few other interesting results worth mentioning. First of all, the effective action becomes a simple expression, with at most two derivatives. This is remarkable, because it comes from a highly complicated expression for the scalar quartic couplings in Type IIB supergravity on $AdS_5 \times S^5$. Another interesting point is that the supergravity result splits into a “free” and a “quantum” part. However, there is no coupling in supergravity that one can put to zero to get a
free theory, unlike in the Yang-Mills theory. This also gives an indication that there is a strong connection between these two theories.

The outline of this thesis is as follows. In chapter 2, I will recall the basic ingredients of the AdS/CFT correspondence: conformal field theory, supersymmetry and Type IIB strings. In chapter 3, the AdS/CFT correspondence is stated, and it is explained how to use it to calculate correlation functions. Chapter 4 concerns the main result: the effective Lagrangian and four-point function is obtained. Chapter 5 contains my conclusions, and suggestions for further research. Technical results are delegated to the appendices, in particular the $C$-algebra (Appendix A) and calculation of exchange diagrams (Appendix B). The results obtained in Chapter 4 and the extension of the method to calculate exchange diagrams in Appendix B are original work.
Chapter 2

Preliminaries

In this chapter the basic ingredients needed for the formulation of the AdS/CFT correspondence are discussed. The emphasis will be on the main ideas, and in particular applications thereof to the main problem of this thesis: finding the four-point correlator of two weight-2 and two weight-3 fields in supersymmetric Yang-Mills theory in four dimensions, by using the AdS/CFT correspondence with supergravity on $\text{AdS}_5 \times S^5$. Much of the material in this chapter with applications to the AdS/CFT correspondence can also be found in e.g. [5, 6, 7].

2.1 Conformal Field Theory

If one wants to consider a scale invariant theory, one possibility is to impose conformal invariance (in fact, each 2-dimensional scale invariant theory is conformally invariant). One example is the $d = 4, \mathcal{N} = 4$ supersymmetric Yang-Mills theory, described below. Other applications are e.g. in statistical mechanics and condensed matter physics. For an introduction to conformal field theory, see e.g. [8, 9].

The conformal group is the group of coordinate transformations that keep the metric invariant up to an arbitrary scale factor $\Omega(x)$,

$$ g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). $$

Since Poincaré transformations leave the metric invariant, the Poincaré group is a subgroup of the conformal group. In addition to the Poincaré transformations, one also has scale transformation (sometimes called dilatation)

$$ x_{\mu} \rightarrow x'_{\mu} = \lambda x_{\mu}, $$

and the special conformal transformation

$$ x_{\mu} \rightarrow x'_{\mu} = \frac{x_{\mu} + b_{\mu} x^2}{1 + 2b^\mu x_{\mu} + b^2 x^2}. $$
The indices \( \mu \) run from \( \mu = 0, \ldots, d-1 \).

Conformal symmetry imposes strong conditions on for example the correlators in the theory. For example, the structure of two- and three-point correlators is determined up to a constant. Starting from four-point correlators, the dynamics aren’t completely determined by conformal symmetry anymore. However, for \( n = 4 \), consider four coordinates \( x_1, \ldots, x_4 \). Then there are two independent combinations which are conformally invariant, namely the cross-ratios

\[
\begin{align*}
    s &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \\
t &= \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2},
\end{align*}
\]

(2.1)

where \( x_{ij}^2 = |x_i - x_j|^2 \). Hence, the four-point function can be described in terms of \( s \) and \( t \). Under exchange of \( x_1 \leftrightarrow x_2 \), we have \( s \to s/t, t \to 1/t \), and \( x_1 \leftrightarrow x_3 \) amounts to \( s \leftrightarrow t \).

Now consider \( d \) dimensional Minkowski space.\(^2\) Let \( M_{\mu \nu} \) denote the generators of Lorentz transformations, and \( P_\mu \) those of translations. The remaining generators are those of the special conformal transformations \( K_\mu \) and the dilatation operator \( D \). The conformal algebra is then given by

\[
\begin{align*}
    [M_{\mu \nu}, M_{\rho \sigma}] &= (\eta_{\mu \rho} M_{\nu \sigma} - \eta_{\mu \sigma} M_{\nu \rho} - \eta_{\nu \rho} M_{\mu \sigma} + \eta_{\nu \sigma} M_{\mu \rho}), \\
    [M_{\mu \nu}, P_\rho] &= i(\eta_{\mu \rho} P_\nu - \eta_{\nu \rho} P_\mu), \\
    [M_{\mu \nu}, K_\rho] &= i(\eta_{\mu \rho} K_\nu - \eta_{\nu \rho} K_\mu), \\
    [D, P_\mu] &= iP_\mu, \\
    [D, K_\mu] &= -iK_\mu, \\
    [K_\mu, P_\nu] &= -2iM_{\mu \nu} - 2i\eta_{\mu \nu} D.
\end{align*}
\]

(2.2)

Other commutators now shown are zero. This algebra is in fact isomorphic to \( \mathfrak{so}(d, 2) \), where we take the signature of the metric to be \((-,-,\ldots,-)\). By making linear combinations of the generators we can put (2.2) in standard form, by defining the generators \( J_{ab} \) by

\[
\begin{align*}
    J_{\mu \nu} &= M_{\mu \nu}, \\
    J_{\mu d} &= \frac{1}{2}(K_\mu - P_\mu), \\
    J_{\mu(d+1)} &= \frac{1}{2}(K_\mu + P_\mu), \\
    J_{(d+1)d} &= D
\end{align*}
\]

where \( a, b = 0, \ldots, d+1 \).

2.1.1 Primary fields

In physics we are usually interested in the representations of the conformal group that involve fields (or operators) which are eigenfunctions of the dilatation operator \( D \), with eigenvalue \( i\Delta \). The number \( \Delta \) is called the scaling dimension (or conformal dimension). Such fields transform as \( \phi(x) \to \phi'(x) = \lambda^\Delta \phi(\lambda x) \) under scale transformations.

From the commutation relations in the third line of (2.2), it follows that \( P_\mu \) raises the conformal dimension by one, and \( K_\mu \) lowers it. In unitary field theories,
there are lower bounds for the possible values of $\Delta$, e.g. $\Delta \geq (d - 2)/2$ for scalar fields. Hence, for each representation of the conformal group there must be some operator of lowest dimension, which is then annihilated by $K_\mu$. Such operators are called (quasi) primary operators.

### 2.2 Supersymmetry

Another generalization of ordinary Poincaré symmetry is supersymmetry (susy). Supersymmetry introduces transformations that relate bosons and fermions. Although exact supersymmetry is not observed in nature, it is believed that the supersymmetry may be spontaneously broken. Because of this, there is great interest in supersymmetric extensions of the standard model. This is generally seen as the way to go to unify the fundamental forces, although it also introduces new problems.

One can introduce symmetry by extending the Poincaré algebra by including $\mathcal{N}$ spinor supercharges, where $\mathcal{N}$ is the number of independent supersymmetries. The generators are

$$I = 1, \ldots, \mathcal{N} \quad \begin{cases} Q_\alpha^I & \alpha = 1, 2 \\ \bar{Q}_{\dot{\alpha}} I & = (Q_\alpha^I)^\dagger \end{cases}$$

Here $Q_\alpha$ transforms as a left Weyl spinor, and $\bar{Q}_{\dot{\alpha}}$ as a right Weyl spinor.

A natural question is if it is possible to have both conformal and supersymmetry. This is indeed possible for some values of the dimension $d \leq 6$ and $\mathcal{N}$. The resulting algebra is called the superconformal algebra. The case of interest to us is the $\mathcal{N} = 4$ case in $d = 4$ dimensions, which has the maximum amount of supersymmetry in four dimensions. In addition to the supercharges, one also has to include their superconformal partners $S_\alpha^i$ and $\bar{S}_{\dot{\alpha}}^i$. Finally there is a $U(\mathcal{N})$ R-symmetry, which rotates the supercharges into each other. The generators of this symmetry are denoted by $R^i_j$, and they satisfy the commutation relations

$$[R^i_j, R^k_l] = \delta^i_k R^j_l - \delta^i_l R^j_k.$$ 

In the case $\mathcal{N} = 4$ it is possible to impose $R^i_i = 0$, and the R-symmetry algebra reduces to $\mathfrak{su}(4)$.

Adding these new generators enlarges the algebra. This gives the $\mathfrak{su}(2, 2|\mathcal{N})$ superalgebra. Because there are fermionic degrees of freedom now, this is no ordinary Lie algebra, but a graded algebra. See [10] for an overview of superalgebras. Schematically, the new additional algebra relations can be written as

$$[D, Q] = \frac{i}{2}Q, \quad [D, S] = -\frac{i}{2}S, \quad [K, Q] \simeq S, \quad [P, S] \simeq Q \quad \{Q, Q\} \simeq P, \quad \{S, S\} \simeq K, \quad \{Q, S\} \simeq M - D + R$$

For the precise relations one can refer to an introduction to superalgebras.
2.2.1 Superconformal representations

Superconformal primary states can be completely characterised by six numbers: the scaling dimension $\Delta$, Lorentz spins $s_1, s_2$, and three Dynkin labels $[k_1, k_2, k_3]$. These three Dynkin labels determine the $\mathfrak{su}(4)$ irreps. The dimension of the corresponding irreps is

$$\dim([k_1, k_2, k_3]) = \frac{1}{12} (k_1 + k_2 + k_3 + 3)(k_1 + k_2 + 2)(k_2 + k_3 + 2)(k_1 + 1)(k_2 + 1)(k_3 + 1).$$

In the literature it is also common to denote the irreps by the ir dimension. For example, $20$ is used to denote the irrep $[0, 2, 0]$. For a superconformal primary operator $O$ we have

$$K_\mu O = 0, \quad D O = i \Delta O, \quad \tilde{S}^\alpha O = \bar{S}^{\dot{\alpha}} O = 0.$$

One can generate the supermultiplets by acting with the supercharges $Q^a_i$ and $\bar{Q}^{\dot{a}}_{\dot{i}}$ on the superconformal primary states. The number of states in such a generic (long) multiplet is

$$2^{16} \dim([k_1, k_2, k_3])(2s_1 + 1)(2s_2 + 1).$$

There are some special primary operators. These are called chiral primary operators (CPOs). These operators satisfy BPS-like conditions (after Bogomol’nyi, Prasad and Sommerfeld). That is, they are annihilated by some of the supercharges. Using the supersymmetry algebra, one can show that there are two cases: $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS operators. They are annihilated by $1/2$ and $1/4$ of the supercharges respectively. These conditions impose some restrictions on the corresponding quantum numbers. In particular, they have no spin. In the case of $\frac{1}{2}$-BPS operators, the Dynkin labels are $[0, k, 0]$ and the dimension $\Delta = k$. For $\frac{1}{4}$-BPS these are $[q, p, q], \Delta = p + 2q$ respectively. The corresponding supermultiplets have less states than the ordinary long multiplets [11]. For this reason, they are called short multiplets. The dimensions for multiplets for which the conformal primary state is a $\frac{1}{2}$-BPS operator are protected from quantum corrections.

2.2.2 $\mathcal{N} = 4$ supersymmetric Yang-Mills

In this section the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in $d = 4$ (Euclidean) dimensions is described. For a more elaborate introduction, see e.g. [12]. This theory is superconformal, and has 32 susy generators. We consider the gauge group $SU(N)$, where we note that we are interested in the limit where $N$ is large.

The action, with coupling constant $g_{YM}$, is then

$$S = \frac{1}{g_{YM}^2} \int d^4x \left\{ \frac{1}{4} (F^a_{\mu\nu})^2 + \frac{1}{2} (D_\mu \phi^a)^2 + \frac{1}{4} f^{abc} f^{ade} \phi^b_i \phi^d_j \phi^e_i \phi^j \right. \left. + \frac{1}{2} \bar{\psi}^a \Gamma^\mu D_\mu \psi^a + \frac{1}{2} f^{abc} \bar{\psi}^a \Gamma^i \phi^b_i \psi^c \right\},$$

(2.3)
where $\Gamma^\mu$ and $\Gamma^i$ are the ten dimensional gamma matrices in the Majorana-Weyl representation. The field strength and covariant derivative are given by the expected expressions

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^a A_\nu^b, \quad D_\mu \phi_i^a = \partial_\mu \phi_i^a + f^{abc} A_\mu^b \phi_i^c.$$

The interesting operators in this theory are the $\frac{1}{2}$-BPS operators mentioned above. By the correspondence, they correspond to scalar fields $s_k$ in the supergravity theory. On the gauge side, these operators are given by

$$O^I_k = C^I_{i_1 \ldots i_k} \text{tr} \left( \phi^{(i_1} \ldots \phi^{i_k)} \right),$$

where the trace is over $SU(N)$ indices. The tensor $C^I_{i_1 \ldots i_k}$ is a traceless symmetric $SO(6)$ tensor (see Appendix A for more details), and $I$ runs over the basis of the corresponding $SU(4)$ representation with Dynkin labels $[0, k, 0]$. The main part of this thesis is to approximate the four-point correlator

$$\langle O^I_1(x_1) O^I_2(x_2) O^I_3(x_3) O^I_4(x_4) \rangle,$$

in the supergravity approximation.

### 2.3 Type IIB strings

Type IIB is a specific string theory that arises when one adds fermionic degrees of freedom and supersymmetry to the bosonic string living in 26 dimensions [13, 14]. Starting point is the superstring, which has to live in $d = 10$ dimensions to be consistent. The simplest description is in the superconformal gauge, where the action becomes

$$S = \frac{1}{4\pi \alpha'} \int d^2 \sigma \left( \partial_+ X^\mu \partial_- X_\mu - i \psi_L \cdot \partial_+ \psi_L - i \psi_R \cdot \partial_- \psi_R \right),$$

where light-cone coordinates $\sigma_\pm = \tau \pm \sigma$ are used. The fields $\psi_{L,R}$ denote the left- and right-moving fermionic degrees of freedom. The coordinate $\sigma$ has range $[0, 2\pi]$ and is periodic. Thus, we consider the closed superstring.

There are two consistent boundary conditions for the closed superstring, namely periodic and anti-periodic:

$$\psi_{L,R}(\sigma) = \pm \psi_{L,R}(\sigma + 2\pi).$$

Periodic boundary conditions are called Ramond (R) boundary conditions, while the anti-periodic boundary conditions are called Neveu-Schwarz (NS) boundary conditions.
conditions. The corresponding states are said to be in the R-sector and NS-sector respectively. The conditions can be imposed separately on the left- and right-movers, so that there are a total of four possibilities

\[(R,R), \ (NS,NS), \ (R,NS), \ (NS,R).\]

The states in the first two sectors are bosonic, those in the later two fermionic.

The theory can be quantized in the usual way, for example by doing the canonical quantization by substituting Poisson brackets with (anti)-commutators, and obtain an oscillator description with annihilation and creation operators. The spectrum can then be obtained by acting with the creation operators on the ground state for the NS- and R-sectors separately. The (R,R), (NS,NS) etc. sectors are then obtained by tensoring. One finds [14] for the lowest levels a tachyon with \(\alpha' m^2\) negative, a massless multiplet and massive states. The tachyon is troublesome, because it signals an instability. Another issue is that the number fermionic and bosonic states do not match, which is what we would expect from a supersymmetric theory. These problems however can be solved.

To remove the problematic states and get a consistent theory, one must perform the so-called GSO projection, after Gliozzi, Scherk and Olive. This restores modular invariance, removes tachyons from the theory, and provides space-time supersymmetry. In the Ramond sector, there are two consistent ways to do this. Doing the same projection on both left- and right-moving modes gives the Type IIB superstring. This is a chiral \(\mathcal{N} = (2,0)\) theory. The other consistent way is to impose opposite GSO conditions, which gives the Type IIA superstring. This is a non-chiral \(\mathcal{N} = (1,1)\) theory.

### 2.4 Supergravity

Type IIB supergravity is the low-energy limit of Type IIB string theory, which corresponds to the limit \(\alpha' \to 0\). Since the states in Type IIB theory have mass squared of order \(1/\alpha'\), the supergravity limit is given by the massless states: the massive states decouple. The representation content is given by these representations of the little group \(SO(8)_L\).

\[
\begin{align*}
1 & \oplus 28 + 35_v + 1 + 28 + 35_c \\
8_s + 8_s + 56_s + 56_s.
\end{align*}
\]

The first line corresponds to the bosonic part, the second line to the fermions. Hence, there are 128 bosonic and 128 fermionic states.

The bosonic supergravity states have specific names:

- The \(1\) (from the (NS,NS)-sector) is the dilaton \(\phi\)
- The \(28\) is a anti-symmetric rank-two tensor \(B_{\mu\nu}\)

\[\text{Actually one should consider the universal cover, since there are also spinor representations.}\]
The $35_v$ is the graviton $g_{\mu\nu}$

The $1$ (from the (R,R)-sector) is the axion $\chi$

The $28$ is an anti-symmetric rank-two tensor $B_{\mu\nu}$

The $35_c$ corresponds to a four-form $A^{(4)}$ with self-dual field strength $F^{(5)} = dA^{(4)}, \ast F = F$

Sometimes the two real scalars $\phi$ and $\chi$ are combined into a complex field $\tau$. The most important fields in our case are the graviton and the self-dual tensor. We will compute the four-point function of scalar fields $s_k$, which arise after compactification, that are a mix of the trace of the graviton and the four-form. This is explained in the next section.

2.5 Compactification on $\text{AdS}_5 \times S^5$

The next step is to find the supergravity solution on a $\text{AdS}_5 \times S^5$ background. This process is called compactification: 5 dimensions get compactified on the sphere $S^5$. This has been carried out in [15, 16] to find the resulting mass spectrum. The basic procedure is as follows: consider the equations of motion, and expand them in fluctuations around the $\text{AdS}_5 \times S^5$ background. These fluctuations can then be expanded in spherical harmonics on $S^5$. From the resulting equations the arising spectrum and corresponding masses can be read off.

We now carry out this procedure for the bosonic degrees of freedom. For the fermionic degrees, a similar approach can be made, but they do not contribute to the specific four-point function we want to compute. The covariant equations of motion of the bosonic part are

\begin{equation}
R_{mn} = \frac{1}{6} F_{mijkl} F^{ijkl} \tag{2.6}
\end{equation}

\begin{equation}
F_{mnopq} = \frac{1}{5!} \varepsilon_{mnopqabcde} F^{abcde} \tag{2.7}
\end{equation}

Equation (2.7) is the condition $\ast F = F$, i.e. it expresses the self-duality of the field strength $F$. This field strength is given by

$$F_{mnopq} = 5 \partial_{[m} A_{nopq]} = \partial_m A_{nopq} + 4 \text{ other.}$$

The background solution on $\text{AdS}_5 \times S^5$ is (in units where the radius of $S^5$, $R_0$, is set to unity)

$$ds^2 = \frac{1}{z^2_0} (dz_0^2 + \eta_{\mu\nu} dx^\mu dx^\nu) + d\Omega_5^2 \equiv g_{mn} dx^m dx^n$$

$$R_{\mu\nu\sigma} = -(g_{\mu\nu} g_{\lambda\sigma} - g_{\mu\sigma} g_{\nu\lambda}); R_{\mu\nu} = -4 g_{\mu\nu} \tag{2.8}$$

$$R_{\alpha\beta\gamma\delta} = (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\gamma\beta}); R_{\alpha\beta} = 4 g_{\mu\nu}$$

$$F_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = \varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}, \quad F_{\mu_1\mu_2\mu_3\mu_4\mu_5} = \varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}$$

13
Latin indices are indices of the ten-dimensional space, while $\alpha, \beta, \ldots$ are $S^5$ indices and $\mu, \nu, \ldots$ are AdS$_5$ indices. This specific choice for $F$ is called the Freund-Rubin ansatz.

We then consider fluctuations around the background solution (2.8). Write

$$ G_{mn} = g_{mn} + h_{mn}, \quad A_{mnpq} = A_{mnpq} + a_{mnpq}, \quad F = \mathcal{F} + f, $$

where $\mathcal{A}$ is the 4-form potential that gives the field strength $\mathcal{F}$. Note that $f$ depends on $a_{mnpq}$. To remove the gauge symmetry, one can impose the de Donder gauge

$$ \nabla^\mu h_{\alpha\mu} = \nabla^\mu h_{(\mu\nu)} = \nabla^\mu a_{\mu\nu\rho\sigma} = 0, \quad h_{(\mu\nu)} \equiv h_{\mu\nu} - \frac{1}{5} g_{\mu\nu} h^\sigma_{\sigma}, $$

together with

$$ a_{\mu\nu\rho\sigma} = \varepsilon_{\mu\nu\rho\sigma} \nabla^\tau b; \quad a_{\mu\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma\delta} \nabla^{\delta} \phi_{\mu}, $$

One can also introduce the dual forms for $a_{\mu\nu\rho\sigma}$ and $a_{\mu\nu\rho\alpha}$,

$$ a_{\mu\nu\rho\sigma} = -\varepsilon_{\mu\nu\rho\sigma} Q^\tau; \quad a_{\mu\nu\rho\alpha} = -\varepsilon_{\mu\nu\rho\alpha} \phi^{\sigma\tau}. $$

Considering that the compact space is $S^5$, it’s natural to decompose the fluctuations in spherical harmonics (see Appendix A). To illustrate this, an example of the expansion of some of the fields:

$$ \pi(x, y) \equiv h_{\sigma\sigma}(x, y) = \sum \pi_{I}^{I_1}(x) Y_{k}^{I_1}(y) $$

$$ b(x, y) = \sum b_{I}^{I_1}(x) Y_{k}^{I_1}(y), $$

where $x$ are coordinates on the AdS space, and $y$ on the sphere. The indices $I_1$ are summed over, and run over the basis of the corresponding representations. There are more fields, which are also expanded into spherical harmonics. These are not written here for the sake of brevity.

After expanding the equations of motion (2.6) and (2.7) up to linear order in the fluctuations, and decompose them into the spherical harmonics, one is left with systems of coupled differential equations, for instance

$$ \square_x \begin{pmatrix} \pi \\ b \end{pmatrix} + \begin{pmatrix} \square_y - \frac{32}{5} & 80 \square_y \\ -\frac{4}{5} & \square_y \end{pmatrix} \begin{pmatrix} \pi \\ b \end{pmatrix} = 0. \quad (2.9) $$

After plugging in the expansions in spherical harmonics, the system can be diagonalized by finding the eigenfunctions, and use that $\square_y Y_k = -k(k + 4) Y_k$. The eigenfunctions are $s_k$ and $t_k$, which are related to $\pi_k$ and $b_k$ by

$$ \pi_k = 10k s_k + 10(k + 1) t_k $$

$$ b_k = -s_k + t_k. $$

---

Footnote: Inessential representation indices are omitted for clarity.
The corresponding eigenvalues are $m_{s_k}^2 = k(k - 4), k \geq 2$ and $m_{t_k}^2 = (k + 4)(k + 8), k \geq 0$. The notation of the eigenvalues as a mass squared, is natural, since in terms of the eigenfunctions the system becomes

$$(\nabla^\mu \nabla_\mu - m_{s_k}^2)s_k = 0, \quad (\nabla^\mu \nabla_\mu - m_{t_k}^2)t_k = 0.$$  

The representations obtained after the compactification are grouped into *Kaluza-Klein (KK) multiplets*. The scalar fields $s_k$ are primary states of these multiplets, and the full multiplets can be obtained by acting with the supersymmetry generators. The AdS-masses of the fields in a multiplet are generally different. This is related to the fact that the mass operator is not a Casimir of the supersymmetry group. For example, the KK multiplet obtained from the scalar field $s_2$ contains a massless symmetric tensor, i.e. the *graviton*. This is why this multiplet is called the *massless multiplet*. The scalar fields $s_k$ are commonly called KK modes.

The diagonalization of the other equations is done in a similar fashion, and allows finding the complete mass spectrum of the theory. The decomposition in terms of spherical harmonics also gives information about the representation of $SU(4)$ the fields transform as, namely those associated with the corresponding spherical harmonics. The resulting mass spectrum of all fields that contribute to four-point functions of $\frac{1}{2}$-BPS operators are listed in Table 2.1.

<table>
<thead>
<tr>
<th>Spin</th>
<th>Field</th>
<th>$m^2$</th>
<th>$SU(4)$ irrep.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$s_k$</td>
<td>$k(k - 4)$</td>
<td>$[0, k, 0]$</td>
</tr>
<tr>
<td>0</td>
<td>$t_k$</td>
<td>$(k + 4)(k + 8)$</td>
<td>$[0, k, 0]$</td>
</tr>
<tr>
<td>0</td>
<td>$\varphi_k$</td>
<td>$k(k + 4)$</td>
<td>$[0, k, 0]$</td>
</tr>
<tr>
<td>1</td>
<td>$A_{\mu,k}$</td>
<td>$k^2 - 1$</td>
<td>$[1, k - 1, 1]$</td>
</tr>
<tr>
<td>1</td>
<td>$C_{\mu,k}$</td>
<td>$(k + 3)(k + 5)$</td>
<td>$[1, k - 1, 1]$</td>
</tr>
<tr>
<td>2</td>
<td>$\phi_{\mu\nu,k}$</td>
<td>$k(k + 4)$</td>
<td>$[0, k, 0]$</td>
</tr>
</tbody>
</table>

Table 2.1: Mass spectrum of Type IIB supergravity on $\text{AdS}_5 \times S^5$ contributing to four-point functions corresponding to $\frac{1}{2}$-BPS operators (from [15]).

---

We have $SO(6) \simeq SU(4)$, or more precisely, $SU(4)$ is the covering group of $SO(6)$. We will be sloppy and identify $SO(6)$ and $SU(4)$ representations in this thesis. The difference only matters for representations for fermions, which we do not need.
Chapter 3

The AdS/CFT correspondence

The AdS/CFT correspondence was first conjectured by Maldacena [1]. Mathematical details were filled in shortly after this paper, by Witten [17] and Gubser, Klebanov and Polyakov [18]. It should be stressed that the correspondence still has the status of a conjecture, although non-trivial checks have been performed that support the conjecture.

This chapter tries to introduce the main ingredients of the conjecture, to get a bit of feeling what is going on. It will provide the necessary background for the main results in this thesis, namely the calculation of the four-point function. During the years after the proposal by Maldacena, a large number of reviews and lectures on the subject appeared. For more background material one can refer to these publications, e.g. [5, 6, 19, 20]. This list is somewhat arbitrary, and many publications on the subject are not included. This chapter is based largely on these reviews, as well as the papers by Maldacena and Witten.

3.1 Anti-de Sitter space

Anti-de Sitter space (AdS for short) is the maximally symmetric solution of the Einstein equations with a negative cosmological constant \( \Lambda \). There are several ways to describe AdS space. The simplest way to visualise it is to consider it as a space embedded in flat \((d+2)\)-dimensional space with signature \((2,d-1)\), where it is the hyperboloid

\[
X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2. \tag{3.1}
\]

The parameter \( R \) is the radius of the AdS space. From this description the \( SO(2,d) \) isometry is obvious.

It is possible to solve (3.1) by

\[
X_0 = R \cosh \rho \cos \tau, \quad X_{d+1} = R \cosh \rho \sin \tau, \quad X_i = R \sinh \rho \Omega_i, \quad (i = 1, \ldots, d; \sum_i \Omega_i^2 = 1), \tag{3.2}
\]
where \( \rho \geq 0 \) and \( 0 \leq \tau < 2\pi \) cover the hyperboloid once. Substituting this in the \((d+1)\) dimensional metric gives the metric on \( \text{AdS}_{d+1} \)

\[
ds^2 = R^2 \left( - \cosh^2 \rho \tau^2 + d\rho^2 + \sinh^2 \rho d\Omega^2 \right).
\]

Near \( \rho = 0 \), the metric behaves like \( ds^2 \simeq -d\tau^2 + d\rho^2 + \rho^2 d\Omega^2 \), which is the metric of \( S^1 \times \mathbb{R}^d \). Hence, there are closed timelike curves in the \( \tau \) direction. To solve this problem, one can consider the universal covering of \( \text{AdS}_{d+1} \), obtained by unwrapping to \( \tau \) coordinate to \( -\infty < \tau < \infty \). In the remaining part we usually refer to this universal cover when we consider AdS space.

Another set of coordinates are \((u,t,x)\), where \( u > 0 \) and \( x \in \mathbb{R}^{d-1} \). The solution of (3.1) in these coordinates is

\[
X_0 = \frac{1}{2u} \left( 1 + u^2 (R^2 + x^2 - t^2) \right),
\]

\[
X_i = Rux_i \quad (i = 1, \ldots, d-1),
\]

\[
X_d = \frac{1}{2u} \left( 1 - u^2 (R^2 - x^2 + t^2) \right),
\]

\[
X_{d+1} = Ru t.
\]

The associated metric is

\[
ds^2 = R^2 \left( \frac{du^2}{u^2} + u^2 (-dt^2 + dx^i dx^i) \right).
\]

These coordinates are called Poincaré coordinates.

### 3.1.1 Euclidean continuation

In calculations it’s easier to consider the Euclidean continuation of \( AdS_{d+1} \). This is obtained by performing the Wick rotation \( \tau \rightarrow \tau_E = -i\tau \), and \( X_{d+1} \rightarrow X_E = -iX_{d+1} \) in equations (3.1). It is then possible to solve these equations (c.f. (3.2)), by doing the Wick rotation in the solutions (3.2) and (3.4). In the case of Poincaré coordinates, this leads to the metric

\[
ds_E^2 = R^2 \left( \frac{du^2}{u^2} + u^2 (d\tau_E^2 + dx_i^2) \right).
\]

This simplifies a bit by setting \( u = 1/z_0 \). This way we get the metric

\[
ds^2 = \sum_{\mu,\nu=0}^d g_{\mu\nu} dz_\mu dz_\nu = \frac{1}{z_0^2} \left( dz_0^2 + \sum_{i=1}^d dz_i^2 \right),
\]

where the radius \( R \) is set to unity. This space can also be obtained from doing an Euclidean continuation of the Poincaré coordinates, even though they cover only part of the AdS hyperboloid.
In this description, the flat space boundary can be identified as follows [17]: at \( z_0 = 0 \), there is a copy of \( \mathbb{R}^d \), while for \( z_0 \to \infty \), the metric (3.5) becomes that of a point. Hence, the boundary of the Euclidean description of AdS_{d+1} is \( \mathbb{R}^d \) plus a point. But this is the conformal compactification \( S^d \) of \( \mathbb{R}^d \), since it is well known that this is obtained by adding a point at infinity to \( \mathbb{R}^d \).

The Christoffel symbols in this Euclidean coordinate basis are given by
\[
\Gamma^k_{\mu\nu} = \frac{1}{z^0} \left( \delta^0_\delta \delta^k_\mu - \delta^0_\delta \delta^k_\nu - \delta^0_\delta \delta^k_\mu \right).
\] (3.6)

From the Christoffel symbols we can find the Riemann tensor
\[
R^\alpha_{\mu\nu\sigma} = \frac{1}{z^2} \left( \delta^\alpha_\delta \delta^\mu_\nu - \delta^\alpha_\delta \delta^\mu_\sigma \right).
\] (3.7)

From this expression it’s easy to see that the Ricci tensor is \( R_{\mu\nu} = dg_{\mu\nu} \) and the scalar curvature is \( R = d(d+1) \), hence constant.

An important quantity is the AdS-invariant chordal distance \( u \), defined by
\[
u = \frac{(z-w)^2}{2z_0 w_0}, \quad (z-w)^2 = \delta_{\mu\nu}(z-w)_\mu(z-w)_\nu.
\] (3.8)

AdS-invariant functions such as propagators can be expressed in terms of this chordal distance, due to homogeneity and isotropy of the AdS space.

### 3.2 The conjecture

A key rôle in the AdS/CFT correspondence is played by D3-branes (see e.g. [21] for an introduction). Basically, a Dp-brane is a \((p+1)\)-dimensional hyperplane in spacetime, where open strings can end. It is also a source of closed strings, and can therefore carry \((R, R)\) charges. If one puts \( N \) Dp-branes on top of each other, the resulting hyperplane carries \( N \) units of the corresponding \((p+1)\)-form charge. One can show that in Type IIB theory, a D3-brane breaks one half of the supersymmetry, that is, it is a \( \frac{1}{2} \)-BPS object. A key ingredient is that the world-volume effective action of a configuration of \( N \) coincident D3-branes, that is we take the distance \( r \) between them small, is a \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory in \( d = 4 \), with a \( SU(N) \) gauge group.

A D3 brane solution of supergravity is given by the metric
\[
ds^2 = f^{1/2}(dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + r^2d\Omega_5^2)
\]
\[
f = 1 + \frac{R^4}{r^4}.
\] (3.9)

Here \( R \) is a length scale, and \( d\Omega_5^2 \) is the metric of the 5-sphere. In a D3-brane configuration with \( N \) units of flux of the 5-form field strength \( F \), the length scale \( R \) is given by
\[
R^4 = 4\pi g_s N \alpha'^2,
\] (3.10)
where $g_s$ is the string coupling constant.

Now consider the “near horizon” limit of the metric (3.9). In this limit $r \ll R$, the 1 in the definition of $f$ can be neglected, and the metric reduces to

$$ds^2 = \frac{r^2}{R^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2 \frac{dr^2}{r^2} + R^2 d\Omega_5^2. \quad (3.11)$$

It is convenient to make the coordinate substitution $u = r/R^2$. The metric then becomes in these new coordinates

$$ds^2 = u^2 R^2 (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2 \frac{du^2}{u^2} + R^2 d\Omega_5^2.$$

But this is precisely the metric of AdS$_5 \times$S$^5$. Maldacena argued that this region $r \ll R$ should be identified with the low energy Yang-Mills description from the first part of this section [1]. Hence, he conjectured that $\mathcal{N} = 4$ SYM$_4$ in four dimensions is the same as Type IIB string theory on AdS$_5 \times$S$^5$.

In the argument a few approximations are made. It is good to keep this in mind, and note the conditions when these approximations are valid. The limit Maldacena considered, which is the one we use, is the large $N$ limit of the Yang-Mills theory. The classical supergravity solution when $g_s \ll 1$ is valid at large length scale with respect to string scale, that is, $\alpha' \ll R^2$. This requires $N \to \infty$ with $4\pi g_s N$ fixed and large, by equation (3.10). This relates to the ’t Hooft limit in field theory, because the correspondence relates the coupling constants in the two theories by

$$g_s = \frac{g_{YM}^2}{4\pi}, \quad \chi_0 = \frac{\theta_{YM}}{2\pi}.$$

Here, $\chi_0$ is the value of the RR scalar, which is constant in Type IIB theory, and $\theta_{YM}$ is the Yang-Mills vacuum angle. Hence, $4\pi g_s N$ large but fixed corresponds to $\lambda = g_{YM}^2 N$ large and fixed. The interesting thing is that we have a strong/weak duality, in the sense that the strong coupling limit in SYM$_4$ (where calculations cannot be done perturbatively) corresponds to a low energy approximation, i.e. the supergravity approximation, and vice versa.

There are a few comments to be made on the conjecture. First of all, the location of the SYM$_4$ is conjectured to be the AdS boundary, located at $u \to 0$, as was mentioned in the previous section. There, the boundary values of the bulk supergravity fields act as sources for gauge-invariant composite operators of the Yang-Mills theory. In the next section this will be made more precise.

It should be stressed that the above derivation is not a proof. For this, one needs to treat the string theory non-perturbatively. This is at the moment not accessible; one of the reasons is that the full string spectrum of Type IIB theory on AdS$_5 \times$S$^5$ is not known. That said, however, it should be noted that there have been a large number of checks that support the conjecture, see for example Section 3.2 in [5] for an (incomplete) list.

Another aspect to be mentioned is that the conjecture has been generalised to other spacetimes involving Anti-de Sitter spaces and some compact manifold,
see one of the review papers mentioned earlier. There have also been attempts to make similar conjectures for non-conformal field theories like QCD, see e.g. [22] for a review.

3.3 Correlation functions

The conjecture by Maldecena does not specify how to compute e.g. correlation functions in the correspondence. This was addressed by Witten [17] and Gubser, Klebanov and Polyakov [18]. In these papers, a method to calculate correlation functions was proposed. In this section the general idea of the prescription is described. There are some technicalities involved, for more details one can refer to the cited papers. At the end of the section the four-point function is described from the field theory point of view. These considerations lead to constraints on the form of the four-point function.

3.3.1 Correlators in the correspondence

The most natural objects on the CFT side are operators $O$. The gravity side however is described in terms of fields $\phi(x)$. The CFT theory is seen as living on the boundary of AdS, hence it is natural to assume that correlation functions depend on the boundary value $\phi_0(x)$ of $\phi(x)$. The assumption is then that this boundary field couples to the operator $O(x)$, by a term $\int d^4x O(x)\phi_0(x)$. More precisely, it is argued that the generating functional is

$$\langle e^{\int d^4x \phi_0(x)O(x)} \rangle_{\text{CFT}} = Z_{\text{string}}[\phi(x, z) = \phi_0(x)]. \quad (3.12)$$

Here, $Z_{\text{string}}$ is the full string partition function. This is the so called strong form of the conjecture, which holds for arbitrary values of $\lambda$ and $N$. In the large $N$ limit where classical supergravity is applicable, the string partition function can be approximated by

$$Z_{\text{string}}(\phi_0) = \exp(-I_S(\phi)).$$

Here, $I_S$ is the classical supergravity action, evaluated on $\phi$, with the boundary condition $\phi_0$. Since we now have an expression for the generating functional, we can use standard techniques to calculate correlation functions. In particular, to find a correlator of an operator $O$, one has to take functional derivatives of the generating functional with respect to the corresponding boundary value $\phi_0(x)$, and in the end set $\phi_0(x) = 0$. It should be noted that this description is not entirely complete: in some cases boundary terms have to be added. The origin of these boundary terms is explained in [23]. For the case under consideration here however, these terms play no role.

\footnote{For simplicity we consider a scalar field $\phi(x)$. The argument is similar for vectors and tensors.}

\footnote{These boundary terms arise when doing UV an IR regularization, to fix the breaking of conformal invariance.
Figure 3.1: Witten diagram for a three-point function of scalar fields

The correct boundary conditions on-shell supergravity action can be obtained by introducing the \textit{bulk-to-boundary propagator} $K_{\Delta}$, that propagates the boundary field $\phi_0$ into the AdS bulk. That is, one has schematically

$$\phi(z) = \int d^4x K_{\Delta}(z;x)\phi_0(x),$$

where $z$ is a point in the AdS$_5$ bulk, and $x$ is a point on the (4-dimensional) boundary. This way the contributions to correlators can be described graphically by so-called \textit{Witten diagrams}. For example, a three-point function can be described by the diagram in Figure 3.3.1. The circle represents the AdS$_5$ boundary, and the lines represent propagators that propagate the boundary fields into the AdS$_5$ bulk. The interaction point in the middle should be integrated over the AdS$_5$ bulk, just as for ordinary Feynman diagrams. In the supergravity approximation, only tree diagrams contribute. If higher order corrections are necessary, one also has to include loop diagrams.

As mentioned: for each operator in the gauge theory, there is a corresponding field propagating from the boundary to the bulk on the string side. An important question arises then: how do we find the corresponding field for an operator, and vice versa. For a scalar field with AdS$_{d+1}$ mass $m^2$ (in units where the radius of curvature is 1), the scaling dimension of the corresponding operator is

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}.$$  

This relation can be derived by analysing the behaviour of $\int d^4x \phi_0(x)O(x)$ near the AdS boundary [17]. For fields with spin there are similar relations, for example for vectors

$$\Delta_\pm = \frac{1}{2}(d \pm \sqrt{(d-2)^2 + 4m^2}),$$

and for a massless spin-2 field $\Delta = d$.  

21
\[ \mathcal{N} = 4 \text{ SYM}_4 \quad \text{AdS}_5 \times S^5 \text{ supergravity} \]

<table>
<thead>
<tr>
<th>operators ( \mathcal{O} )</th>
<th>fields on AdS$_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>global symmetry $SU(2, 2</td>
<td>4)$</td>
</tr>
<tr>
<td>$SU(4)$ R-symmetry</td>
<td>$SO(6)$ symmetry of $S^5$</td>
</tr>
<tr>
<td>gauge group $SU(N)$</td>
<td>$S^5$ flux $\int F = N$</td>
</tr>
<tr>
<td>$\langle e^\int d^4x \phi_0(x) \mathcal{O}(x) \rangle$</td>
<td>$Z_{\text{string}}[\phi(x, z)</td>
</tr>
<tr>
<td>CPOs $\mathcal{O}_k$</td>
<td>scalar fields $s_k$</td>
</tr>
<tr>
<td>scaling dimension $\Delta$</td>
<td>mass $m_k^2 = \Delta(\Delta - 4)$</td>
</tr>
<tr>
<td>$\lambda \equiv g_Y^2 N$</td>
<td>$1 \ll g_s N &lt; N$ (in SUGRA)</td>
</tr>
</tbody>
</table>

Table 3.1: Corresponding quantities between the gauge and string theory according to the correspondence

One should also match the quantum numbers corresponding to the global symmetry $SU(2, 2|4)$ of both theories. In particular, one can consider the maximal bosonic subgroup $SO(4, 2) \times SU(4)$. One can then argue that the single-trace CPOs in $\mathcal{O}_k$ in SYM$_4$ correspond to the scalar fields $s_k$ on AdS$_5$ [17]. These are our fields of interest.

The basic statements of the AdS/CFT conjecture can then be summarized in Table 3.1. This table lists the identifications between quantities on the $\mathcal{N} = 4$ SYM$_4$ side and Type IIB supergravity on an AdS$_5 \times S^5$ background.

### 3.3.2 CFT predictions

The conformal structure of SYM$_4$ restricts the form of the four-point function. In this section the general form of the four-point function of two weight-2 and two weight-3 $1/2$-BPS operators is discussed. This section is based on the method that is outlined in Sections 2 and 3 of [24].

The operators we consider are of the form (2.4). A natural basis in the field theory then arises by connecting the four points with the free propagators of the scalar fields $\phi^i$. These are called Wick contractions. The propagators for the $\mathcal{N} = 4$ SYM$_4$ scalar fields are given by

\[ \langle \phi^i(x_1)\phi^j(x_2) \rangle = \frac{\delta^{ij}}{x_{12}}. \tag{3.13} \]

To keep track of the $SO(6)$ indices, it is convenient to introduce harmonic variables. This is a complex vector $u_\iota$, satisfying

\[ u_\iota u_\iota = 0, \quad u_\iota \bar{u}_\iota = 1. \tag{3.14} \]

With this variable we can “project” the tensors $C_{i_1 \ldots i_k}^{J}$, to get

\[ \mathcal{O}_k = u_{i_1} \ldots u_{i_k} \text{tr}(\phi^{i_1} \ldots \phi^{i_k}). \]
From equation (3.14) it follows immediately that the structure multiplying the is indeed traceless and totally symmetric, as expected. We can also project the propagator (3.13)

\[ \langle \phi(x_1) \phi(x_2) \rangle = \frac{u^1 u^2}{x_{12}^2} = \frac{(12)}{x_{12}^2} = \frac{(21)}{x_{12}^2}, \]  

(3.15)

where \( u^i_n \) is the harmonic at point \( x_n \).

To get the most general four-point function one has to consider the possible contractions. These can be conveniently denoted by the diagrams in Figure 3.3.2. At points \( x_1 \) and \( x_2 \) the operators are \( O^{(2)} \), hence there are two indices to contract. If we denote each contraction by a line, this means there are two lines emanating from the points \( x_1 \) and \( x_2 \). Similarly, there are three lines coming from \( x_3 \) and \( x_4 \). Note that we don’t have lines where the begin and end point are connected to the same coordinate \( x_i \), since by equation (3.14), these contributions are zero.\(^3\) All possible contractions are then easily found: these are precisely the diagrams in 3.3.2.

With expression (3.15) it is then very simple to get the expression corresponding to a diagram. Basically, each line (corresponding to a propagator) between point \( i \) and \( j \) is represented by \( (ij)/x_{ij}^2 \). To get the most general conformally invariant form, we can also multiply each structure by a function of the conformal

\(^3\)This is related to the tracelessness of the \( C \)-tensors.
cross-ratios. Following this procedure gives
\[
\langle O^2(x_1)O^2(x_2)O^3(x_3)O^3(x_4) \rangle = \frac{a(s, t)(12)^2(34)^3}{x_{12}^4x_{34}^4} + \frac{b(s, t)(13)(23)(14)(24)(34)}{x_{13}^2x_{14}^2x_{23}^2x_{24}^2x_{34}^4} + \frac{c_1(s, t)(12)(13)(24)(34)^2}{x_{12}^2x_{13}^2x_{24}^2x_{34}^4} + \frac{c_2(s, t)(12)(14)(23)(34)^2}{x_{12}^2x_{13}^2x_{23}^2x_{34}^4} + \frac{d_1(s, t)(13)^2(24)^2(34)}{x_{13}^4x_{24}^2x_{34}^4} + \frac{d_2(s, t)(12)(23)^2(34)}{x_{12}^2x_{14}^2x_{23}^2}
\]  
(3.16)

This is the most general form of the four-point function we consider.

By simple symmetry considerations, relations between the coefficient functions can be derived. Note that when permuting $x_1 \leftrightarrow x_2$, the cross ratios transform as $s \rightarrow s/t$, and $t \rightarrow 1/t$. On the other hand, the result from interchanging the two coordinates in figure 3.3.2 is obvious. Comparing this with the right-hand side of (3.16) it follows that we must have
\[
a(s, t) = a(s/t, 1/t), \quad b(s, t) = b(s/t, 1/t), \\
c_1(s/t, 1/t) = c_2(s, t), \quad d_1(s/t, 1/t) = d_2(s, t).
\]  
(3.17)

These are the crossing symmetry relations.

There is another relation between these coefficient functions. This is based on the insertion procedure, which is a well-known procedure in quantum field theory. The procedure gives additional constraints on the quantum part of the correlator.

According to [24], there is a function $\mathcal{F}(s, t)$, such that
\[
\frac{\partial}{\partial y^2_M} \langle O^2(x_1)O^2(x_2)O^3(x_3)O^3(x_4) \rangle = R^{2222} F^{0011}
\]  
\[
= \frac{(12)^2(34)^4}{x_{12}^4x_{34}^4} s + \frac{(13)(24)^2}{x_{13}^4x_{24}^4} t + \frac{(13)(14)(23)(24)}{x_{13}^2x_{14}^2x_{23}^2x_{24}^2} (s - t - 1)
\]  
\[
+ \frac{(12)(14)(23)(24)}{x_{12}^2x_{14}^2x_{23}^2x_{34}^4} (1 - s - t) + \frac{(12)(13)(24)(34)}{x_{12}^2x_{13}^2x_{24}^2x_{34}^4} (t - s - 1)
\]  
\[
\frac{(34)}{x_{34}^2} \mathcal{F}(s, t)
\]  
(3.18)

Comparing this with (3.16), one sees directly that the coefficients functions can be expressed in terms of a single function $\mathcal{F}(s, t)$, by
\[
a(s, t) = s\mathcal{F}(s, t), \quad d_1(s, t) = \mathcal{F}(s, t), \quad d_2(s, t) = t\mathcal{F}(s, t)
\]  
(3.19)

and the remaining coefficients by
\[
b(s, t) = (s - t - 1)\mathcal{F}(s, t)
\]  
\[
c_1(s, t) = (t - s - 1)\mathcal{F}(s, t)
\]  
\[
c_2(s, t) = (1 - s - t)\mathcal{F}(s, t)
\]  
(3.20)
The crossing symmetry relations (3.17) then imply that \( \mathcal{F}(s/t, 1/t) = t\mathcal{F}(s, t) \).

Hence, all dynamical information in the four-point function is completely determined by the single function \( \mathcal{F}(s, t) \) of the cross-ratios. From the supergravity point of view, there is no apparent reason this should be the case, so verifying that the supergravity-induced four-point function has indeed this structure, can be seen as evidence for the AdS/CFT correspondence.

Finally, we express (3.16) in terms of the \( SO(6) \) tensors from Appendix A. To do this, one has to remove the harmonics \( u^i_n \). This boils down to replacing each factor \( (ij) \) with a contraction of an index of the \( SO(6) \) tensor. For example,

\[
(13)^2(24)^2(34) = C^I_{IJ} C^{J}_{KL} C_{IJ事业}^{I事业} = \Upsilon^{1234}.
\]

In this notation equation (3.16) becomes

\[
\langle O_1^1(x_1)O_2^2(x_2)O_3^3(x_3)O_4^4(x_4) \rangle = a(s, t) \frac{\delta^{12}_{\text{sym}} \delta^{34}_{\text{sym}}}{x_1^2 x_3^2 x_2^2 x_4^2} + b(s, t) \frac{\delta^{1234}_{\text{sym}}}{x_1^2 x_3^2 x_2^2 x_4^2} + c_1(s, t) \frac{C_{1243}}{x_2^2 x_3^2 x_4^2} + c_2(s, t) \frac{C_{1234}}{x_1^2 x_2^2 x_3^2 x_4^2} + d_1(s, t) \frac{\Upsilon^{1234}}{x_1^2 x_2^2 x_3^2 x_4^2} + d_2(s, t) \frac{\Upsilon^{1243}}{x_1^2 x_2^2 x_3^2 x_4^2}.
\]  

(3.21)

This result is purely based on conformal field theory considerations and the insertion procedure. Later we will compare the result from supergravity calculations with this general form.

### 3.3.3 A small digression: free field theory

It is possible to calculate the diagrams in Figure 3.3.2 in the large \( N \) limit in free field theory [25]. See [26] for an alternative method. In this section a heuristic method is presented to determine the free field contributions. A proof of this method will not be presented here.

As an introduction, first consider a two-point function of CPOs. Conformal symmetry dictates that the the weight of the two operators are the same. In free theory, the two-point function is then found by contracting the scalar fields \( \phi^i \) pairwise, by using their propagator. We have to calculate expressions like

\[
\langle \text{tr}(\phi^{i_1}(x_1)\ldots \phi^{j_k}(x_1)) \text{tr}(\phi^{j_1}(x_2)\ldots \phi^{j_k}(x_2)) \rangle.
\]

In the large \( N \) limit, only planar diagrams contribute. These are found by contracting the \( i \) and \( j \) indices in the order they appear, and cyclic permutations. These permutations can be understood to be coming from the cyclic property of the trace. For example, for the two-point function of weight-2 CPOs this leads to

\[
\langle \text{tr}(\phi^{i_1}(x_1)\phi^{i_2}(x_1)) \text{tr}(\phi^{j_1}(x_2)\phi^{j_2}(x_2)) \rangle \propto \delta^{i_1j_1} \delta^{i_2j_2} + \delta^{i_1j_2} \delta^{i_2j_1}.
\]
One can then restore the $C$-tensors in the definition, and contract the indices, according to the expression found by contracting the method indicated above. In this case, we get $2\delta_{12}^2$.

The leading terms in the free field results are of some power $N$. To find this power, one first has to find the corresponding diagram like in Figure 3.3.2. These diagrams can be drawn on a sphere. The power of $N$ is then the number of parts it divides the surface of the sphere in.\(^4\) For example, the second diagram in 3.3.2 is of order $N^2$.\(^5\) Ignoring inessential factors of $g_{YM}^2$ and $2\pi$ (they cancel after changing the normalization), define the canonically normalized operators CPOs

$$O_2^I = \frac{O_1^I}{N\sqrt{2}}$$ and $O_3^I = \frac{O_1^I}{N^3\sqrt{3}}$.

The free-field contributions for the four-point function under consideration in this thesis can then be found by a similar method as described above. First, note that the expression behaves like

$$\langle \text{tr}(\phi^i_1\phi^i_2)(x_1) \text{tr}(\phi^j_1\phi^j_2)(x_2) \text{tr}(\phi^k_1\phi^k_2\phi^k_3)(x_3) \text{tr}(\phi^l_1\phi^l_2\phi^l_3)(x_4) \rangle,$$

where the coordinates $x_i$ are written outside the trace for brevity. The power of $N$ corresponding to the diagram can then be found by the method described above, i.e. by drawing it on the sphere. The combinatorical factor is a bit harder to find. First, one has to pair the fields $\phi^i$ according to the corresponding diagram. For example, if there is a line between $x_1$ and $x_3$, a field $\phi^i(x_1)$ and a field $\phi^i(x_3)$ should be paired, in the same order as they appear in the expression. Then, combinatorical factors can be found by cyclicly permuting the fields inside the trace, and keep the pairing the same, that is, if before permuting the first field inside the trace for $x_1$ and the first field inside the trace $x_2$ were paired, this should be done again after the permutation. The combinatorical factor is then the number of different distinctly different pairings obtained.

As an example, consider the third diagram in Figure 3.3.2. Doing the pairing, we find a result proportional to

$$\delta^{i_1k_1}\delta^{i_2k_2}\delta^{j_1l_1}\delta^{j_2l_2}\delta^{k_3l_3}.$$ 

Now we have to permute the fields in the trace. If for example we do a cyclic permutation of the second trace term, and do the pairing again in the same order, we find

$$\delta^{i_1k_1}\delta^{i_2k_2}\delta^{j_2l_1}\delta^{j_1l_2}\delta^{k_3l_3}.$$ 

Considering all permutations, this leads to 36 distinct pairings. Hence, taking into account the canonical normalization, the free field value of the diagram is $6/N^2$.

There is one more thing to consider: the crossing symmetries. For the $\Upsilon^{1234}$ and $C^{1234}$ the only crossing symmetry that leaves these diagrams invariant are

\(^4\)For disconnected diagrams, one has to consider a sphere for each connected component. The total power of $N$ is then the sum of the powers for each individual component.

\(^5\)Note that the lines in the middle do not intersect, hence the diagram consists of two “loops”.
$x_1 \leftrightarrow x_2$ and $x_3 \leftrightarrow x_4$ simultaneously. But doing this gives the same pairings as already found in the normal case. The same is true for the disconnected diagram.

For the $S^{1234}$ tensor however, the change $x_1 \leftrightarrow x_2$ does give new pairings, in fact another 36. Hence, the total combinatorical factor for this diagram is 72. Taking into account the canonical normalization, this becomes $12/N^2$.

Summarizing, the free field contributions we find this way are (after taking into account the canonical normalization)

$$a(s,t) = 1, \quad b(s,t) = \frac{12}{N^2}, \quad c_1(s,t) = \frac{6}{N^2}, \quad d_1(s,t) = \frac{6}{N^2}, \quad (3.22)$$

and similar for $c_2(s,t)$ and $d_2(s,t)$.

### 3.4 Lagrangian formulation

When one wants to compute four-point functions in the Type IIB supergravity on $\text{AdS}_5 \times S^5$, one faces the problem that the covariant equations of motion are essentially non-Lagrangian. This is due to the presence of the self-dual field strength $F$ [27]. By removing unphysical degrees of freedom, and introducing auxiliary non-propagating fields it is possible to obtain a manifestly covariant action [28, 29].

This method has been used to find the quadratic part of the Lagrangian on an $\text{AdS}_5 \times S^5$ background [30]. However, when one needs higher order corrections to calculate three- and four-point functions, this method becomes very cumbersome, and it is easier to work directly with the equations of motion.

The procedure is basically the same as the one used to obtain the compactification of Type IIB theory in Section 2.5. However, instead of using the linearized equations of motion, one has to consider them up to quartic corrections for the cubic vertices, and up to cubic corrections for quartic vertices. After integrating the equations of motion, these give the cubic and quartic couplings in the Lagrangian respectively. The problem can be simplified a bit by including only the relevant corrections. For example, we are only interested in the quartic terms of fields that are dual to the $1/2$-BPS operators in SYM$_4$.

The main problem with the method is that the equations of motion that are obtained in this way are non-Lagrangian, that is, they cannot be derived from an action. To solve this problem, one has to do a complicated analysis, to perform chain of field redefinitions that cast the equations of motion into a Lagrangian form. This procedure has been carried out in [31]. First, one removes terms with six derivatives. The equations of motion thus obtained still aren’t Lagrangian, and further field redefinitions are necessary. By doing these further redefinitions and using symmetry properties of integrals of spherical harmonics, finally the complete Lagrangian of the quartic couplings of scalar fields $s_k$ was obtained. Cubic couplings were found earlier in Refs [25, 32, 33].

The expressions for the couplings look very complicated. One thing for example is the presence of four-derivative terms. It is not possible to remove this terms with
an additional field transformation. However, in each case considered so far it was
found that these terms vanish, by explicitly specifying the representation content,
instead of using the integrals of spherical harmonics. Because of these results, it
is suggested that there might be a $\sigma$-model type description of the supergravity
theory.

There is also a side effect of the field redefinitions: because of these redefin-
itions, the scalar fields $s_k$ don’t correspond to (single-trace) CPOs anymore, but
to extended CPOs. This was already argued in [32], because there it was found
that so-called extremal cubic couplings vanish. This means that the extremal
three-point functions vanish as well, which certainly isn’t true for ordinary CPOs
in SYM$_4$. The extended CPOs that correspond to the scalar fields $s_k$ can be
obtained by adding a proper combination of multi-trace CPOs

$$\hat{O}^{I_1} = O^{I_1} + \frac{1}{2N} \sum_{I_2+I_3=I_1} C^{I_1I_2I_3} O^{I_2} O^{I_3}.$$ 

For the cases of interest to us, where $k = 2, 3$, the extended CPOs and CPOs
coincide. In fact, in the large $N$ limit this operator mixing is surpressed, for
regular four-point correlators [34]. Regular four-point functions are four-point
functions which are neither extremal ($k_1 = k_2 + k_3 + k_4$) nor sub-extremal ($k_1 =
k_2 + k_3 + k_4 - 2$).

One might ask the question if it is possible to find a Lagrangian in terms
of fields that do correspond to single-trace CPOs. This seems to be possible in
principle, but it requires complicated field redefinitions, not only on the scalar
fields $s_k$, but also on other fields in the Lagrangian [34].
Chapter 4

Correlation functions in the supergravity approximation

In this chapter we proceed to calculate, using the AdS/CFT correspondence, a specific four-point function of \( \frac{1}{2} \)-BPS operators in \( \mathcal{N} = 4 \) SYM. We will work in the supergravity approximation, that is, in the large \( N \) limit.

The four-point function we consider is

\[
\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4) \rangle. \tag{4.1}
\]

As mentioned before, the operators \( O_2 \) and \( O_3 \) are dual to the supergravity scalar fields \( s_2 \) and \( s_3 \). Hence, we will need the Lagrangian for all terms which contribute to a four-point function with two \( s_2 \) and two \( s_3 \) fields. Finding this Lagrangian and its on-shell value constitutes the main part of this chapter.

A novelty of this calculation is that it features operators of different conformal dimension. Up to now, only four-point correlators of operators of equal weight have been considered, as well as certain examples of so-called extremal correlators, which vanish [30, 35, 24, 36].

This chapter is organised as follows: first the relevant Lagrangian on the supergravity side is presented. Then its on-shell value is found, by computing various exchange diagrams. Using this on-shell value, the four-point function is found by the procedure outlined in the previous chapter. Finally, the result obtained is compared with predictions from the CFT side.

4.1 Relevant supergravity Lagrangian

To compute the four-point functions, we have to differentiate the partition function on the string side with respect to the boundary conditions of the fields on the AdS space. This shows that we must now the supergravity action up to the quartic order in the fields \( s_k \). As described in Section 3.4, finding the Lagrangian is highly non-trivial. This section is devoted to finding the relevant terms in the complicated expression for this Lagrangian. It turns out that the resulting Lagrangian is
remarkably simple, a feature that was seen in other calculations of four-point functions dual to CPOs as well.

We will use the action given in [31] as our starting point. Schematically, this can be written as

\[ S(s) = \frac{4N^2}{(2\pi)^5} \int d^5 z \sqrt{g_a} (L_2 + L_3 + L_4), \]  

(4.2)

where \( g_a \) is the determinant of the Euclidean metric of \( AdS_5 \), given by \( ds^2 = \frac{1}{z_0^2}(dz_0^2 + dx_a dx_a) \), and \( a = 1, 2, 3, 4 \). The terms \( L_2, L_3 \) and \( L_4 \) represent the terms quadratic, cubic and quartic in the fields respectively. Note that this is the Euclidean action. This gives an overall minus sign in comparison with the action used in [31].

To find the complete relevant action, we need to examine the cubic vertices to see what kind of tree-level diagrams can be made with two \( s_2 \) and two \( s_3 \) fields on the external legs. This will tell us what fields are involved, so we can add the kinetic parts for these fields to the Lagrangian. By this procedure one can identify which fields have to be included in the Lagrangian.

### 4.1.1 Quadratic part

In comparison with the action given in [31], we first perform some field rescalings, to simplify the action. The scalar fields \( s \) are rescaled as

\[ s_2 \rightarrow \frac{3\pi^{3/2}}{2\sqrt{2}} s_2, \quad s_3 \rightarrow \frac{\pi^{3/2}}{2^{5/2}3^{1/2}} s_3. \]

Furthermore, we rescale the symmetric tensors \( \phi_{\mu\nu,k} \) as

\[ \phi_{\mu\nu,0} \rightarrow 3^{1/2} \pi^{3/2} \phi_{\mu\nu,0}, \quad \phi_{\mu\nu,1} \rightarrow 3^{1/2} \phi_{\mu\nu,1}, \]

and finally the vectors \( A_{\mu,k} \) as

\[ A_{\mu,1} \rightarrow \pi^{3/2} 6^{1/2} A_{\mu,1}, \quad A_{\mu,2} \rightarrow 4 \pi^{3/2} \frac{3^{1/2}}{2} A_{\mu,2}. \]

Due to these field rescalings, the action will be, when we again write \( L_2 \) etc. for the rescaled parts of the Lagrangian,

\[ S(s) = \frac{N^2}{8\pi^2} \int d^5 z \sqrt{g_a} (L_2 + L_3 + L_4). \]
The rescaled quadratic part of the Lagrangian is then
\[
\mathcal{L}_2 = \frac{1}{4} (\nabla_\mu s_2^1 \nabla^\mu s_2^1 - 4 s_2^1 s_2^1) + \frac{1}{4} (\nabla_\mu s_3^1 \nabla^\mu s_3^1 - 3 s_3^1 s_3^1)
\]
\[
+ \frac{1}{2} (F_{\mu_1,1}^{1})^2 + \frac{1}{2} (F_{\mu_2,2}^{1})^2 + 3(A_{\mu,2}^1)^2
\]
\[
+ \frac{1}{2} \nabla_\rho \phi_{\mu_0,0} \nabla^\rho \phi^{\mu_0} - \frac{1}{2} \nabla_\mu \phi_{\rho,0} \nabla^\nu \phi_{\nu,0} - \frac{1}{4} \nabla_\rho \phi_{\mu,0} \nabla^\nu \phi_{\nu,0}
\]
\[
- \frac{1}{2} \phi_{\mu_0,0} \phi^{\mu_0} + \frac{1}{2} (\phi_{\nu,0}^{\nu})^2
\]
\[
+ \frac{1}{4} \nabla_\rho \phi_{\mu_1,1} \nabla^\rho \phi^{\mu_1} - \frac{1}{2} \nabla_\mu \phi_{\rho,1} \nabla^\nu \phi_{\nu,1} + \frac{1}{2} \nabla_\mu \phi_{\rho,1} \nabla^\nu \phi_{\nu,1} - \frac{1}{4} \nabla_\rho \phi_{\mu,1} \nabla^\nu \phi_{\nu,1}
\]
\[
+ \frac{3}{4} \phi_{\mu_1,1} \phi^{\mu_1} - \frac{7}{4} (\phi_{\nu,1}^{\nu})^2,
\]
where \(F_{\mu,2} = \partial_\mu A_{\nu,2} - \partial_\nu A_{\mu,2}\), and only the fields that are involved in the cubic and quartic terms are written, and a summation over the upper indices, which run over the basis of the representation corresponding with the field, is implied.

### 4.1.2 Cubic couplings

First we consider the cubic couplings involving three \(s\) fields, by looking at equation (2.20) of [31]. The result is expressed in terms of so-called \(C\)-tensors, see Appendix A for more details. Note that in this expression a sum over all possible indices \(k_1, k_2\) and \(k_3\) is implied. We first note that the expression is invariant under permutations of \(k_1, k_2\) and \(k_3\), so we can consider the cases \(k_1 = k_2 = 2, k_1 = k_3 = 3\) and \(k_1 = 2, k_2 = 3\). Using the selection rules for \(a_{123}\), this implies that only a finite set of values for \(k_3\) is possible.

In the case \(k_1 = k_2 = 2\), we can easily read of the cubic coupling from [35], giving \(-\frac{1}{4} (C_1^1 C_2^2 C_3^3) s_2^1 s_2^1 s_3^3\). The second case is calculated in [24], and gives a contribution of \(-\frac{3}{4} (C_1^1 C_2^2 C_3^3) s_2^1 s_2^1 s_3^3\). Note that we have not included the term involving the \(s_4\) field, since there is no term \(s_2^2 s_4\), so it is impossible to create an \(s_4\) exchange diagram, containing two \(s_2\) and two \(s_3\) fields on the external legs. Finally, the case \(k_1 = 2, k_2 = 3\) does not contribute new terms to the action: the selection rules imply \(k_3 = 1, 3, 5\), but in the first and last case, the prefactor \(S_{123}\) vanishes, and the case \(k_3 = 3\) is already accounted for in the \(s_3 s_3 s_2\) term. Summarizing, we have

\[
\mathcal{L}_3(s) = -\frac{1}{4} (C_1^1 C_2^2 C_3^3) s_2^1 s_2^1 s_3^3 - 3 (C_1^1 C_2^2 C_3^3) s_3^1 s_3^1 s_2^3 - 3 (C_1^1 C_2^2 C_3^3) s_3^1 s_3^3 s_2^3.
\]  

(4.3)

The scalar field \(t\), from expression (2.21) in [35], does not appear. For each possible value of \(k_3\), dictated by the selection rules for \(a_{123}\), the term multiplying \(a_{123}\) vanishes. The same holds for the scalar field \(\phi\): for the values of \(k_3\) allowed by the selection rules for the tensor \(p_{123}\), the term multiplying \(p_{123}\) vanishes.

Next are the symmetric tensors \(\phi_{\mu\nu,k_3}\). From \(k_1 = k_2 = 2\), we get a coupling of two \(s_2\) fields with \(\phi_{\mu\nu,0}\), the graviton. Note that in this case, we have the tensor
The coupling of two fields \( \mathcal{L}_2 \), \( \mathcal{L}_3 \), is a crucial aspect of the understanding of various physical phenomena. In particular, the coupling of two fields \( \phi^\mu \) enters in a vertex with the proper external legs. These two cases hence give the following terms:

\[
\mathcal{L}_2^{\text{MF}}(\phi_{\mu\nu}) = -\frac{1}{4} \left\langle C_2^1 C_2^2 C_3^3 \right\rangle \left( \nabla^\mu s_2^1 \nabla^\nu s_2^2 \phi_{\mu\nu0} - \frac{1}{2} \left( \nabla^\mu s_2^1 \nabla^\nu s_2^3 - 4s_2^1 s_2^3 \right) \phi_{\nu\rho0} \right) \\
\mathcal{L}_3^{\text{MF}}(\phi_{\mu\nu}) = -\frac{1}{4} \left\langle C_3^1 C_2^2 C_3^3 \right\rangle \left( \nabla^\mu s_3^1 \nabla^\nu s_3^2 \phi_{\mu\nu0} - \frac{1}{2} \left( \nabla^\mu s_3^1 \nabla^\nu s_3^3 - 3s_3^1 s_3^3 \right) \phi_{\nu\rho0} \right),
\]

where the summation over the indices 1 and 2 is done by our earlier observation about \( \delta^{12} \).

Now consider the case \( k_1 = 2 \), \( k_2 = 3 \). From the selection rules and the explicit value of \( G_{12} \), in equation (2.23) of [31], we see that the only possible for \( k_3 \) is one. This gives the following terms:

\[
\mathcal{L}_3^{\text{MF}}(\phi_{\mu\nu}) = -\frac{1}{4} \left\langle C_2^1 C_2^2 C_3^3 \right\rangle \left( \nabla^\mu s_2^1 \nabla^\nu s_2^2 \phi_{\mu\nu1} - \frac{1}{2} \left( \nabla^\mu s_2^1 \nabla^\nu s_2^3 - 6s_2^1 s_2^3 \right) \phi_{\nu\rho1} \right).
\]

Since \( \phi_{\mu\nu} \) is symmetric, it is easy to see by renaming the summation variables, that \( k_1 \leftrightarrow k_2 \) gives exactly the same term.

From [35, 24] we see that the vector \( A_{\mu,1} \) couples to two \( s_2 \) fields, as well as two \( s_3 \) fields, corresponding to the cases \( k_1 = k_2 = 2 \) and \( k_1 = k_2 = 3 \) respectively. The coupling of two \( s_3 \) fields with \( A_{\mu,3} \) is not relevant here, since the \( A_{\mu,3} \) field does not couple to \( s_2 \). So these terms give a contribution

\[
\mathcal{L}_3^{\text{MF}}(A_{\mu,1}) = -\frac{1}{4} \left\langle C_2^1 C_2^2 C_3^3 \right\rangle s_1^1 A_{\mu,1} - \frac{3}{2} \left\langle C_3^1 C_2^2 C_3^3 \right\rangle s_2^1 A_{\mu,1}.
\]

The remaining non-zero contributions allowed by the selection rules, come from the cases \( k_3 = 2 \), \( k_1 = 2, k_2 = 3 \) or \( k_1 = 3, k_3 = 2 \). These give the contributions

\[
\mathcal{L}_3^{\text{MF}}(A_{\mu,2}) + \mathcal{L}_3^{\text{MF}}(A_{\mu,2}) = -\sqrt{3} \left\langle C_3^1 C_2^2 C_3^3 \right\rangle s_1^1 A_{\mu,2} - \sqrt{3} \left\langle C_3^1 C_2^2 C_3^3 \right\rangle s_2^1 A_{\mu,2},
\]

which can be easily rewritten as

\[
\mathcal{L}_3^{\text{MF}}(A_{\mu,2}) + \mathcal{L}_3^{\text{MF}}(A_{\mu,2}) = -\sqrt{3} \left\langle C_3^1 C_2^2 C_3^3 \right\rangle \left( s_1^1 \nabla^\mu s_2^2 - s_2^1 \nabla^\mu s_1^2 \right) A_{\mu,2},
\]

by using that \([1, 1, 1]\) is anti-symmetric in \( 1 \leftrightarrow 2 \).

Finally, for the vectors \( C_{\alpha} \), the couplings vanish for the cases allowed by the selection rules for the tensor \( t_{123} \), hence these fields do not enter.

An interesting thing to note about these calculations, is the appearance of the fields \( \phi_{\mu,1} \) and \( A_{\mu,1} \). These do not enter in the calculations of the correlation functions of four operators of equal weight, mentioned earlier (see also [36]). The reason for this, is that in the case under consideration here, also the possibility where \( k_1 \) is not equal to \( k_2 \) has to be considered.
4.1.3 Quartic couplings

The most involved part of finding the relevant Lagrangian, is the determination of the quartic vertices. These vertices correspond to contact diagrams. These vertices consist of four scalar fields, with either zero, two or four derivatives. As can be seen in [31], the explicit values of these couplings look, a priori, highly complicated. It should also be noted, that since we’re dealing with fields from different multiplets, we should consider all possible permutations of $k_i$, with two $k_i$ equal to two, and two equal to three.

The vertices are given in terms of the $a_{123}$, $t_{123}$ and $p_{123}$ tensors. It turns out to be convenient to use the formulas in Appendix A to write these in terms of the basis of $C$-tensors, to get simpler results.

Four-derivative terms

A priori, the four-derivative terms look very complicated. However, with integration by parts and using the symmetry relations for the vertices, we can in fact completely remove the four-derivative terms, just like in the cases where correlation functions of operators of equal weight [24, 36, 35]. These results suggest that the problem may in fact be described by some $\sigma$-model.

In this section, for the sake of brevity we will write $[2233]$ for $k_1 = 2, k_2 = 2, k_3 = 3, k_4 = 4$, etc. The first terms we consider are the symmetric part of the $[2233]$ and $[3322]$ vertices. These are given by

$$-\frac{9}{5 \cdot 2^{18}} (6C^{1234} + 6C^{1243} + 12S^{1234} + \delta_2^{12} \delta_3^{34} + 3\Upsilon^{1234} + 3\Upsilon^{1243})$$

(4.4)

multiplied by the fields

$$s_2^1 \nabla_\mu s_2^3 \nabla \cdot \nabla (s_3^3 \nabla^\mu s_3^4) + s_3^3 \nabla_\mu s_3^4 \nabla \cdot \nabla (s_2^1 \nabla^\mu s_2^2).$$

(4.5)

The next terms we consider are the $[2323]$ and $[3232]$ vertices. These are given by

$$\frac{3}{5 \cdot 2^{18}} (182C^{1234} - 146C^{1243} + 36S^{1234} + 3\delta_2^{12} \delta_3^{34} + 133\Upsilon^{1234} - 115\Upsilon^{1243}),$$

(4.6)

multiplied by the fields

$$s_2^1 \nabla_\mu s_3^3 \nabla \cdot \nabla (s_2^3 \nabla^\mu s_3^4) + s_3^3 \nabla_\mu s_2^4 \nabla \cdot \nabla (s_3^4 \nabla^\mu s_2^2).$$

(4.7)

To rewrite these terms, we first note that on an AdS background, we have the following important formula, obtained by writing out the $\nabla \cdot \nabla$ derivative in the quartic derivative terms. One has to be a bit careful with this: for scalar functions $f$ we have $\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f$, since the covariant derivative is torsion free. However, for vectors $V^\mu$, we have [37]

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho_{\sigma \mu \nu} V^\sigma,$$

(4.8)
where \( R^\mu_{\sigma \mu \nu} \) is the Riemann tensor from equation (3.7). Applying this formula leads to \( \nabla^2 \nabla^k s^k = \nabla \mu (\nabla^2 - 4) s^k \). Using this result we find

\[
\begin{align*}
& \sum_2^1 \nabla^2 \nabla^3 \nabla (s^3_2 \nabla^\mu s^3_2) = \\
& (m^2_{k_1} + m^2_{k_2} - 4) s^3_2 \nabla^\mu s^3_2 \nabla^\nu s^3_2 + 2 s^3_1 \nabla^\mu s^3_2 \nabla^\nu s^3_2, \\
& \sum_3^2 \nabla^2 \nabla^3 \nabla (s^3_2 \nabla^\mu s^3_2) = \\
& 2 s^3_2 \nabla^\mu s^3_2 \nabla^\nu s^3_2.
\end{align*}
\]

(4.9)

where the equations of motion for the \( s \)-fields on an AdS-background are used.\(^1\)

If we now apply formula (4.9), and similar expressions, to equations (4.5) and (4.7), we see that four-derivative parts of these equations are in fact equal. So for the four-derivative part, we can add the tensor structures in equations (4.4) and (4.6), and the four-derivative term becomes

\[
\frac{3}{5 \cdot 2^{16}} \left( 41C^{1234} - 41C^{1243} + 31Y^{1234} - 31Y^{1243} \right) \times \\
(2s^2_2 \nabla^\mu s^2_2 \nabla^\nu s^2_2 \nabla^\mu s^2_2 + 2s^2_3 \nabla^\nu s^3_2 \nabla^\nu s^3_2). \\
\] (4.10)

It is remarkable to see that the tensor multiplying the fields is anti-symmetric under both \( 1 \leftrightarrow 2 \) and \( 3 \leftrightarrow 4 \) separately.

It turns out that the remaining terms, the \([3223]\) and \([2332]\) couplings and the anti-symmetric part of the \([2233]\) and \([3322]\) terms, have the same structure. These give the contribution

\[
- \frac{3}{5 \cdot 2^{16}} \left( 41C^{1234} - 41C^{1243} + 31Y^{1234} - 31Y^{1243} \right), \\
\] (4.11)

multiplied by the term with the fields

\[
\sum_2^1 \nabla^2 \nabla^3 \nabla (s^3_2 \nabla^\mu s^3_2) + s^3_2 \nabla^\mu s^3_2 \nabla^\nu s^3_2 + \nabla (s^3_2 \nabla^\nu s^3_2) + \\
\sum_3^2 \nabla^2 \nabla^3 \nabla (s^3_2 \nabla^\mu s^3_2) .
\] (4.12)

If we now use again equation (4.9) to rewrite these terms, and add the result with (4.10), we see the total quartic-derivative term becomes (4.11) multiplied by the fields

\[
\sum_2^1 \nabla^2 \nabla^3 \nabla^\mu s^2_2 \nabla^\nu s^2_2 + s^2_3 \nabla^\mu s^2_2 \nabla^\nu s^2_2 \nabla^\mu s^2_2 .
\] (4.13)

But the first term is symmetric under \( 3 \leftrightarrow 4 \), and the second under \( 1 \leftrightarrow 2 \). But (4.11) is anti-symmetric under these symmetries, hence the result is zero, and we conclude that all quartic derivative terms vanish, and the effective Lagrangian is of \( \sigma \)-model type.\(^2\)

\(^1\)The equation of motion we use is \( (\nabla^2 - m^2) s_k = 0 \), that is, we do not include the correction terms, since they do not give contributions to the four-point function. This is because these correction terms include the fields on AdS\(_5\), hence this would lead to terms with five fields or more.

\(^2\)After the completion of this calculation, I learned of [34], where it is proved that these particular four-derivative couplings have to vanish for the AdS/CFT correspondence to be consistent. This is a so-called sub-sub-extremal case, where \( k_1 = k_2 + k_3 + k_4 - 4 \). If this coupling were non-zero, the associated contact diagram would lead to divergences. The calculation done there is in the same spirit as this one.
Two-derivative terms

We first collect the two-derivative terms that come from applying equation (4.9) in the previous part. The symmetric terms from the $[2233]$ and $[3322]$ coupling contribute with the tensor (4.4) multiplied by the fields

$$2(m^2_2 + m^2_3 - 4)s^1_2 \nabla_{\mu}s^2_3 s^3_3 \nabla^\mu s^4_3.$$

The $[2323]$ and $[3232]$ terms contribute by (4.6) multiplied by

$$(m^2_2 + m^2_3 - 4)(s^1_2 \nabla_{\mu}s^2_3 s^3_3 \nabla^\mu s^4_3 + s^1_3 \nabla_{\mu}s^2_2 s^4_3 \nabla^\mu s^3_2)$$

But this term is symmetric in $1 \leftrightarrow 2$, and $3 \leftrightarrow 4$, hence the anti-symmetric part of the tensor (4.6) vanishes.

Finally, the remaining terms contribute by (4.11) multiplied by

$$2(m^2_2 + m^2_3 - 4)s^1_2 \nabla_{\mu}s^2_2 s^3_3 \nabla^\mu s^4_3 + 2(m^2_2 + m^2_3 - 4)s^1_3 \nabla_{\mu}s^2_2 s^4_3 \nabla^\mu s^3_2.$$  

But this is symmetric under $3 \leftrightarrow 4$, while (4.11) is anti-symmetric, hence this term vanishes.

Proceeding as before, we collect the $[2233]$, $[3322]$, $[2332]$ and $[3223]$ terms, together with the relevant two-derivative terms coming from the four-derivative terms. This results in

$$\left( - \frac{1}{4} C^{1234} + \frac{1}{165150720} \left[ 296486086(C^{1234} + C^{1243}) + 800970364 S^{1234} 
- 24940595 C^{1234}_2 \delta^{34} + 62185183(Y^{1234} + Y^{1243}) \right] \right) s^1_2 s^2_3 s^3_3 \nabla_{\mu} s^4_3 (4.14)$$

But the part in square brackets is symmetric under $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. We can rewrite this part.

Indeed, let $\chi^{1234}$ be a tensor that is symmetric under the permutations $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. By a partial integration, we then find

$$\chi^{1234} s^1_2 s^2_3 \nabla_{\mu} s^3_3 \nabla^\mu s^4_3 = -\chi^{1234} s^1_2 s^3_3 s^4_3 \nabla^\mu s^3_3 - \chi^{1234} s^1_2 s^3_3 \nabla_{\mu} s^4_3$$

But using the symmetry, we see the first term is the same as the left-hand side. Hence, we can see we can rewrite the term in square brackets above as

$$\frac{-1}{3303014400} \left( 592972172 C^{1234} + 800970364 S^{1234} 
- 24940595 C^{1234}_2 \delta^{34} + 124370366 Y^{1243} \right) s^1_2 s^2_3 s^3_3 \nabla^\mu s^4_3 (4.15)$$

plus a term without derivatives. We used the symmetries of the field term to collect the $C$ and $Y$-tensors together.

The $[2323]$ term, together with the contribution coming from the four-derivative term, is given by

$$\frac{-1}{660602580} \left( 62581198 C^{1234} + 530282110 C^{1243} + 902149180 S^{1234} 
- 24949667 C^{1234}_2 \delta^{34} + 180758071 Y^{1243} - 56442137 Y^{1234} \right) s^1_2 s^2_3 s^3_3 \nabla^\mu s^4_3.$$  

35
Note that the expression of the fields is symmetric under $3 \leftrightarrow 4$, hence the antisymmetric part vanishes. The $[3232]$ term is similar. By doing two partial integrations, we can rewrite this term such that the derivatives will be on the scalar fields of weight 3. Indeed, if $\chi^{1234}$ is again symmetric under $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, we find

$$\chi^{1234} \nabla_\mu s_2^1 \nabla_\nu s_2^2 \nabla_\rho s_3^3 s_4^4 = \chi^{1234} s_1^1 s_2^2 \nabla_\mu s_3^3 s_4^4 + \chi^{1234} (m_3^2 - m_2^2) s_2^1 s_2^2 s_3^3 s_4^4,$$

hence we can take this term and (4.15) together.

In the end after adding all terms, we get to the very simple expression

$$-\frac{1}{4} (C^{1234} - S^{1234}) s_1^1 \nabla_\mu s_2^2 s_3^3 \nabla_\nu s_4^4$$

for the two-derivative contribution in the quartic couplings. In this expression, we did a partial integration again to rewrite $s_1^1 s_2^2 \nabla_\mu s_3^3 \nabla_\nu s_4^4$, multiplying the $S^{1234}$ tensor.

**No-derivative terms**

The contribution coming from the partial integrations of the two-derivative couplings is given by

$$-\frac{1}{94371840} (592863308 C^{1234} + 902149180 S^{1234} - 24949667 \delta_2^{12} \delta_3^{34}$$

$$+ 124315934 \Upsilon^{1234}) s_2^1 s_2^2 s_3^3 s_4^4.$$

The no-derivative vertices contribute, by using symmetry in the representation indices, by

$$\frac{1}{94371840} (911368268 C^{1234} + 1079096380 S^{1234} - 60339107 \delta_2^{12} \delta_3^{34}$$

$$+ 18147614 \Upsilon^{1234}) s_2^1 s_3^3 s_4^4.$$

Summing these two terms, we get the remarkably simple result

$$\frac{3}{8} \left( 9C^{1234} + 5S^{1234} - \delta_2^{12} \delta_3^{34} - 3 \Upsilon^{1234} \right) s_2^1 s_2^2 s_3^3 s_4^4.$$  

So by using the summation relations for $C$-tensors, and the symmetries in the quartic vertices, we have been able to reduce the a priori complicated looking expressions for the quartic couplings to a very simple contribution

$$\mathcal{L}_4 = -\frac{1}{4} (C^{1234} - S^{1234}) s_2^1 \nabla_\mu s_2^2 s_3^3 \nabla_\nu s_4^4$$

$$+ \frac{3}{8} (9C^{1234} + 5S^{1234} - \delta_2^{12} \delta_3^{34} - 3 \Upsilon^{1234}) s_2^1 s_2^2 s_3^3 s_4^4.$$

This reduction to a simple expression was also observed in the calculation of the effective Lagrangian for four-point functions of equal weight ($k = 2, 3, 4$) operators. This indicates that the supergravity theory can in fact be described by a simpler model.
4.2 Equations of motion

According to the procedure outlined in Section 3.3, we have to find the on-shell value of the effective action, to compute the correlation functions. To this end, we will first derive the equations of motion, and then use perturbation theory to find the on-shell value. In determining the equations of motion, we will only include terms that give a contribution to four-point functions. That is, only terms with two scalar fields. To see this, note that we only need terms with four boundary fields. Higher order corrections to the equations of motion will lead to terms with five or more boundary fields.

For the scalar fields, the Lagrangian in Section 4.1 leads to the following equations of motion:

\[
\nabla_\mu \nabla^\mu + 4)s^3_2 = -2(C^1_2 C^2_3 C^3_0) s^1_2 s^2_2 - 6(C^1_2 C^2_3 C^3_0) s^1_3 s^2_3
\]

(4.19)

\[
\nabla_\mu \nabla^\mu + 3)s^3_3 = -12(C^1_2 C^2_3 C^3_0) s^1_2 s^2_3.
\]

(4.20)

To simplify notations a bit, introduce the following currents

\[
J^3_{\mu,2} = \langle C^1_2 C^2_3 C^3_{[1,0,1]} \rangle(s^1_2 \nabla_\mu s^2_2 - s^2_2 \nabla_\mu s^1_2)
\]

(4.21)

\[
J^3_{\mu,3} = \langle C^1_3 C^2_3 C^3_{[1,0,1]} \rangle(s^1_3 \nabla_\mu s^3_2 - s^2_3 \nabla_\mu s^1_3)
\]

(4.22)

\[
J^3_{\mu,23} = \langle C^1_2 C^3_3 C^3_{[1,2,1]} \rangle(s^1_2 \nabla_\mu s^3_2 - s^3_2 \nabla_\mu s^1_2).
\]

The first two currents are conserved on-shell, \(\nabla_\mu J^3_{\mu,2} = \nabla_\mu J^3_{\mu,3} = 0\), however the last one is not. With these notations, the equations of motion of the vector fields are

\[
\nabla^\nu(\nabla_\nu A^3_{\mu,1} - \nabla_\mu A^3_{\nu,1}) = -\frac{1}{4} J^3_{\mu,2} - \frac{3}{8} J^3_{\mu,3}
\]

(4.23)

\[
\nabla^\nu(\nabla_\nu A^3_{\mu,2} - \nabla_\mu A^3_{\nu,2}) = -\frac{\sqrt{3}}{2} J^3_{\mu,23}.
\]

The equations of motion for the symmetric tensors are a bit more involved. Consider the action for a symmetric tensor of mass \(f\), coupled to a current \(T_{\mu\nu}\),

\[
L_{gr} = -\frac{1}{4} \nabla_\rho \phi_{\mu\nu} \nabla^\rho \phi^{\mu\nu} + \frac{1}{2} \nabla_\mu \phi_{\rho\nu} \nabla^\nu \nabla_\rho - \frac{1}{2} \nabla_\nu \phi^\rho_{\rho\mu} \nabla^\nu \phi^{\mu\nu} + \frac{1}{4} \nabla_\rho \phi^{\mu\nu} \nabla_\nu \phi^{\mu\nu} + \frac{1}{4} (2 - f) \phi_{\mu\nu} \phi^{\mu\nu} + \frac{1}{4} (2 + f)(\phi^\mu_\mu)^2 + \alpha T_{\mu\nu} \phi^{\mu\nu}.
\]

(4.24)

It is convenient to separate the trace part of the symmetric tensor,

\[
\phi_{\mu\nu} = \tilde{\phi}_{\mu\nu} + \frac{g_{\mu\nu}}{d+1} \phi^\lambda_\lambda,
\]

where \(\tilde{\phi}_{\mu\nu}\) is traceless. The Lagrangian for the trace part is then

\[
\frac{d(d-1)}{4(d+1)^2} \nabla^\rho \phi^\lambda_\lambda \nabla_\rho \phi_\mu^\mu - \frac{1}{2} \frac{d-1}{d+1} \nabla^\rho \phi^\lambda_\lambda \nabla_\mu \tilde{\phi}_\rho^\mu
\]

+ \frac{1}{4} \frac{d+2d+df}{d+1} (\phi^\lambda_\lambda)^2 + \frac{\alpha}{d+1} T^\mu_\mu \phi^\lambda_\lambda.
\]

(4.25)
The equation of motion for the trace from this Lagrangian is
\[
\nabla^\rho \nabla_\rho \phi^\lambda = \nabla^\rho \nabla_\rho \phi^{\mu\nu} - \frac{4 + 2d + df}{1 - d} - \frac{2\alpha}{1 - d} \phi^\lambda T^\mu_{\nu'},
\]
(4.26)
after using (4.24) to rewrite the result. Using this equation, we find the equation
\[
W_{\mu\nu}^{\rho\lambda} \phi_{\rho\lambda} = \left( g_{\mu\nu'} g_{\nu'\nu'} + g_{\mu\nu} g_{\nu'\nu'} + \frac{2}{2 - d} g_{\mu\nu} g_{\nu'\nu'} \right) T^{\mu'\nu'},
\]
(4.27)
where \( W_{\mu\nu}^{\rho\lambda} \) is the modified Ricci operator, defined by
\[
W_{\mu\nu}^{\rho\lambda} \phi_{\rho\lambda} = - \nabla_\rho \nabla^\rho \phi_{\mu\nu} + \nabla_\mu \nabla^\rho \phi_{\rho\nu} + \nabla_\nu \nabla^\rho \phi_{\rho\mu} - \nabla_\mu \nabla_\nu \phi_{\rho} - \left( (2 - f) \phi_{\mu\nu} + \frac{6 - f}{1 - d} g_{\mu\nu} \phi^\rho \right).
\]
(4.28)
Now introduce the currents
\[
T_2^{\mu\nu} = \nabla^\mu s_1^\nu \nabla^\nu s_1^\mu - \frac{1}{2} g^{\mu\nu} \left( \nabla^\rho s_1^\mu \nabla_\rho s_1^\nu - 4 s_1^\mu s_1^\nu \right)
\]
\[
T_3^{\mu\nu} = \nabla^\mu s_2^\nu \nabla^\nu s_2^\mu - \frac{1}{2} g^{\mu\nu} \left( \nabla^\rho s_2^\mu \nabla_\rho s_2^\nu - 3 s_2^\mu s_2^\nu \right)
\]
\[
T_{23}^{\mu\nu} = \nabla^\mu s_3^\nu \nabla^\nu s_3^\mu - \frac{1}{2} g^{\mu\nu} \left( \nabla^\rho s_3^\mu \nabla_\rho s_3^\nu - 2 s_3^\mu s_3^\nu \right).
\]
With these notations, the current in the case of the graviton \((k = 0, f = 0)\), is
\[
\alpha T^{\mu\nu} = - \frac{1}{2} T_2^{\mu\nu} - \frac{1}{4} T_3^{\mu\nu},
\]
and in the case of the massive symmetric tensor \((k = 1, f = 5)\), \(\alpha T^{\mu\nu} = - \frac{1}{2} \langle C_1^2 C_2^2 C_3^3 \rangle T_{23}^{\mu\nu} \).

### 4.3 On-shell Lagrangian

To find the on-shell value of the action, we will use perturbation theory. Write the solutions of the equations of motion in the form
\[
s_k = s_k^{1,0} + s_k^{1,1}, \quad A_{\mu, k} = A_{\mu, k}^{1,0} + A_{\mu, k}^{1,1}, \quad \phi_{\mu, k} = \phi_{\mu, k}^{1,0} + \phi_{\mu, k}^{1,1},
\]
(4.29)
where \( s_k^{1,0}, A_{\mu, k}^{1,0} \) and \( \phi_{\mu, k}^{1,0} \) are solutions of the linearized equations of motion with fixed boundary conditions, and the other terms are corrections with vanishing boundary conditions. Note that to find the four-point correlators we’re interested in, we have to vary the scalar fields \( s_k \) with respect to the boundary conditions. Hence, only the terms in the on-shell Lagrangian with two \( s_k^{1,0} \) and two \( s_k^{1,0} \) are relevant.

Let us now introduce Green’s functions for the linearized differential equation for the scalar [38], vector and tensor [39] fields, satisfying
\[
(\nabla^\mu \nabla_\mu - m_k^2) G_k = - \delta(z, w)
\]
(4.30)
\[
\nabla^\mu (\nabla_\mu G_{\mu\nu', k} - \nabla_\nu G_{\mu\nu', k}) - m_k^2 G_{\mu\nu', k}(w, z) = - g_{\mu\nu'} \delta(z, w)
\]
(4.31)
\[
W_{\mu\nu}^{\rho\lambda} G_{\rho\mu\nu', k}(w, z) = \left( g_{\mu\nu'} g_{\nu'\nu'} + g_{\mu\nu} g_{\nu'\nu'} + \frac{2}{2 - d} g_{\mu\nu} g_{\nu'\nu'} \right) \delta(z, w).
\]
(4.32)
where \( u \) is the invariant AdS distance from (3.8), and \( k \) is used to indicate the corresponding \( SO(6) \) representation. These Green’s functions are in fact bulk-to-bulk propagators that propagate the fields in the AdS bulk. The precise forms of these propagators are not necessary here, but it should be noted that these are known.

With perturbation theory it is now easy to find the correction terms \( s_{k}^{I,1}, A_{\mu,k}^{I,1} \) and \( \phi_{\mu,\nu,k}^{I} \). To avoid the cumbersome notation, the superscript zero is omitted from now on, when it’s clear from the context. This gives

\[
\begin{align*}
    s_{2}^{3,1}(w) &= 2 \langle C_{2}^{I} C_{2}^{I} C_{[0,2,0]}^{3} \rangle \int \frac{d^{5}\zeta}{z_{0}^{5}} G_{2}(u)s_{2}^{I}(z)s_{2}^{I}(z) \\
    &+ 6 \langle C_{3}^{I} C_{3}^{I} C_{[0,2,0]}^{3} \rangle \int \frac{d^{5}\zeta}{z_{0}^{5}} G_{2}(u)s_{3}^{I}(z)s_{3}^{I}(z) \\
    s_{3}^{3,1}(w) &= 12 \langle C_{2}^{I} C_{3}^{I} C_{[0,3,0]}^{3} \rangle \int \frac{d^{5}\zeta}{z_{0}^{5}} G_{3}(u)s_{2}^{I}(z)s_{3}^{I}(z) \\
    A_{\mu,1}^{3,1}(w) &= \frac{1}{4} \int \frac{d^{5}\zeta}{z_{0}^{5}} G_{\mu,1}(u)J_{\nu,2}^{3}(z) + \frac{3}{8} \int \frac{d^{5}\zeta}{z_{0}^{5}} G_{\mu,1}(u)J_{\nu,3}^{3}(z) \\
    A_{\mu,2}^{3,1}(w) &= \frac{\sqrt{3}}{2} \int \frac{d^{5}\zeta}{z_{0}^{5}} G_{\mu,2}(u)J_{\nu,23}^{3}(z) \\
    \phi_{\mu,\nu}^{3,1,0}(w) &= \frac{1}{4} \int \frac{d^{5}\zeta}{z_{0}^{5}} G_{\mu,\nu}(u) \left( T_{2}^{\mu,\nu} + T_{3}^{\mu,\nu} \right) \\
    \phi_{\mu,\nu}^{3,1}(w) &= \frac{1}{2} \langle C_{2}^{I} C_{3}^{I} C_{[0,1,0]}^{3} \rangle \int \frac{d^{5}\zeta}{z_{0}^{5}} G_{\mu,\nu}(u)T_{\nu}^{\mu,\nu},
\end{align*}
\]

where in the last two lines it should be noted that the appropriate propagator for the graviton and massive symmetric tensor should be used respectively.

As mentioned before, we will write the Lagrangians in terms of the solutions of the linearized equations of motion and their correction terms, and keep only the terms with two \( s_{2}^{I,0} \) and two \( s_{3}^{I,0} \) terms. It is interesting to see that followin this procedure, the quadratic part of the Lagrangian also gives a contribution to the four-point function. To illustrate this, consider

\[
\begin{align*}
    \int \frac{d^{5}\zeta}{z_{0}^{5}} \nabla_{\mu}s_{2}^{I} \nabla_{\mu}s_{2}^{I} + m_{2}^{2}s_{2}^{I} s_{2}^{I} &\approx 2 \langle C_{2}^{I} C_{2}^{I} C_{[0,2,0]}^{I} \rangle \int \frac{d^{5}\zeta}{z_{0}^{5}} s_{2}^{10} s_{2}^{20} s_{2}^{I,1},
\end{align*}
\]

where in the first line we did a partial integration, and in the second line we substituted \( s_{2}^{I} = s_{2}^{10} + s_{2}^{I,1} \). Finally, in the last line we used (4.30) under the integral sign in \( s_{2}^{I,1} \), and kept only the relevant terms.
The calculation for the contribution of the tensors is a bit more involved. By doing a partial integration and some clever relabeling of the indices, we can rewrite the quadratic part of the Lagrangian as

\[ \mathcal{L}_{\text{quad}} = \frac{1}{2} (C_1^2 C_2 C_3^{[0,2,0]}) s_2 s_2 s_2^{3,1} + \frac{3}{2} (C_1^3 C_2^2 C_3^{[0,0,0]}) s_1 s_2 s_2^{3,1} + 3 (C_1^2 C_2 C_3^{[0,2,0]}) s_2^{3,1} + \frac{1}{4} A_{\mu,1}^{\phi_{\mu,3}} + \frac{3}{8} A_{\mu,1}^{\phi_{\mu,3}} + \frac{\sqrt{3}}{2} A_{\mu,1}^{\phi_{\mu,3}} \]  

(4.34)

where from now on it will be understood that the currents \( T_{\mu\nu} \) and \( J_\mu \) depend only on the scalar fields \( s_k^{J_\mu} \), i.e. the solutions of the linearized equations of motion with given boundary conditions.

For the cubic couplings we can also find the on-shell value by inserting (4.29) and keeping the relevant terms. For the scalar couplings this gives

\[ \mathcal{L}_{\text{scal}} = - (C_1^1 C_2^2 C_3^{[0,2,0]}) s_2 s_2 s_2^{3,1} - 3 (C_1^3 C_2^2 C_3^{[0,0,0]}) s_1 s_2 s_2^{3,1} - 6 (C_1^2 C_2 C_3^{[0,2,0]}) s_2^{3,1} \]

The contribution from the vector couplings is

\[ \mathcal{L}_{\text{vec}} = - \frac{1}{2} J_2^{\mu,3} A_{\mu,1}^{\phi_{\mu,3}} - \frac{3}{4} J_3^{\mu,3} A_{\mu,1}^{\phi_{\mu,3}} - \sqrt{3} J_2^{\mu,3} A_{\mu,1}^{\phi_{\mu,3}} \]

and for the graviton and massive symmetric tensor

\[ \mathcal{L}_{\text{tens}} = - \frac{1}{4} \phi_{\mu,0}^{\phi_{\mu,3}} T_{2}^{\mu\nu} - \frac{1}{4} \phi_{\mu,0}^{\phi_{\mu,3}} T_{3}^{\mu\nu} - \frac{1}{2} (C_1^1 C_2^2 C_3^{[0,2,0]}) T_{23}^{\mu\nu} \phi_{\mu,1}^{\phi_{\mu,3}} \]

The terms arising from the quadratic part and the cubic couplings\(^3\) can be described in the language of Feynman diagrams. To see this, note that in the correction terms appearing, integrals over bulk-to-bulk propagators are appearing. Hence, the expressions appearing can in fact be considered as exchange diagrams.

Finally, recall that the four-point function is obtained by varying the action with respect to the boundary conditions of the scalar fields. Denote the value of

\(^3\)Note that on-shell we have \( \mathcal{L}_2 + \mathcal{L}_3 = \frac{1}{2} \mathcal{L}_4 \). This is in fact a general rule to compute the on-shell contribution from the quadratic and cubic part.
a scalar field $s_k$ on the AdS boundary by $\tilde{s}_\Delta(x)$. The solution of the linearized equation of motion with these boundary conditions is then [17, 40]

$$s^{I,0}_\Delta(z_0, z) = \int d^4x \tilde{K}_\Delta(z_0, z;x)s^{I}_\Delta(x),$$

(4.35)

where $\tilde{K}_\Delta$ is the normalized bulk-to-boundary propagator, given by

$$\tilde{K}_\Delta(z, x) = \begin{cases} \frac{1}{\pi^\Delta K_\Delta(z, x)} \quad \text{for } \Delta = 2 \\ \frac{\Gamma(\Delta)}{\pi^{\frac{\Delta}{2}}(\Delta - 2)} K_\Delta(z, x) \quad \text{for } \Delta > 2, \end{cases}$$

where $z$ is a point in the AdS bulk and $x$ a point on the boundary. The function $K_\Delta(z, x)$ is given by

$$K_\Delta(z, x) = \left( \frac{z_0}{z_0^2 + (z - x)^2} \right)^\Delta.$$

Using these solutions, the on-shell value of the action becomes

$$S = \frac{N^2}{8\pi^2} \frac{6}{4\pi^8} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \tilde{s}_1^1(x_1) \tilde{s}_2^2(x_2) \tilde{s}_3^3(x_3) \tilde{s}_4^4(x_4)(L_{\text{ex}} + L_{\text{cont}}),$$

where $L_{\text{ex}}$ and $L_{\text{cont}}$ are the contributions from the exchange and contact diagrams, calculated below.

### 4.3.1 Scalar exchange diagrams

To get the exchange diagrams, substitute the expressions for the corrections (4.33) and keep the relevant terms. First note that

$$\langle C^1_2 C^2_2 C^3_{[0,2,0]} \rangle \tilde{s}_2^1 \tilde{s}_2^2 \tilde{s}_2^3 = 3 \langle C^1_3 C^2_3 C^3_{[0,2,0]} \rangle \tilde{s}_3^3 \tilde{s}_2^3 \tilde{s}_2^3,$$

Figure 4.1: Exchange diagrams of $\Delta = 2$ and $\Delta = 3$ scalars contributing to the four-point function.
which can be seen by writing out each side, note that \( u \) is invariant under \( w \leftrightarrow z \), and do a relabeling of the representation indices. Thus there are only two scalar exchange diagrams we have to calculate. These correspond to the Witten diagrams in Figure 4.3.1.

The scalar exchange diagrams can be easily calculated by the technique described in [41]. For the exchange of the \( \Delta = 2 \) scalar, the \( z \)-integral is

\[
\int \frac{d^5 z}{z_0^5} G_2(u) K_2(z, x_1) K_2(z, x_2) = \frac{1}{4x_{12}^2} K_1(w, x_1) K_1(w, x_2),
\]

and hence the \( \Delta = 2 \) exchange diagram is

\[
-\frac{3}{2x_{12}^2} \langle C_2^1 C_2^2 C_0^5 \rangle \langle C_3^3 C_3^4 C_0^5 \rangle D_{1133}.
\]

The other diagram appearing is the exchange of a \( \Delta = 3 \) scalar. The result of the \( z \)-integral is

\[
\int \frac{d^5 z}{z_0^5} G_3(u) K_2(z, x_1) K_3(z, x_3) = \frac{1}{8x_{13}^2} K_1(w, x_1) K_2(w, x_3),
\]

and hence the exchange diagram is

\[
-\frac{9}{2x_{13}^2} \langle C_2^1 C_3^3 C_0^5 \rangle \langle C_3^2 C_3^4 C_0^5 \rangle D_{1223}.
\]

By using the summation formulae from Appendix A, we find that the scalar exchange diagrams contribute by

\[
L_{\text{scal}} = -\frac{1}{4x_{12}^2} (3C_1^{1234} + 3C_1^{1243} - \delta_1^{1234}) D_{1133} - \frac{3}{4x_{13}^2} (2C_1^{1243} + 4S_1^{1234} - \Upsilon_1^{1234}) D_{1223}.
\]

4.3.2 Vector exchange diagrams

There are two types of vector exchange diagrams appearing, depicted in figure 4.3.2. A massless vector exchange coupling to a conserved current, and a massive vector exchange, where the current is not conserved. The massless vector exchange diagram is given by

\[
-\frac{3}{16} \int \frac{d^5 w}{w_0^5} \int \frac{d^5 z}{z_0^5} J_5^{\mu 5}(w) G_{\mu 1}^\nu(u) J_5^{\nu 2}(z).
\]

\footnote{Note that for the bottom coefficient, the denominator in equation (2.24) from [41] becomes zero. However, one has to keep in mind that it comes from \((k - 2)(k - 1)u_k = (k - 1)(k - 2)u_{k-1}\), hence for \(k = 2\) we can consistently choose \(u_1 = 0\), to get a solution with the required asymptotics for \(t \to 0\).}
Figure 4.2: Exchange diagrams of the $k = 1$ massless vector and the $k = 2$ massive vector.

The $z$-integral was calculated in [35]. There it was found that

\[
\int \frac{d^5z}{z_0^6} G_{\mu\nu}(u)K_2(z,x_1) \nabla^\nu K_2(z,x_2) = \frac{1}{4x_{12}^2}(K_1(w,x_1)\nabla_\mu K_1(w,x_2) - K_1(w,x_2)\nabla_\mu K_1(w,x_1)).
\]

The remaining $w$-integral is then easily found, by applying (A.24), and we get

\[
\frac{9}{16x_{12}^2} \left( C^{1243} - C^{1234} \right) \left( x_{24}^2D_{1234} - x_{14}^2D_{2134} - x_{23}^2D_{1243} + x_{13}^2D_{1243} \right).
\]

The other diagrams is the exchange of a massive tensor. For this we will use the technique developped in Appendix B.1. The $z$-integral we need to calculate is

\[
I_\mu(w) = \int \frac{d^5z}{z_0^6} G_{\mu\,3}(u) (K_2(z,x_1)\nabla_\nu K_3(z,x_3) - K_3(z,x_3)\nabla_\nu K_2(z,x_1)).
\]

Using this procedure in the appendix, one readily finds that the $z$-integral is given by

\[
I_\mu(t) = -\frac{t^2}{3w_0^2} \frac{w_\mu}{w^2}.
\]

Recall that we did a coordinate transformation to calculate the $z$-integral. To transform back to the original coordinates, the following equation is useful

\[
\frac{1}{w^2} J_{\mu\nu}(w) \left( \frac{w' - x'}{w' - x''} \right)^2 = \left( \frac{w - x}{w - x''} \right)^2 \frac{w_\mu}{w^2}.
\]

We also make the transformations

\[
w'_0 \to \frac{w_0}{w_0^2 + (w - x_1)^2} \quad w_0 \to \frac{w_0}{w_0^2 + (w - x_1)^2} \quad w_0 \to \frac{w_0}{w_0^2 + (w - x_3)^2}
\]

\[43\]
Transforming back to the original coordinates, we get

\[ A_\mu(w, x_1, x_3) = \frac{1}{3x_{13}^2} K_1(w, x_1) K_2(w, x_3) \left[ \frac{(w - x_1)_\mu}{(w - x_1)^2} - \frac{(w - x_3)_\mu}{(w - x_3)^2} \right]. \]

Now note that we have the following identity

\[ \frac{(w - x_1)_\mu}{(w - x_1)^2} K_\Delta(w, x_1) = -\frac{1}{2\Delta} \nabla_\mu K_\Delta(w, x_1) + \frac{1}{2} P_\mu K_\Delta(w, x_1), \]

which follows straightforwardly from applying the derivative. So the z-integral is equal to

\[ A_\mu(w, x_1, x_3) = \frac{1}{6x_{13}^2} \left( \frac{1}{2} K_1(w, x_1) \nabla_\mu K_2(w, x_3) - K_2(w, x_3) \nabla_\mu K_1(w, x_1) \right). \]

The w-integral is then easily found, and we find that the total contribution from the massive vector exchange is

\[ -\frac{3}{40x_{13}^4} \left( 5C^{1243} - 5S^{1234} - \Upsilon^{1234} \right) \times \]

\[ \left( 3(x_{14}^2 D_{2224} - x_{34}^2 D_{1234}) + 2(x_{23}^2 D_{1333} - x_{12}^2 D_{2323}) \right). \]

### 4.3.3 Tensor exchange diagrams

There are two tensor exchange diagrams, one where the graviton is exchanged, and another where a massive tensor is exchanged (see Figure 4.3.3). The z-integral for the graviton exchange (we take the z-integral of the two \( s_3 \) fields) is given by

\[ I_{\mu\nu}(t) = \frac{3}{4} \left( \frac{1}{3} g_{\mu\nu} - P_\mu P_\nu \right) (t + t^2), \]
where \( P_\mu = \delta_\mu^0/w_0 \). Transforming back to the original coordinates, we find
\[
A_{\mu\nu}(w, x_3, x_4) = \frac{3}{4x_{34}^3} \left[ \frac{1}{4} \nabla_\mu K_1(x_3) \nabla_\nu K_1(x_4) + \frac{1}{4} \nabla_\nu K_1(x_3) \nabla_\mu K_1(x_4) - \frac{g_{\mu\nu}}{6} K_1(x_3) K_1(x_4) \right] \\
+ \frac{3}{16x_{34}^3} \left[ \frac{1}{4} \nabla_\mu K_2(x_3) \nabla_\nu K_2(x_4) + \frac{1}{4} \nabla_\nu K_2(x_3) \nabla_\mu K_2(x_4) \right].
\]

The integral for the no-derivative term is by definition equal to
\[
\int \frac{d^5w}{w_0^2} K_2(w, x_1) \nabla_\mu K_2(w, x_2) K_3(w, x_3) \nabla_\mu K_3(w, x_4) = 6D_{2233} - 12x_{34}^2 D_{2334}.
\]

**Contact diagrams**

The quartic couplings in (4.18) give rise to contact diagrams, shown in Figure 4.3.4. The integral for the no-derivative term is by definition equal to \( D_{2233} \). For the derivative term, we can use the identity (A.24), to find
\[
\int \frac{d^5w}{w_0^2} K_2(w, x_1) \nabla_\mu K_2(w, x_2) K_3(w, x_3) \nabla_\mu K_3(w, x_4) = 6D_{2233} - 12x_{34}^2 D_{2334}.
\]

Note that this is indeed symmetric under \( x_1 \leftrightarrow x_2 \) and \( x_3 \leftrightarrow x_4 \), as expected.

Remains the exchange of the massive tensor. The \( z \)-integral for the massive tensor exchange is calculated in the Appendix, and is given by (B.38). By transforming back to the original coordinates, we find that it is equal to
\[
A_{\mu\nu}(w, x_1, x_3) = -\frac{3}{5x_{13}^3} K_2(w, x_1) K_1(w, x_3) Q_\mu Q_\nu, \quad (4.38)
\]
where \( Q_\mu = \frac{(w-x_1)_\mu}{(w-x_1)^2} - \frac{(w-x_3)_\mu}{(w-x_3)^2} \). The result in the appendix was calculated for \( \Delta_1 = 3 \) and \( \Delta_3 = 2 \). In the rest of the action, the convention was \( \Delta_1 = 2 \) and \( \Delta_3 = 3 \), so we have to interchange \( x_1 \leftrightarrow x_3 \).

The \( w \)-integral can be found using the identity (A.23), for example
\[
\nabla_\mu K_2(x_2) \nabla^\nu K_3(x_4) Q_\mu Q_\nu K_1(x_1) K_2(x_3) = \frac{3}{x_{3}^3} \left( x_{13}^2 x_{24}^2 D_{2334} - x_{12}^2 x_{34}^2 D_{2334} - x_{12}^2 x_{34}^2 D_{2334} + x_{12}^2 x_{34}^2 D_{2334} \right),
\]
and similar for the other contributions. Collecting terms, the final contribution from the massive tensor exchange is then given by
\[
-\frac{18}{5x_{13}^3} \Gamma^{1234} \left( x_{13} x_{24} D_{2334} + x_{23} x_{34}^2 D_{1344} - x_{12}^2 x_{34}^2 D_{2334} - x_{12}^2 x_{34}^2 D_{2334} + x_{12}^2 x_{34}^2 D_{2334} \right).
\]

With this result all the contributing exchange diagrams are calculated.
With this result it follows directly that

\[ L_{\text{cont}} = \frac{3}{8} \left( C^{1234}(5D_{2233} + 8x_{24}^2 D_{2334}) + 8x_{24}^2 S^{1234} D_{2233} + (9S^{1234} - \delta_2^{12} \delta_3^{34} - 3Y^{1234}) D_{2233} \right) \]  

(4.39)

This completes the computation of the value of the on-shell action.

### 4.4 Four-point function

Now that the generating functional is known, one can use the standard procedure to calculate the correlation function. According to the prescription, we have to vary the on-shell action with respect to the boundary conditions. This is particularly simple, since these boundary fields appear as an overall factor in the on-shell action. It’s easy to see that taking the functional derivatives amounts to interchanging the coordinates and corresponding representation labels. There are four different permutations: the trivial permutation, \( x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_4 \) and both \( x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_4 \) together. For example, a term \( x_{24}^2 D_{2334} C^{1234} \) contributes to the four-point function by

\[ (x_{24}^2 D_{2334} + x_{13}^2 D_{3243}) C^{1234} + (x_{14}^2 D_{3234} + x_{23}^2 D_{2343}) C^{1243}, \]

where the symmetry properties of the \( C \)-tensor are used.

To get to the canonical normalization of the four-point function, consider the two-point functions calculated from this action. There are some subtleties in this calculation, involving a cut-off \( \varepsilon \) that has to be introduced near the AdS boundary, see for example Section 3.3.1 of [5]. The case \( \Delta = 2 \) has to be treated seperately, this is done in Appendix B of [35]. The results are

\[ \langle O_2^1(x_1) O_2^2(x_2) \rangle = \frac{N^2}{2\pi^4} \frac{\delta^{12}}{x_{12}^4}, \quad \langle O_3^1(x_1) O_3^2(x_2) \rangle = \frac{N^2}{4\pi^4} \frac{\delta^{12}}{x_{12}^4}. \]
The canonically normalized CPOs are then \( O_2^l(x_1) = \frac{2^5 \pi^2}{N} O_2^l(x_1) \) and \( O_3^l(x_1) = \frac{2^5 \pi^2}{N} O_3^l(x_1) \).

After finding the permutations of the on-shell value, as described above, it is convenient to express the four-point function in terms of the conformal cross-ratios \( s \) and \( t \). To this end, one has to pass from \( D \)-functions to \( \overline{D} \)-functions, using formula (A.26) from the Appendix. The four-point function of the canonically normalized CPO’s has indeed the structure predicted by the conformal field theory

\[
(\overline{O}_2^1(x_1)\overline{O}_2^3(x_2)\overline{O}_3^4(x_3)\overline{O}_3^3(x_4)) = a(s,t) \frac{\delta_{12}^{12} \delta_{13}^{34}}{x_{12}^2 x_{34}^2 x_{13}^4 x_{24}^2} + b(s,t) \frac{S_{1234}^{34}}{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2}
+ c_1(s,t) \frac{C_{1243}^{1243}}{x_{12}^2 x_{13}^4 x_{14}^2 x_{23}^4} + c_2(s,t) \frac{C_{1234}^{1234}}{x_{12}^2 x_{13}^4 x_{14}^2 x_{23}^4}
+ d_1(s,t) \frac{\gamma_{1234}^{1234}}{x_{13}^4 x_{24}^2 x_{23}^4} + d_2(s,t) \frac{\gamma_{1234}^{1234}}{x_{13}^4 x_{24}^2 x_{23}^4}.
\]

The coefficient functions are described by rather big expressions. In principle it is possible to use the various relations for \( \overline{D} \)-functions to simplify these expressions, however this is not necessary for our purpose. The factor multiplying the \( \delta_{12}^{12} \delta_{13}^{34} \) tensor is given by

\[
a(s,t) = \frac{3}{N^2} (-s \overline{D}_{1133}(s,t) + 4s^2 \overline{D}_{2233}(s,t) + (1 - s + t)s \overline{D}_{2244}(s,t)). \quad (4.40)
\]

Note that we have indeed the crossing symmetry \( a(s,t) = a(s/t, 1/t) \). The coefficient \( b(s,t) \) is (the coordinates of the \( \overline{D} \)-functions are omitted)

\[
b(s,t) = \frac{3}{2N^2} (8t(\overline{D}_{1223} + \overline{D}_{1232}) + t(\overline{D}_{1234} + \overline{D}_{1243}) - t(t + 1)\overline{D}_{1333}
+ 8(\overline{D}_{2123} + t\overline{D}_{2132}) + (t + 1)\overline{D}_{2134} - 2(t + 1)\overline{D}_{2224} - 2t(t + 1)\overline{D}_{2242}
- 36t\overline{D}_{2333} + 2st(\overline{D}_{2323} + \overline{D}_{2332}) + 4t(\overline{D}_{2334} + t\overline{D}_{2343}) - (t + 1)\overline{D}_{3133}
+ 2s(\overline{D}_{3223} + t\overline{D}_{3232}) + 4t(\overline{D}_{3234} + \overline{D}_{3243})). \quad (4.41)
\]

This coefficient also has the crossing symmetry \( b(s/t, 1/t) = b(s,t) \). The coefficient \( c_1(s,t) \) is

\[
c_1(s,t) = \frac{3}{2N^2} (4\overline{D}_{1133} + 4s\overline{D}_{1232} - (2 + s)\overline{D}_{1234}
+ 2t\overline{D}_{1243} + 2\overline{D}_{2134} - 2D_{2143} - 4s\overline{D}_{3234}
+ s(4\overline{D}_{2132} - 2\overline{D}_{2143} + 2\overline{D}_{2224} - 10\overline{D}_{2233})
+ st(\overline{D}_{1333} + 2\overline{D}_{2242} - 4\overline{D}_{2343} + \overline{D}_{3133}) - 2s^2(\overline{D}_{2323} + \overline{D}_{3232})). \quad (4.42)
\]

The coefficient \( c_2(s,t) \) is similar, and obeys \( c_2(s,t) = c_1(s/t, 1/t) \). Finally, \( d_1(s,t) \)
is equal to
\[ d_1(s, t) = -\frac{3}{10N_T} (10(\mathcal{D}_{1223} + \mathcal{D}_{2132}) - (\mathcal{D}_{1234} + \mathcal{D}_{2143})) + 2\mathcal{D}_{2224} \]
\[ + 30\mathcal{D}_{2233} + 3\mathcal{D}_{2334} + \mathcal{D}_{3133} + \mathcal{D}_{3144} + 3\mathcal{D}_{3243} \]
\[ + s(2(\mathcal{D}_{2233} + \mathcal{D}_{3322}) + 3(\mathcal{D}_{2334} + \mathcal{D}_{3243}) - 3\mathcal{D}_{3324}) \]
\[ + t(\mathcal{D}_{1333} + \mathcal{D}_{1344} + 2\mathcal{D}_{2242} - 3(\mathcal{D}_{2334} + \mathcal{D}_{3243} - s\mathcal{D}_{3342})) \]  

Again, the coefficient \( d_2(s, t) \) is similar and has the crossing symmetry \( d_2(s, t) = d_1(s/t, 1/t) \).

### 4.5 Verifying CFT predictions

In Section 3.3 a number of relations were derived for the coefficient functions, based on conformal field theory considerations and the insertion procedure. There are two ways to verify these relations. First of all, one could work with the \( \mathcal{D} \)-functions, and use the identities in Appendix D.2 of [24]. There crossing symmetry and other relations are listed for \( \mathcal{D} \)-functions. An advantage of this method is that it is possible to obtain a simple form of the coefficient functions.

In the present situation however, the above method becomes rather cumbersome and error-prone, due to the large number of different \( \mathcal{D} \)-functions involved. A more straightforward method is to express \( \mathcal{D} \)-functions into differential operators \( \mathcal{D} \), such that
\[ \mathcal{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(s, t) = \mathcal{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}\Phi(s, t), \]  
where \( \Phi(s, t) \) is the one-loop box integral. Details are listed in the appendix, in particular the differential operators which are listed on page 61. Using equations (A.31), it is then possible to write the coefficients completely in terms of \( \ln s, \ln t \) and \( \Phi(s, t) \). Using that \( \Phi(s/t, 1/t) = t\Phi(s, t) \), one finds that the crossing symmetries (3.17) are indeed obeyed. This however is a rather trivial check, as these crossing symmetries follow automatically if one considers the correct permutations when calculating the individual contributions to the four-point function.

First we read of the single function \( \mathcal{F}(s, t) \) from the coefficient \( a(s, t) \).\(^5\) In terms of the \( \mathcal{D} \)-functions it is given by
\[ \mathcal{F}(s, t) = \frac{3}{N^2} \left( -\mathcal{D}_{1133}(s, t) + 4s\mathcal{D}_{2233}(s, t) + (1 - s + t)\mathcal{D}_{2244}(s, t) \right). \]  

\(^5\)The reason to read it of from \( a(s, t) \) is that there is no free field contribution to the connected four-point function for the \( \delta_{1234}^{i} \) tensor.
Then, by using the method described above, one finds

\[ b(s, t) - (s - t - 1)F(s, t) = \frac{12}{N^2} \]

\[ c_1(s, t) - (t - s - 1)F(s, t) = \frac{6}{N^2} \]

\[ c_2(s, t) - (1 - s - t)F(s, t) = \frac{6}{N^2} \]

\[ d_1(s, t) - F(s, t) = 0 \]

\[ d_2(s, t) - tF(s, t) = 0. \]  

Comparing these results with the restrictions for the quantum parts predicted by equations (3.19) and (3.20) shows that these are indeed satisfied by the supergravity-induced four-point function, up to some constants. But for \( c_1 \), \( c_2 \) and \( b \) these constants are precisely the free field contributions (3.22)! Hence, we conclude that the coefficients split into a quantum and a free field part. The reason there is no free field contribution found for the \( \delta \) and \( \Upsilon \) tensors is discussed in the next chapter.
Chapter 5

Conclusions and discussion

In the previous chapter the four-point function was found in the supergravity approximation. It was shown that the quantum part can be described by a single function \( F(s, t) \), in precisely the way that was predicted by the field-theoretical insertion procedure. This is the first time this has been checked for four-point functions dual to \( \frac{1}{2} \)-BPS operators where not all four weights are equal. This can be seen as further support for the AdS/CFT correspondence, as there is no obvious reason why the supergravity result should have this property.

There is also a splitting observed into a “free” and a “quantum” part. The question arises why there is only a free field contribution for the \( C_{1234} \) and \( S_{1234} \) tensor. For the \( \delta_{2}^{12} \delta_{3}^{34} \) tensor this is obvious: the free field diagram corresponds to a disconnected diagram, but since we take the log of the generating functional, we only calculate the connected diagrams.

The absence of a free-field contribution in the \( \Upsilon \)-tensor case is a bit more subtle. From Figure 3.3.2, it is not obvious that these give no contribution in the supergravity approximation. Comparing with the \( C \) and \( S \)-tensors, the difference is that the \( \Upsilon \) diagrams are not one-particle irreducible (1PI), that is, we can make the diagram disconnected by removing one propagator. For the \( C \)- and \( S \)-tensors this is not possible. The generating functional for 1PI diagrams is the effective action \([42]\). The AdS/CFT conjecture precisely states that the CFT effective action is equal to the on-shell supergravity solution (in the large \( N \) limit) \([17]\). This shows nicely in the fact that we don’t find a connected but one-particle reducible contribution.

On the other hand, one can consider the four-point function from a field theory point of view. That is, calculate the coefficient functions in perturbation theory on the field theory side. This has been done up to order \( g_{YM}^4 \) for the correlator we consider here \([43]\). This should not necessarily give the same result, as the supergravity result corresponds to the strong coupling limit, while the perturbative result is in the weak coupling limit.

Even though we found the four-point function, there are still a lot of questions. For instance, it would be interesting to study the operator product expansion. This
can be used to check unitarity and e.g. find anomalous dimensions.

Although there have been a number of checks in the AdS/CFT correspondence of the relations found by the insertion procedure, the precise mechanisms are still ill-understood. The method that is used to calculate the four-point functions is rather cumbersome, since the expressions for the Lagrangian are very complicated. Since in each case considered so far the Lagrangian reduces to a simple expression with at most two derivatives, it is very well possible that there is a much simpler model which describes the supergravity side. For example, there might be more “hidden” symmetries that can be used to find a simpler Lagrangian. The restrictions obtained by the insertion procedure might give some hints on this: one can assume that these would lead to similar restrictions on the form of the supergravity Lagrangian.
Appendix A

Spherical harmonics and $C$-tensors

In the expressions we encountered, heavy use is made of spherical harmonics on $S^5$, and $C$-tensors associated with them. In this section, we consider some properties of these spherical harmonics, following the appendices of [25, 32], and $C$-tensors [24], and list some useful identities which can be used in the calculations.

A.1 Spherical harmonics

We describe the spherical harmonics $S^5$, but it should be noted that the theory can also be developed for higher or lower dimensional spheres.

A.1.1 Scalar spherical harmonics

The set of scalar functions on $S^5$ form a vector space. The scalar spherical harmonics form a complete basis on this vector space. To give a description of this basis, it is convenient to consider $S^5$ embedded in $\mathbb{R}^6$. In the embedding space, we can expand each $C^\infty$ function in a power series in the coordinates $x^i$ of $\mathbb{R}^6$. Thus, it is enough to construct a basis for polynomials on $\mathbb{R}^6$ which are homogeneous of degree $k$, for each integer $k \geq 0$.

We have to be careful here however, since we work on the embedding space. Indeed, the function $r^2 x^{i_1} \cdots x^{i_k}$ is homogeneous of degree $k + 2$, but restricted to the sphere $S^5$, it is equal to $x^{i_1} \cdots x^{i_k}$, which is homogeneous of degree $k$. Hence care must be taken if we want all spherical harmonics to be linearly independent. We can do this by making sure that for each degree $k$, the spherical harmonics of that degree are linearly independent of those of lower degree. This can be done by considering, at degree $k$, only functions $C_{i_1 \cdots i_k} x^{i_1} \cdots x^{i_k}$, where $C_{i_1 \cdots i_k} \delta^{i_m i_n} = 0$ for each $1 \leq m, n \leq k$, hence we see that the $C$-tensors are traceless. Without loss of generality, we can assume that the tensor $C_{i_1 \cdots i_k}$ is symmetric in $i_1 \cdots i_k$. 

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The tensor $C$ transforms according to the $[0, k, 0]$ irrep of $SO(6)$, hence we can decompose it as $C_{i_1 \ldots i_k}^I$, where $I$ runs over the basis of the irrep $[0, k, 0]$. Note that $C$ also depends on $k$, but we sometimes omit this in the notation. Otherwise, they are denoted by $C_k$ or $C_{[0,k,0]}$, to explicitly indicate the tensor transforms according to the $[0, k, 0]$ irrep. Now the scalar spherical harmonics can be defined as

$$Y^I = z^{-1/2}(k)C_{i_1 \ldots i_k}^I x^{i_1} \ldots x^{i_k}, \quad (A.1)$$

where we employ the same normalization as in [32], and the normalization factor $z(k)$ is given by

$$z(k) = \frac{\pi^{3}}{2^{k-1}(k + 1)(k + 2)}.$$

The $C$ tensors are orthonormal: $C_{i_1 \ldots i_k}^I C^J_{i_1 \ldots i_k} = \delta^{IJ}$, and for the spherical harmonics, we have with this normalization

$$\int_{S^5} Y^I Y^J = \delta^{IJ},$$

so we conclude that the scalar spherical harmonics form a complete orthonormal basis for scalar functions on $S^5$.

Another important property is that the spherical harmonics of the Laplacian $\nabla^2$. That is, we have $\nabla^2 Y^I = -k(k + 4)Y^I$. An indication of how to prove this, can be found in [25].

### A.1.2 Vector and tensor spherical harmonics

Vectors and rank two tensors on the 5-sphere can also be expanded into spherical harmonics. For vectors, one can define the vector spherical harmonics as the tangent components of the vector

$$Y^I_\alpha = z(k)^{-1/2}C_{\alpha i_1 \ldots i_k}^I x^{i_1} \ldots x^{i_k},$$

where $\alpha$ is a vector index, and $I$ is again as above. The tensor $C$ is again symmetric and traceless with respect to the indices $i_1, \ldots, i_k$, and the normalization condition is now given by

$$C_{\alpha i_1 \ldots i_k}^I C_{\beta i_1 \ldots i_k}^J = \delta^{IJ} \delta_{\alpha \beta}.$$

The vector spherical harmonics form again a complete orthonormal basis for the vector functions on $S^5$.

Finally, the tensor spherical harmonics are projections of the following tensor on the 5-sphere:

$$Y^{I\alpha \beta} = z(k)^{-1/2}C_{\alpha \beta i_1 \ldots i_k}^I x^{i_1} \ldots x^{i_k}.$$

Here $\alpha$ and $\beta$ are tensor indices, and the tensor $C_{\alpha \beta i_1 \ldots i_k}^I$ is again symmetric and traceless with respect to $i_1, \ldots, i_k$, and $\alpha, \beta$, and it’s symmetric part vanishes:

$$C_{\alpha \beta i_1 \ldots i_k}^I + C_{\alpha i_1 \beta \ldots i_k}^I + \cdots + C_{\alpha i_k \beta i_1 \ldots i_{k-1}}^I \beta = 0.$$
The tensors are again orthonormal,

\[ C^I_{\alpha_1 \beta_1 ; i_1 \ldots i_k} C^J_{\alpha_2 \beta_2 ; i_1 \ldots i_k} = \delta^{IJ} \delta_{\alpha_1 \beta_1 ; \alpha_2 \beta_2}, \]

and the tensor spherical harmonics form a complete basis.

### A.1.3 Integrals of spherical harmonics

Working with the equations, we often encounter integrals of different kinds of spherical harmonics. Let us first make a few definitions:

\[ \int Y^{I_1} Y^{I_2} Y^{I_3} = a_{123} \tag{A.2} \]

\[ \int \nabla^a Y^{I_1} Y^{I_2} Y^{I_3} = t_{123} \tag{A.3} \]

\[ \int \nabla^a Y^{I_1} \nabla^\beta Y^{I_2} Y^{I_3(\alpha \beta)} = p_{123} \tag{A.4} \]

With the normalization introduced above, equation (A.2) can be computed as

\[ a_{123} = (z(k_1)z(k_2)z(k_3))^{-1/2} \frac{\pi^3}{k_1!k_2!k_3!} \frac{k_1!k_2!k_3!}{(\Sigma + 2)!} \frac{1}{2^{\Sigma-2} \alpha_1!\alpha_2!\alpha_3!} \langle C^{I_1} C^{I_2} C^{I_3} \rangle_{[0,k_3,0]} \tag{A.5} \]

where \( \alpha_i = \frac{1}{2}(k_i - k_i - k_i) \), where \( j \neq l \neq i \), \( \Sigma = k_1 + k_2 + k_3 \), and \( \langle C^{I_1} C^{I_2} C^{I_3} \rangle_{[0,k_3,0]} \) is the unique \( SO(6) \) tensor, obtained by contracting \( \alpha_1 \) indices between \( C^{I_2} \) and \( C^{I_3} \), \( \alpha_2 \) indices between \( C^{I_2} \) and \( C^{I_3} \), and \( \alpha_3 \) indices between \( C^{I_2} \) and \( C^{I_1} \). Explicitly,

\[ \langle C^{I_1} C^{I_2} C^{I_3} \rangle_{[0,k_3,0]} = C^{I_1}_{i_1 \ldots i_3} C^{I_2}_{j_1 \ldots j_3} C^{I_3}_{l_1 \ldots l_3} \]

\[ = \sum_{i_1 \ldots i_3, \Sigma} \left( \begin{array}{c} k_1 \\ \Sigma \end{array} \right) \left( \begin{array}{c} k_2 \\ \Sigma \end{array} \right) \left( \begin{array}{c} k_3 \\ \Sigma \end{array} \right) \]

\[ \sum_{i_1 \ldots i_3, \Sigma} \frac{1}{(\Sigma + 2)!} \frac{1}{2^{\Sigma-2}} \alpha_1!\alpha_2!\alpha_3! \langle C^{I_1} C^{I_2} C^{I_3} \rangle_{[1,k_3-1,1]} \tag{A.6} \]

Note that this construction allows us to formulate some selection rules for the tensors \( a_{123} \). Indeed, the construction of the invariant \( SO(6) \) tensor is only possible if \( k_1, k_2 \) and \( k_3 \) are such that each \( a_i \) is non-negative and integer-valued. Hence, in other cases \( a_{123} \) must vanish. This implies, for example, that for each \( k \), there are only finitely many coupling terms containing the field \( s_k \), which can be seen from the explicit couplings involving the \( a_{123} \) tensors in [31].

Equation (A.3) can be calculated as

\[ t_{123} = \frac{\pi^3}{k_3 + 1} \left( \frac{1}{2}(\Sigma + 3)! \right) \left( \frac{1}{2}(\Sigma - 3)! \right) \frac{k_1!k_2!k_3!}{(\alpha_1 - 1)!(\alpha_2 - 1)!(\alpha_3 - 1)!} \langle C^{I_1} C^{I_2} C^{I_3} \rangle_{[1,k_3-1,1]} \tag{A.6} \]

where the tensor \( \langle C^{I_1} C^{I_2} C^{I_3} \rangle_{[1,k_3-1,1]} \) is also constructed by contracting certain indices:

\[ \langle C^{I_1} C^{I_2} C^{I_3} \rangle_{[1,k_3-1,1]} = C^{I_1}_{m_1 \ldots m_3 j_1 \ldots j_3} C^{I_2}_{j_1 \ldots j_3 l_1 \ldots l_3} C^{I_3}_{m_1 \ldots m_3 l_1 \ldots l_3 j_1 \ldots j_3} - C^{I_1}_{l_1 \ldots l_3 j_1 \ldots j_3} C^{I_2}_{j_1 \ldots j_3 l_1 \ldots l_3} C^{I_3}_{m_1 \ldots m_3 j_1 \ldots j_3 l_1 \ldots l_3} \]
Table A.1: Selection rules for tensors, when $k_1$ and $k_2$ are given.

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{123}$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0,2,4,6</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1,3,5</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1,3,5</td>
</tr>
<tr>
<td>$p_{123}$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2,4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$t_{123}$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1,3,5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2,4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2,4</td>
</tr>
</tbody>
</table>

where $p_1 = \alpha_1 + \frac{1}{2}$, $p_2 = \alpha_2 - \frac{1}{2}$ and $p_3 = \alpha_3 - \frac{1}{2}$. Again, this construction implies certain selection rules for the $k_1$, $k_2$ and $k_3$.

Finally, equation (A.4) can be expressed as

$$p_{123} = \left(z(k_1)z(k_2)z(k_3)\right)^{-1/2} \frac{\pi^3}{(\frac{1}{2}\Sigma + 1)! \cdot 2^{\frac{1}{2}\Sigma} \alpha_1!\alpha_2!(\alpha_3 - 1)!} \langle C_1^I C_2^J C_3^K \rangle_{[2,k_3-2,2]}$$

where $\langle C_1^I C_2^J C_3^K \rangle_{[2,k_3-2,2]}$ is given by

$$\langle C_1^I C_2^J C_3^K \rangle_{[2,k_3-2,2]} = C_{m_1 \ldots i_2 j_1 \ldots j_3}^{I_1} C_{n_1 \ldots j_2 i_1 \ldots i_3}^{I_2} C_{l_1 \ldots l_2}^{I_3} C_{m n l_1 \ldots l_2 i_1 \ldots i_3}^{I_3}$$

where $p_1 = \alpha_1$, $p_2 = \alpha_2$ and $p_3 = \alpha_3 - 1$, and once again we can derive selection rules for the $k_i$ for these tensors. The selection rules for this and the other tensors, for the values of $k_i$ that are of interest to us, are summarized in Table A.1.

### A.2 Summation of $C$-tensors

In the expressions for the Lagrangian and the four-point functions, we frequently encounter expressions of the form

$$\langle C_1^I C_2^J C_3^K \rangle_{[0,k,0]} \langle C_3^I C_4^J C_5^K \rangle_{[0,k,0]}$$

where there is a summation over the index $I_5$. We can actually perform the summation, to rewrite the expression in terms of a basis of independent tensor structures. This can be done using the completeness relation for tensors.

The first step is to rewrite tensor $C_{i_1 \ldots i_k}^{I_1 \ldots I_k}$, where there is a summation over $I$, in terms of Kronecker deltas. There are several ways to do this. The first method is quite natural and straightforward, but has the disadvantage that the number of calculations involved grows rapidly with increasing $k$. It consists of the following step.

1. Write down all possible products of Kronecker deltas, involving all the indices $i_1 \ldots i_k$ and $j_1 \ldots j_k$, with arbitrary coefficients. For example, for $k = 2$, this would lead to

$$\alpha_1 \delta_{i_1 i_2} \delta_{j_1 j_2} + \alpha_2 \delta_{i_1 j_2} \delta_{i_2 j_1} + \alpha_3 \delta_{i_1 j_1} \delta_{i_2 j_2}.$$
2. The tensor $C_{i_1\ldots i_k}$ is totally symmetric in the indices $i_1, \ldots, i_k$, so the total expression is as well. This can be used to solve some of the coefficients, by imposing the symmetry conditions. The same holds for the indices $j_1, \ldots, j_k$.

3. The expression should also be traceless with respect to any pair of the indices $i_1, \ldots, i_k$, which allows us to solve for even more coefficients. The same holds again for the $j$ indices.

4. Finally, an overall normalization factor can be fixed by imposing the normalization for $C$-tensors,

$$ C_{i_1\ldots i_k}^I C_{j_1\ldots j_k}^I = \text{dim}[0, k, 0]. $$

As mentioned before, this method requires a lot of computations, and is not very efficient if the algorithm is programmed on a computer. Another approach, which is less insightful, but requires much less computation, is to use the general formula (B.8) given in [24]. This is given by

$$ C_{i_1\ldots i_n}^I C_{j_1\ldots j_n}^I = \left[ \frac{2}{n!} \right] \sum_{k=0}^n \theta_k \sum_{(l_1, \ldots, l_{2k})} \delta_{i_1 l_1} \delta_{i_2 l_2} \cdots \delta_{i_{2k-1} l_{2k}} \delta_{l_{2k} i_{2k}} \delta_{j_1 l_1} \delta_{j_2 l_2} \cdots \delta_{j_{2k-1} l_{2k}} \delta_{j_{2k+1} \ldots j_n} \delta_{l_{2k+1} \ldots l_{2k}}. $$

(A.8)

In this expression, $\delta_{i_1\ldots i_p j_1\ldots j_p}$ denotes total symmetrization of the indices. For fixed $k$, the internal sum runs over all tuples $(l_1, \ldots, l_{2k}) \in (1 \ldots n)$, which lead to different products $\delta_{i_1 l_1} \delta_{i_2 l_2} \cdots \delta_{i_{2k-1} l_{2k}} \delta_{i_{2k} l_{2k}}$. Finally, the factor $\theta_k$ are given by

$$ \theta_0 = 1, \quad \theta_k = \frac{(-1)^k}{2^k(n+1) \cdots (n+2-k)}. $$

As an example, we find

$$ C_{i_1 i_2}^I C_{j_1 j_2}^I = \frac{1}{2} (\delta_{i_1 j_1} \delta_{i_2 j_2} + \delta_{i_1 j_2} \delta_{i_2 j_1}) - \frac{1}{6} \delta_{i_1 i_2} \delta_{j_1 j_2}. $$

In the calculation we encounter four different kinds of tensors. One should take notice that in each tensor, there are two $C$-tensors transforming as $[0, 2, 0]$, and two as $[0, 3, 0]$, involved. This should be taken into account carefully when permuting the representation indices. The tensor structures involved are given by

$$ \delta_{ij}^{12} \delta_{34}^{34} = C_{ij}^1 C_{ij}^2 C_{klm}^{3} C_{klm}^{4} \quad \text{(A.9)} $$
$$ C_{1234}^{1234} = C_{ij}^1 C_{jk}^2 C_{klm}^{3} C_{klm}^{4} \quad \text{(A.10)} $$
$$ \Upsilon_{1234}^{1234} = C_{ij}^1 C_{lm}^{2} C_{ij}^{3} C_{lm}^{4} \quad \text{(A.11)} $$
$$ S_{1234}^{1234} = C_{ik}^1 C_{jl}^{2} C_{ikm}^{3} C_{ijm}^{4} \quad \text{(A.12)} $$

Unfortunately there is a misprint in the formula given in the article.
The tensors $\delta_2^{12}\delta_3^{34}$ and $S^{3234}$ are symmetric under $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ separately, while $C^{1234}$ and $\Upsilon^{1234}$ have the following symmetry relations:

$$C^{1234} = C^{2143}, \quad \Upsilon^{1234} = \Upsilon^{2143}.$$  \hspace{1cm} (A.13)

We can now plug (A.8) in the expressions like (A.7), contract the indices, and use equations (A.9)- (A.12) to put the result in terms of the tensor basis.

For the case $k_1 = k_2 = 2, k_3 = k_4 = 3$, this gives the following results.

$$\langle C_1^1C_2^2C_{[0,0,0]}^5 | C_{3}^{3}C_{4}^{4}C_{[0,0,0]}^{5} \rangle = \delta_2^{12}\delta_3^{34}$$

$$\langle C_1^1C_2^2C_{[0,2,0]}^5 | C_{3}^{3}C_{4}^{4}C_{[0,2,0]}^{5} \rangle = \frac{1}{2} C^{1234} + \frac{1}{2} C^{1243} - \frac{1}{6} \delta_2^{12}\delta_3^{34}$$

$$\langle C_1^1C_2^2C_{[0,4,0]}^5 | C_{3}^{3}C_{4}^{4}C_{[0,4,0]}^{5} \rangle = -\frac{2}{15} C^{1234} - \frac{2}{15} C^{1243} + \frac{2}{3} S^{1234}$$

$$\hspace{1cm} + \frac{1}{6} \Upsilon^{1234} + \frac{1}{6} \Upsilon^{1243} - \frac{1}{60} \delta_2^{12}\delta_3^{34}$$  \hspace{1cm} (A.14)

It is interesting to see that they are of the same form as those found in the cases where $C^1, C^2, C^3$ and $C^4$ are all in the same representation [36, 24], if one makes the proper identifications.

For the summation over the vector representations, we use the same technique as in [24]. First we note the following important relations, proved in [31]:

$$t_{125t_{345}} = -\frac{(f_1 - f_2)(f_3 - f_4)}{4f_5}a_{125a_{345}} + \frac{1}{4} f_5(a_{145a_{235}} - a_{245a_{135}}),$$

$$\frac{1}{4} - f_5 \frac{1}{4} t_{125t_{345}} = \frac{1}{4} (f_2^2 - f_3(f_1 + f_2 + f_3 + f_4 - 4))(a_{145a_{235}} - a_{135a_{245}})$$

$$\hspace{1cm} - \frac{4 - f_5}{4f_5}(f_1 - f_2)(f_3 - f_4)a_{125a_{345}},$$

where $f_k = k(k + 4)$, and one should take notice that there is a summation over $k_5$ implied in both lines. One can now use equations (A.5) and (A.6) and substitute the results of (A.14). This leaves us with two equations with two unknowns, from which we can solve $\langle C_1^1C_2^2C_{[1,0,1]}^5 | C_{3}^{3}C_{4}^{4}C_{[1,0,1]}^{5} \rangle$ and $\langle C_1^1C_2^2C_{[1,2,1]}^5 | C_{3}^{3}C_{4}^{4}C_{[1,2,1]}^{5} \rangle$.

This leads to

$$\langle C_1^1C_2^2C_{[1,0,1]}^5 | C_{3}^{3}C_{4}^{4}C_{[1,0,1]}^{5} \rangle = 2(C^{1243} - C^{1234})$$

$$\langle C_1^1C_2^2C_{[1,2,1]}^5 | C_{3}^{3}C_{4}^{4}C_{[1,2,1]}^{5} \rangle = \frac{1}{3}(C^{1234} - C^{1243}) + \frac{2}{3}(\Upsilon^{1234} - \Upsilon^{1243}).$$  \hspace{1cm} (A.15)

Finally, to determine $\langle C_1^1C_2^2C_{[2,0,2]}^5 | C_{3}^{3}C_{4}^{4}C_{[2,0,2]}^{5} \rangle$ we can use equation (A.4) with equation (B.10) from [31]. Since the selection rules allow for only one value of $k_5$, it is not necessary to use equation (B.11) and solve the two equations in two unknowns. This results in

$$\langle C_1^1C_2^2C_{[2,0,2]}^5 | C_{3}^{3}C_{4}^{4}C_{[2,0,2]}^{5} \rangle = -\frac{3}{2}(C^{1234} + C^{1243}) + \frac{4}{3}(\Upsilon^{1234} + \Upsilon^{1243})$$

$$- \frac{8}{3} S^{1234} + \frac{2}{15} \delta_2^{12}\delta_3^{34},$$  \hspace{1cm} (A.16)

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and we see again the same structure as the result for the case of equal weight. The results for the case $k_1 = k_2 = 3, k_3 = k_4 = 2$ are the same, if one replaces the corresponding representation indices accordingly.

For $k_1 = k_3 = 2, k_2 = k_4 = 3$ the results are

$$
\langle C_1^A C_2^B C_3^C | C_1^D C_2^E C_3^F \rangle \langle C_2^A C_3^B C_4^C | C_2^D C_3^E C_4^F \rangle = \Upsilon^{1243}
$$

for $k_1 = k_2 = 3, k_3 = k_4 = 2$.

For $k_1 = k_3 = 2, k_2 = k_4 = 3$ the results are

$$
\langle C_1^A C_2^B C_3^C | C_1^D C_2^E C_3^F \rangle \langle C_2^A C_3^B C_4^C | C_2^D C_3^E C_4^F \rangle = \frac{1}{3} C_{1342}^{1324} + \frac{2}{3} S_{1342}^{1324} - \frac{1}{6} \Upsilon^{1324}
$$

for $k_1 = k_2 = 3, k_3 = k_4 = 2$.

and using the same method as indicated above we also find

$$
\langle C_1^A C_2^B C_3^C | C_1^D C_2^E C_3^F \rangle \langle C_2^A C_3^B C_4^C | C_2^D C_3^E C_4^F \rangle = \frac{3}{2} C_{1342}^{1324} - \frac{3}{2} S_{1342}^{1324} - \frac{3}{10} \Upsilon^{1324}
$$

for $k_1 = k_2 = 3, k_3 = k_4 = 2$.

and for the $p$ tensors

$$
\langle C_1^A C_2^B C_3^C | C_1^D C_2^E C_3^F \rangle \langle C_2^A C_3^B C_4^C | C_2^D C_3^E C_4^F \rangle = -\frac{16}{9} C_{1342}^{1324} - \frac{40}{63} C_{1342}^{1324} - \frac{16}{63} S_{1342}^{1324}
$$

for $k_1 = k_2 = 3, k_3 = k_4 = 2$.

The results of the remaining cases are the same, if one changes the representation labels accordingly, except for equation (A.18), which acquires an additional minus sign in the cases $k_1 = k_4 = 3, k_2 = k_3 = 2$ and $k_1 = k_4 = 2, k_2 = k_3 = 3$.

### A.3 Pairings and projectors

It is possible to explicitly calculate pairings of $C$-tensors. That is, expressions like $C_{[1,1,1]}^{1243} \Upsilon^{1243}$, where there is a summation over the upper representation indices. This is done by first writing the expressions in term of the $SO(6)$ tensors $C_{[i_1,...,i_k]}^{I_k}$ and then use the result (A.8). The remaining summation over the $SO(6)$ indices can then be done easily on a computer. The results are listed in Table A.2.

The pairings can be used to calculate the normalized projectors. These operators project onto the irreps. appearing in the tensor decomposition

$$
[0, 2, 0]_{20} \otimes [0, 3, 0]_{50} = [0, 1, 0]_{6} + [0, 3, 0]_{50} + [0, 5, 0]_{196} + [1, 1, 1]_{64} + [1, 3, 1]_{384} + [2, 1, 2]_{300},
$$

where the subscripts denote the dimension of the corresponding representation. The projectors are not needed in this thesis. When considering the operator product expansion of the four-point function however, they are necessary, which is why
the results are included here. The problem of finding the normalized projectors is actually almost solved already, they are proportional to equations (A.17), (A.18) and (A.19). The pairing table can then be used to fix proper normalization.

### A.4 D-functions

In the evaluation of the diagrams in Section 4.3, an important rôle is played by the so-called $D$-functions. These $D$-functions correspond to a quartic interaction of scalar fields [44]. The $D$-function of scalar fields of conformal dimension $\Delta_i$ related to $\text{AdS}_5$ are defined by

$$D_{\Delta_{1}\Delta_{2}\Delta_{3}\Delta_{4}}(x_1, x_2, x_3, x_4) = \int \frac{d^5 z}{z_0} K_{\Delta_1}(z, x_1) K_{\Delta_2}(z, x_2) K_{\Delta_3}(z, x_3) K_{\Delta_4}(z, x_4),$$

(A.20)

where $K_\Delta$ is the bulk-to-boundary propagator for scalar fields, defined by

$$K_\Delta(z, x) = \left( \frac{z_0}{z_0^2 + (z-x)^2} \right)^\Delta.$$

To avoid having to write the arguments to the $D$-functions every time, we will use the convention that the order is always $x_1, x_2, x_3, x_4$.

In calculating exchange diagrams a number of identities regarding the propagators $K_\Delta$ are very useful. First introduce the vectors

$$P_\mu = \frac{\delta_\mu 0}{w_0}, \quad Q_\mu = \frac{(w-x_1)_\mu}{(w-x_1)^2} - \frac{(w-x_3)_\mu}{(w-x_3)^2},$$

with squares $P^\mu P_\mu = 1$ and $Q^\mu Q_\mu = x_{13}^2 K_1(w, x_1) K_1(w, x_3)$. The following identities, which are easy to prove by writing out the definitions and doing the derivatives, are then useful in calculations:

$$K_{\Delta_1}(w, x_1) K_{\Delta_2}(w, x_1) = K_{\Delta_1+\Delta_2}(w, x_1),$$

(A.21)

$$\nabla_\mu K_{\Delta}(w, x_1) = \Delta P_\mu K_{\Delta}(w, x_1) - 2\Delta \frac{(w-x_1)_\mu}{(w-x_1)^2} K_{\Delta}(w, x_1),$$

(A.22)

$$\nabla^\mu K_{\Delta}(w, x_2) Q_\mu = \Delta K_{\Delta+1}(w, x_2) (x_{12}^2 K_1(w, x_1) - x_{23}^2 K_1(w, x_3)).$$

(A.23)
There are numerous identities for these \( D \)-functions, which can be used to simplify the final result. The derivatives appearing in the contact diagrams, can be written in terms of \( D \)-functions by using the identity \cite{45}

\[
\nabla_\mu K_{\Delta_1}(w, x_1) \nabla_\nu K_{\Delta_2}(w, x_2) = \Delta_1 \Delta_2 (K_{\Delta_1}(w, x_1)K_{\Delta_2}(w, x_2) - 2x_1^2 K_{\Delta_1+1}(w, x_1)K_{\Delta_2+1}(w, x_2)).
\]

(A.24)

Sums of \( D \)-functions which are symmetric with respect \( x_1 \leftrightarrow x_2 \) and \( x_3 \leftrightarrow x_4 \) can be written in a way where this symmetry is obvious, by using the identity

\[
x_{13}^2 D_{\Delta+1\Delta+1\Delta} + x_{14}^2 D_{\Delta+1\Delta\Delta+1} = \frac{2\Delta + 2\Delta - d}{2\Delta} D_{\Delta\Delta\Delta} - \frac{\Delta}{\Delta} x_{12}^2 D_{\Delta+1\Delta+1\Delta},
\]

(A.25)

and similar expressions for permutations of the indices.

It is possible to express the \( D \)-functions in terms of the conformal cross-ratios \( s \) and \( t \). Introducing the notation \( D \), we have based on the conformal symmetries

\[
\frac{\prod_{i=1}^{4} \Gamma(\Delta_i)}{\Gamma(\Sigma - \frac{d}{2})} \frac{2}{\pi^2} D_{\Delta_1\Delta_2\Delta_3\Delta_4} = \frac{\Gamma^2(x_{13})^{-\Sigma-\Delta_1-\Delta_4} \Gamma^2(x_{24})^{-\Sigma-\Delta_3-\Delta_4}}{(x_{13})^2 (x_{24})^2} D_{\Delta_1\Delta_2\Delta_3\Delta_4}(s, t),
\]

(A.26)

where \( \Sigma = \frac{1}{2} \sum_{i=1}^{4} \Delta_i \). For \( \Delta_i = 1 \), this expression becomes

\[
D_{1111}(s, t) = \Phi(s, t),
\]

(A.27)

where \( \Phi(s, t) \) is the one-loop box integral. From the permutation symmetry of (A.20), the following relations can be deduced

\[
D_{\Delta_1\Delta_2\Delta_3\Delta_4} = t^{-\Delta_2} D_{\Delta_1\Delta_2\Delta_4\Delta_3}(s/t, 1/t)
\]

\[
= t^{\Delta_4-\Sigma} D_{\Delta_2\Delta_3\Delta_4\Delta_1}(s/t, 1/t)
\]

\[
= D_{\Delta_1\Delta_2\Delta_3\Delta_4}(t, s)
\]

\[
= t^{\Delta_1+\Delta_3-\Sigma} D_{\Delta_2\Delta_3\Delta_4\Delta_1}(s, t)
\]

\[
= s^{\Delta_3+\Delta_4-\Sigma} D_{\Delta_4\Delta_3\Delta_2\Delta_1}(s, t).
\]

(A.28)

These relations can be used to simplify expressions of \( D \)-functions, and to check crossing symmetry relations. There are other relations, for example for the sum of specific \( D \)-functions, which can be used to simplify expressions. We do not need them here, but they are listed in \cite{24}.

By considering the derivative relations of the original \( D \)-functions, it is possible to derive the following relations for the \( D \)-functions \cite{24}

\[
D_{\Delta_1+1\Delta_2+1\Delta_3\Delta_4} = -\partial_s D_{\Delta_1\Delta_2\Delta_3\Delta_4}
\]

\[
D_{\Delta_1\Delta_2\Delta_3+1\Delta_4+1} = (\Delta_3 + \Delta_4 - \Sigma - s\partial_s) D_{\Delta_1\Delta_2\Delta_3\Delta_4}
\]

\[
D_{\Delta_1\Delta_2+1\Delta_3+1\Delta_4} = -\partial_t D_{\Delta_1\Delta_2\Delta_3\Delta_4}
\]

\[
D_{\Delta_1+1\Delta_2\Delta_3\Delta_4+1} = (\Delta_1 + \Delta_4 - \Sigma - t\partial_t) D_{\Delta_1\Delta_2\Delta_3\Delta_4}
\]

\[
D_{\Delta_1\Delta_2+1\Delta_3\Delta_4+1} = (\Delta_2 + s\partial_s + t\partial_t) D_{\Delta_1\Delta_2\Delta_3\Delta_4}
\]

\[
D_{\Delta_1+1\Delta_2\Delta_3+1\Delta_4} = (\Sigma - \Delta_4 + s\partial_s + t\partial_t) D_{\Delta_1\Delta_2\Delta_3\Delta_4}.
\]

(A.29)
Starting with (A.27) and subsequently applying these relations, it is possible to assign to each $D$-function a differential operator $\mathbf{D}$, such that

$$\mathbf{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} (s, t) = \mathbf{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \Phi(s, t), \quad (A.30)$$

as long as each $\Delta_1$ is an integer, and their sum is even. Note that there is some arbitrariness in defining $\mathbf{D}$, because there are different combinations of the relations in (A.29) that one can use to find a particular differential operator.

The action of the partial derivatives on $\Phi(s, t)$ is known [46], they are given by

$$\partial_s \Phi(s, t) = \frac{1}{\lambda^2} \left( \Phi(s, t)(1 - s + t) + 2 \ln s - \frac{s + t - 1}{s} \ln t \right),$$

$$\partial_t \Phi(s, t) = \frac{1}{\lambda^2} \left( \Phi(s, t)(1 - t + s) + 2 \ln t - \frac{s + t - 1}{t} \ln s \right), \quad (A.31)$$

where $\lambda = \sqrt{(1 - s - t)^2 - 4st}$. Using this together with (A.30), it is possible to express each combination of $D$-functions into an expression involving only $\Phi(s, t)$.

Finally, this is a list of the differential operators that we need:

$$\begin{align*}
\mathbf{D}_{1133} & = -(1 - s \partial_s)s \partial_s s \\
\mathbf{D}_{1223} & = -(1 + s \partial_s + t \partial_t)s \partial_s s \\
\mathbf{D}_{1232} & = \partial_t s \partial_s s \\
\mathbf{D}_{1234} & = -(1 - s \partial_s)(1 + s \partial_s + t \partial_t)s \partial_s s \\
\mathbf{D}_{1243} & = (1 - s \partial_s)\partial_t s \partial_s s \\
\mathbf{D}_{1333} & = -(2 + s \partial_s + t \partial_t)(1 + s \partial_s + t \partial_t)s \partial_s s \\
\mathbf{D}_{1344} & = (2 + s \partial_s + t \partial_t)(1 - s \partial_s)\partial_t s \partial_s s \\
\mathbf{D}_{2123} & = t \partial_t s \partial_s s \\
\mathbf{D}_{2132} & = -(1 + s \partial_s + t \partial_t)s \partial_s s \\
\mathbf{D}_{2134} & = (1 - s \partial_s)t \partial_t s \partial_s s \\
\mathbf{D}_{2143} & = -(1 + s \partial_s + t \partial_t)(1 - s \partial_s)s \partial_s s \\
\mathbf{D}_{2224} & = t \partial_t (1 + s \partial_s + t \partial_t)s \partial_s s \\
\mathbf{D}_{2233} & = -s \partial_s^2 s \partial_s s \\
\mathbf{D}_{2242} & = (2 + s \partial_s + t \partial_t)\partial_t s \partial_s s.
\end{align*}$$

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Appendix B

Exchange diagrams

In the calculation of 4-point correlation functions in the AdS/CFT correspondence one encounters exchange diagrams, where a bulk-to-bulk propagator is involved. To calculate these diagrams, one has to integrate the two interaction points over $AdS_{d+1}$ space. The first of these integrals is commonly called the $z$-integral. Surprisingly, it turns out that after performing the $z$-integral, the diagrams reduce to a sum of $D$-functions.

One way to do this integrals is to substitute the expressions for the bulk-to-bulk propagators, make a series expansion, and perform the integral. This method has been used to calculate scalar, gauge boson and graviton exchange diagrams [45, 47, 48, 44]. This method is however rather cumbersome. A much simpler method, where the explicit form of the bulk-to-bulk propagators isn’t needed anymore, was then presented in [41]. In this chapter we extend this method for the massive vector and symmetric tensor exchange, to include more general sources where the currents from two scalar bulk-to-boundary propagators aren’t conserved.

Let us first briefly review the basic steps in the method. First we perform a translation and use conformal inversion symmetry to bring the $z$-integral into a simpler form. Next we propose a suitable ansatz for the integral, based on symmetry arguments. The key trick is then to use the wave equation for the bulk-to-bulk propagator under the integral sign. This will lead to a simple expression. On the other hand, we can use the wave operator on the ansatz. Combining these two results, we get an inhomogenous second order differential equation in the variable $t = w_0^2/w^2$, which can be solved recursively, to find the solution for the integral. The final step is to change the coordinates back into the original, and we’re left with a simple expression for the diagram in terms of $D$-functions.

B.1 Massive vector exchange

In this section we consider the general vector exchange diagram, shown in Figure B.1. For the exchange of vectors like in this diagram, the integrals we have to
evaluate are given by

\[ V(x_1, x_2, x_3, x_4) = \int \frac{d^{d+1}w}{w_0^{d+1}} A_\mu(w, x_1, x_3) g^{\mu\nu}(w) K_{\Delta_2}(w, x_2) \overleftarrow{\nabla}_\nu K_{\Delta_4}(w, x_4), \]

where \( x_i \) are points on the AdS boundary, and \( A_\mu \) is the \( z \)-integral of interest, given by

\[ A_\mu(w, x_1, x_3) = \int \frac{d^{d+1}z}{z_0^{d+1}} G_{\mu\nu'}(w, z) g^{\nu'\rho'}(z) K_{\Delta_1}(z, x_1) \overleftarrow{\nabla}_\rho' K_{\Delta_3}(z, x_3) \]

Note that we adopt the same convention as in [41], where the primed indices refer to the \( z \) coordinate, and unprimed indices to the \( w \) coordinate. Note that we do not require that \( \Delta_1 = \Delta_3 \) and \( \Delta_2 = \Delta_4 \), that is, we do not require that the vector couples to a conserved current.

The bulk-to-bulk propagator \( G_{\mu\nu'} \) satisfies the following differential equation [39]

\[ -\nabla^\mu (\nabla_\mu G_{\rho\nu'} - \nabla_\rho G_{\mu\nu'}) + m^2 G_{\rho\nu'}(w, z) = g_{\rho\nu'} \delta(w, z). \]

Note that we have omitted the gauge term, since this is only required when \( m^2 = 0 \). However, the massless vector only couples to conserved currents, and one can refer to [41]. This can be seen from the cubic couplings: they vanish in the case of the massless vector when the two scalar fields are of different weight \( k \).

Equation (B.2) can be simplified by using conformal symmetries [41, 40]. First we use translation symmetry to bring the boundary point \( x_1 \) to zero, so also \( x_3 \rightarrow x_31 \equiv x_3 - x_1 \). This gives

\[ \int \frac{d^{d+1}z}{z_0^{d+1}} G_{\mu\nu'}(w, z) g^{\nu'\rho'}(z) \left( \frac{z_0}{z^2} \right)^{\Delta_1} \overleftarrow{\nabla}_\rho' \left( \frac{z_0}{z_0^2 + (z - x_31)^2} \right)^{\Delta_3} \]

\[ \text{The propagators are of course the Green’s functions from section 4.3.} \]
Next we apply conformal inversion to the coordinates:

\[ z_\mu \rightarrow \frac{z'_\mu}{(z')^2}, \quad w_\mu \rightarrow \frac{w'_\mu}{(w')^2}, \quad x_{31} \rightarrow \frac{x'_{31}}{|x'_{31}|^2}. \]

Under conformal inversion the scalar boundary-to-bulk propagator transforms as

\[ \left( \frac{z_0}{z_0^2 + (z - x)^2} \right)^{\Delta} = \left( \frac{z'_0}{(z'_0)^2 + (z' - x')^2} \right)^{\Delta} \mid x' \mid^{2 \Delta}, \]

and the bulk-to-bulk vector propagator transforms as a bitensor. Performing the inversion we find

\[ I_\mu(\nu, w, x_{13}, x_3) = \frac{1}{|x_{13}|^{2 \Delta_3}} \frac{1}{w^2} J_{\mu\nu}(w) I_\nu(\nu - x_{13}'), \quad (B.4) \]

where \( J_{\mu\nu}(w) = \delta_{\mu\nu} - 2 w_\mu w_\nu / w^2 \). If we finally translate \( z' \rightarrow z' - x'_{13} \), and \( w \) accordingly to keep \( u' \) invariant, we arrive at the expression

\[ A_\mu(w, x_1, x_3) = \frac{1}{|x_{13}|^{2 \Delta_3}} \frac{1}{w^2} J_{\mu\nu}(w) I_\nu(w' - x'_{13}), \quad (B.4) \]

where \( I_\nu(w) \) is given by the integral

\[ I_\mu(w) = \int \frac{d^{d+1}z'}{z_0^{d+1}} G_{\mu'}(w', z) z_0^{\Delta_1} \nabla_{\nu'} \left( \frac{z_0}{z'^2} \right)^{\Delta_3}. \quad (B.5) \]

Based on the \( d \)-dimensional Poincaré symmetry, as well as scaling symmetry, we suggest the following ansatz for the integral:

\[ I_\mu(w) = w_0^{\Delta_{13}} \frac{w_{13}}{w^2} f(t) + w_0^{\Delta_{13}} \frac{\delta_{\mu0}}{w_0} h(t), \quad (B.6) \]

where \( t = w_0^2 / w^2 \), and \( \Delta_{13} = \Delta_1 - \Delta_3 \).

We now apply the Maxwell operator in equation (B.3) to the ansatz. To get a feeling for these kind of calculations we first consider the case \( \Delta_1 = \Delta_3 \). The term containing \( h(t) \) can then be dropped from the ansatz, because of current conservation.\(^2\) Substituting the ansatz we get

\[ \nabla^\mu \left[ \partial_\mu \left( \frac{w_\rho}{w^2} f(t) \right) - \partial_\rho \left( \frac{w_\mu}{w^2} f(t) \right) \right], \quad (B.7) \]

where we do not write the Christoffel symbols from the covariant derivative, since they cancel each other, because of the symmetry in the two lower indices of the Christoffel symbol. Consider first the part between square brackets.

\(^2\)For the full argument, see [41].
The part proportional to $f(t)$ is
\[
\left( \partial_\mu \frac{w_\rho}{w^2} - \partial_\rho \frac{w_\mu}{w^2} \right) f(t) = \left( \frac{\delta_\mu}{w^2} + w_\mu \partial_\mu \frac{1}{w^2} - \frac{\delta_\mu w_\mu}{w^2} \right) f(t) = \left( - \frac{2w_\rho w_\mu}{(w^2)^2} + \frac{2w_\mu w_\rho}{(w^2)^2} \right) f(t) = 0.
\]

For the part proportional to $f'(t)$, note that $\partial_\mu f(t) = (\partial_\mu t) f'(t)$, where $\partial_\mu t = -\frac{2w_\rho w_\mu}{(w^2)^2} + \frac{2w_\mu w_\rho}{w^2}$. Using this formulas gives
\[
\frac{w_\rho}{w^2} \partial_\mu f(t) - \frac{w_\mu}{w^2} \partial_\rho f(t) = - \frac{2w_\rho w_\mu w_0^2}{(w^2)^3} f'(t) + \frac{2w_\rho w_\mu \delta_\mu \delta_\rho}{(w^2)^2} f'(t)
\]
\[
+ \frac{2w_\rho w_\mu \delta_\mu \delta_\rho}{(w^2)^3} f'(t) - \frac{2w_\mu w_\rho \delta_\mu \delta_\rho}{(w^2)^2} f'(t)
\]
\[
= \left( \frac{2w_\mu w_\rho \delta_\mu \delta_\rho}{(w^2)^2} - \frac{2w_\rho w_\mu \delta_\mu \delta_\rho}{(w^2)^2} \right) f'(t) = H_{\mu\rho},
\]
where we introduce the tensor $H_{\mu\rho}$ for convenience. With this calculation, equation (B.7) reduces to
\[
\nabla^\mu H_{\mu\rho} = g^{\sigma\mu} \nabla_\sigma H_{\mu\rho}
\]
\[
= g^{\sigma\mu} \left( \partial_\sigma H_{\mu\rho} - \Gamma^\tau_{\mu\sigma} H_{\tau\rho} - \Gamma^\tau_{\rho\sigma} H_{\mu\tau} \right)
\]
\[
= w_0^2 \left( \partial_\mu H_{\mu\rho} - \Gamma^\tau_{\mu\rho} H_{\tau\rho} - \Gamma^\tau_{\rho\mu} H_{\mu\tau} \right).
\]

Now consider the terms involving the Christoffel symbols. A straightforward computation gives
\[
\Gamma^\tau_{\mu\mu} H_{\tau\rho} = \frac{1}{w_0} \left( \delta_0^\tau \delta_{\mu\mu} - 2 \delta_{0\mu} \delta_0^\tau \right) H_{\tau\rho} = \frac{d-1}{w_0} H_{0\rho},
\]
and
\[
\Gamma^\tau_{\rho\mu} H_{\mu\tau} = \frac{1}{w_0} \left( \delta_0^\tau \delta_{\rho\mu} - \delta_{0\rho} \delta_\mu^\tau - \delta_{\mu0} \delta_0^\tau H_{\mu\rho} \right)
\]
\[
= \frac{1}{w_0} H_{\rho0} - \frac{1}{w_0} \delta_{0\rho} H_{\mu\mu} - \frac{1}{w_0} H_{0\rho}
\]
\[
= - \frac{2}{w_0} H_{0\rho},
\]
where in the last line the anti-symmetry of $H_{\mu\nu}$ in the indices $\mu$ and $\nu$ is used.

For the remaining $\partial_\rho H_{\mu\rho}$ term, again consider the $f'(t)$ and $f''(t)$ terms seper-
ately. The $f''(t)$ term is given by

$$\left( 2w_0 w_\mu \delta_{\mu 0} - \frac{2w_0 w_\mu \delta_{\mu 0}}{(w^2)^2} \right) \partial_\mu f'(t)$$

$$= \left( \frac{2w_0 w_\mu \delta_{\mu 0}}{(w^2)^2} - \frac{2w_0 w_\mu \delta_{\mu 0}}{(w^2)^2} \right) \left( - \frac{2w_0^2 w_\mu}{w^2} + \frac{2w_0 \delta_{\mu 0}}{w^2} \right) f''(t)$$

$$= \left( -4w_0^2 w_\mu w_\nu \delta_{\mu 0} + \frac{4w_0^3 w_\mu w_\mu \delta_{\mu 0}}{(w^2)^4} + \frac{4w_0^2 w_\mu \delta_{\mu 0}}{(w^2)^4} - 4 \frac{w_0^2 w_\mu \delta_{\mu 0}}{(w^2)^4} \right) f''(t)$$

$$= \left( -4w_0^2 w_\mu + \frac{w_0^3 w_\mu}{(w^2)^4} \right) f''(t).$$

And finally, for the term proportional to $f'(t)$, the result is

$$\left( \partial_\mu \frac{2w_0 w_\mu \delta_{\mu 0}}{(w^2)^2} \right) f'(t)$$

$$= \left( \frac{2w_\mu}{(w^2)^2} - \frac{8w_0^2 w_\mu}{(w^2)^3} + \frac{2w_0 \delta_{\mu 0}}{w^2} \right) f'(t)$$

$$= \left( -2(d+1) \frac{w_0 \delta_{\mu 0}}{(w^2)^2} + \frac{8w_0 w_\mu w_\mu \delta_{\mu 0}}{(w^2)^3} \right) f'(t)$$

$$= \left( -2w_\mu - \frac{8w_0^2 w_\mu}{(w^2)^3} - 2(d-3) \frac{w_0 \delta_{\mu 0}}{(w^2)^2} \right) f'(t).$$

Now that we have calculated the individual terms, we can add them together, and identify factors of $t$. If we take into account the factor $w_0^2$ coming from the metric, and the overall minus factor in (B.3), we get

$$4t^2 (t - 1) \frac{w_\mu}{w^2} f''(t) + 4t^2 \frac{w_\mu}{w^2} f'(t).$$

We now return to our original problem. First we apply the wave operator to the integral $I_\rho(w)$. We can use equation (B.3) under the integral sign in (B.5), to get

$$\int \frac{d^{d+1}z}{z_0^{d+1}} g_\rho \partial_\nu \delta(w, z) \Delta_1 w_0 \Delta_3 \frac{z_0}{z^2} \left( \frac{z_0}{z^2} \right)^3$$

$$= \int \frac{d^{d+1}z}{z_0^{d+1}} \delta_\rho \frac{\partial_\nu}{z_0} \delta(w, z) \left( \frac{z_0}{z^2} \Delta_1 w_0 \Delta_3 \frac{z_0}{z^2} \left( \frac{z_0}{z^2} \right)^3 \right)$$

$$= -\delta_\rho \left( \Delta_1 w_0 \Delta_1 \Delta_3 - \Delta_3 w_0 \Delta_1 \right) - \Delta_3 w_0 \Delta_1 \frac{w_\mu}{w^2} \Delta_3$$

$$= -2 \Delta_3 w_0 \Delta_1 \frac{w_\mu}{w^2} \Delta_3 - \frac{\delta_\rho}{w_0} \Delta_1 \Delta_3 w_0 \Delta_3$$

where in the last line we identified the factors of $t$. Note that for $\Delta_1 = \Delta_3$, this equation indeed reduces to the one for reversed currents.
The next step is to calculate the action of the Maxwell operator on our more general ansatz (B.6). First consider the term involving \( h(t) \). The part in square brackets in (B.7) becomes

\[
\partial_\mu \left( \delta_\rho_0 w_0^{\Delta_{13} - 1} h(t) \right) - \partial_\rho \left( \delta_\rho_0 w_0^{\Delta_{13} - 1} h(t) \right) = \delta_\rho_0 w_0^{\Delta_{13} - 1} \left( \frac{2w_0^2 w_\rho}{(w^2)^2} + \frac{2w_0 \delta_\rho_0}{(w^2)} \right) h'(t)
\]

\[
- \delta_\rho_0 w_0^{\Delta_{13} - 1} \left( \frac{2w_0^2 w_\rho}{(w^2)^2} + \frac{2w_0 \delta_\rho_0}{(w^2)} \right) h'(t)
\]

\[
= w_0^{\Delta_{13}} \left( \frac{2w_0 w_\rho \delta_\rho_0}{(w^2)^2} - \frac{2w_0 w_\rho \delta_\rho_0}{(w^2)^2} \right) h'(t).
\]

But this looks just like \( w_0^{\Delta_{13}} H_{\mu\rho} \), only with \( f \) replaced by \( h \). The only extra contribution we get is

\[
-w_0^2 (\partial_\mu w_0^{\Delta_{13}}) \left( \frac{2w_0 w_\rho \delta_\rho_0}{(w^2)^2} - \frac{2w_0 w_\rho \delta_\rho_0}{(w^2)^2} \right) h'(t)
\]

\[
= -\Delta_{13} w_0^{\Delta_{13} + 1} \delta_\rho_0 \left( \frac{2w_0 w_\rho \delta_\rho_0}{(w^2)^2} - \frac{2w_0 w_\rho \delta_\rho_0}{(w^2)^2} \right) h'(t)
\]

\[
= -2\Delta_{13} w_0^{\Delta_{13}} w_\rho h'(t) + 2\Delta_{13} w_0^{\Delta_{13}} t^2 \delta_\rho_0 h'(t).
\]

So the total contribution of the Maxwell operator on the term involving \( h(t) \) is

\[
w_0^{\Delta_{13}} \frac{w_\rho}{w^2} \left( 4t^2(t - 1) h''(t) + 4t \left( 2t + \frac{d - 4 - \Delta_{13}}{2} \right) h'(t) \right) + 2\Delta_{13} w_0^{\Delta_{13} + 1} \delta_\rho_0 w_\rho h'(t).
\]

(B.10)

Finally, we consider the action of the Maxwell operator on \( w_0^{\Delta_{13}} \frac{w_\rho}{w^2} f(t) \). First note

\[
\partial_\mu \left( w_0^{\Delta_{13}} \frac{w_\rho}{w^2} f(t) \right) - \partial_\rho \left( w_0^{\Delta_{13}} \frac{w_\rho}{w^2} f(t) \right) = w_0^{\Delta_{13}} H_{\mu\rho}
\]

\[
+ \Delta_{13} w_0^{\Delta_{13} + 1} \left( \delta_\rho_0 \frac{w_\rho}{w^2} f(t) - \delta_\rho_0 \frac{w_\rho}{w^2} f(t) \right).
\]

The first part we have already calculated. For convenience, denote the terms in the second line by \( K_{\mu\rho} \). As before, we first calculate the contribution coming from the terms with the Christoffel symbols. Note that we already include the overall minus sign, and the contribution coming from the metric in. This gives

\[
w_0^{\Delta_{13}} \Gamma^\tau_{\mu\rho} K_{\tau\rho} = w_0(d - 1) K_{\mu\rho}
\]

\[
= \Delta_{13} w_0^{\Delta_{13}} (d - 1) \left[ \frac{w_\rho}{w^2} - \delta_\rho_0 \frac{w_\rho}{w^2} \right] f(t)
\]

\[
= \Delta_{13} w_0^{\Delta_{13}} (d - 1) \frac{w_\rho}{w^2} f(t) - \Delta_{13} w_0^{\Delta_{13}} (d - 1) \frac{\delta_\rho_0}{w_0} t f(t),
\]

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And also
\[ w_0^2 \Gamma_{\mu \rho} K_{\rho \mu} = w_0 K_{\rho \rho} - w_0 \delta_{\rho \mu} K_{\mu \mu} - w_0 K_{0 \rho} \]
\[ = -2 \Delta_{13} w_0^{\Delta_{13}} \left[ \frac{w_\mu}{w^2} - \delta_{\rho 0} \frac{w_\rho}{w^2} \right] f(t) \]
\[ = -2 \Delta_{13} w_0^{\Delta_{13}} \frac{w_\rho}{w^2} f(t) + 2 \Delta_{13} w_0^{\Delta_{13}} \frac{\delta_{\rho 0}}{w_0} t f(t). \]

Finally, the derivative term gives a contribution of
\[ -\frac{w_0^2}{w^{\Delta_{13}}} \partial_\mu K_{\mu \rho} = -\Delta_{13} (\Delta_{13} - 1) \frac{w_\rho w^2}{(w^2)^2} f(t) - \Delta_{13} (\Delta_{13} - 1) \frac{\delta_{\rho 0} w_0}{w^2} f(t) - \Delta_{13} \frac{\delta_{\rho 0} w_0}{w^2} f(t) \]
\[ + 2 \Delta_{13} \frac{w_\rho w_0^2}{(w^2)^2} f(t) + 2 \Delta_{13} \frac{w_0^4 w_\rho}{(w^2)^3} f'(t) \]
\[ - 2 \Delta_{13} \frac{w_\rho w_0^2}{(w^2)^2} f'(t) + \Delta_{13} (d - 1) \frac{\delta_{\rho 0} w_0}{w^2} f(t) \]
\[ = -\frac{w_\rho}{w^2} \left( (\Delta_{13} (\Delta_{13} - 1) - 2 \Delta_{13} t) f(t) - 2 \Delta_{13} t (t - 1) f'(t) \right) \]
\[ + \Delta_{13} (d - 3 + \Delta_{13}) \frac{\delta_{\rho 0}}{w_0} t f(t). \]

Adding all the terms together we arrive at
\[ w_0^{\Delta_{13}} \frac{w_\rho}{w^2} \left( 4 t^2 (t - 1) f''(t) + 2 t (d - 2 (2 + \Delta_{13}) + t (4 + \Delta_{13})) f'(t) \right) \]
\[ + \Delta_{13} (d - 2 - \Delta_{13} + 2 t) f(t) \]
\[ + w_0^{\Delta_{13}} \frac{\delta_{\rho 0}}{w_0} \left( 2 \Delta_{13} t^2 f'(t) + \Delta_{13}^2 t^3 f(t) \right) \]
\[ (B.11) \]

To get the differential equations, we now combine equations (B.3), (B.9), (B.10) and (B.11). We see that there are two independent structures: one multiplying \( \frac{w_\rho}{w^2} \), and one multiplying \( \frac{\delta_{\rho 0}}{w_0} \) on each side of the equal sign. Equating these terms, and dividing out common factors, we arrive at the following coupled system of differential equations:
\[ \Delta_{13}^2 t f(t) + 2 \Delta_{13} t^2 f'(t) + 2 \Delta_{13} t^2 h'(t) + m^2 h(t) = -\Delta_{13} t^{\Delta_3} \]
\[ 4 t^2 (t - 1) (f''(t) + h''(t)) + 4 t (2 t + \frac{d - 4 - \Delta_{13}}{2}) h'(t) + m^2 f(t) \]
\[ + 2 t (d - 2 (2 + \Delta_{13}) + t (4 + \Delta_{13})) f'(t) + \Delta_{13} (d - 2 - \Delta_{13} + 2 t) f(t) \]
\[ = -2 \Delta_{3} t^{\Delta_3} \]
\[ (B.12) \]

As a consistency check, consider \( \Delta_{13} = 0 \). From (B.12) it follows that \( h(t) = 0 \), since \( m^2 \neq 0 \). In this case, (B.13) reduces to the differential equation that was obtained in [41].

To find the solution, we first assume a power series expansion for \( f \) and \( h \):
\[ f(t) = \sum_k a_k t^k, \quad h(t) = \sum_k b_k t^k. \]

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If we plug this in (B.12), we can solve for $f(t)$ in terms of the coefficients $b_k$. To do this, we assume from now on that $\Delta_{13} \neq 0$. Doing this results in

$$f(t) = -\frac{t^{\Delta_{13}-1}}{\Delta_{13} + 2\Delta_{3} - 2} - \sum_{k} \frac{2\Delta_{13}kb_k + m^2b_{k+1}}{\Delta_{13}^2 + 2\Delta_{13}k} t^k. \quad (B.14)$$

Note that the series for $f(t)$ terminate if and only if $b_k = 0$ for all but a finite number of $k$. We now plug the solution for $f(t)$ in equation (B.13). The contribution of $-\frac{t^{\Delta_{13}-1}}{\Delta_{13} + 2\Delta_{3} - 2}$ can be easily calculated to be

$$-2\Delta_{3}t^{\Delta_{3}} - t^{\Delta_{3}-1} \left( d - \Delta_{13} - 2\Delta_{3} + \frac{m^2}{\Delta_{13} + 2\Delta_{3} - 2} \right).$$

Substituting the power series for $h(t)$ and the remaining part of $f(t)$, equation (B.13) then reduces to

$$\sum_{k} \left[ \frac{-m^2(4k(k + \Delta_{13})b_k}{\Delta_{13}(2k + \Delta_{13})} \right. + \frac{m^2(4k^2 - m^2 + \Delta_{13}(2 - d + \Delta_{13}) + 2k(2 - d + 2\Delta_{13})b_{k+1}}{\Delta_{13}(2k + \Delta_{13})} \left. \right] t^k = \frac{t^{\Delta_{3}-1}}{\Delta_{13} + 2\Delta_{3} - 2} \left( d - \Delta_{13} - 2\Delta_{3} + \frac{m^2}{\Delta_{13} + 2\Delta_{3} - 2} \right).$$

This equation gives a recursion relation that works downwards in $k$ for the coefficients $b_k$. The coefficients are given by

$$b_k = 0 \quad \text{if} \quad k \geq \Delta_{3} \quad (B.15)$$

$$b_{\Delta_{3}-1} = -\frac{\Delta_{13}(m^2 + (\Delta_{1} + \Delta_{3} - 2)(d - \Delta_{1} - \Delta_{3}))}{4m^2(\Delta_{1} - 1)(\Delta_{3} - 1)} \quad (B.16)$$

$$b_k = \frac{-4k^2 + m^2 + 2k(d - 2\Delta_{13} - 2) + (d - 2 - \Delta_{13})\Delta_{13}b_{k+1}}{4k(k + \Delta_{13})} \quad (B.17)$$

Let us now specialise to the case of interest for us, where $d = 4$. In this case the mass of the vector is $m^2 = l^2 - 1$, where the vector transforms according to the $[1, l - 1, 1]$ irrep of SO(6). In this case, the series terminate if $k = \frac{1}{2}(1 \pm l - \Delta_{13})$. To get a well-defined series, we thus need $0 < k_{\min} = \frac{1}{2}(1 \pm l - \Delta_{13}) < \Delta_{3} - 1$. These conditions are indeed satisfied in the supergravity case, where we can choose $k_{\min} = \frac{1}{2}(1 + l - \Delta_{13})$.

First we note that if $\Delta_{1}$ and $\Delta_{3}$ are both even or both odd, $l$ has to be odd, according to the selection rules. If $\Delta_{1}$ is odd (even) and $\Delta_{3}$ is even (odd), $l$ has to be odd. Hence, $k_{\min}$ is always integer. From the criterium for the maximal value of $k_{\min}$, it follows that $l < \Delta_{1} + \Delta_{3} - 3$. From the selection rules, $l = 1, 3, \ldots, \Delta_{1} + \Delta_{3} - 1$. The only problems can occur when $l = \Delta_{1} + \Delta_{3} - 1$ or $l = \Delta_{1} + \Delta_{3} - 3$. In the former case, the supergravity coupling to the $A_\mu$ vector...
vanishes\footnote{This is another example where the couplings vanish in the extremal case}. In the other case, the coefficient $b_{\Delta_3} - 1$ is already zero, hence there is also no problem in this case. The lower bound requires that $l > \Delta_1 - \Delta_3 - 2$. If this condition is not satisfied, one can reverse the role of $\Delta_1$ and $\Delta_3$, at the expense of an additional minus sign, according to (B.5).

With the series expansion for $h(t)$, we can now find $f(t)$ from (B.14). To find the final contribution from the diagram, one has to transform the result back to the original coordinates, and write the remaining integral in terms of $D$-functions.

\section{Symmetric tensor exchange}

We will now use the same procedure to calculate the exchange diagram for a massive symmetric tensor, shown in Figure B.2. We do not restrict ourselves to couplings to conserved currents. The exchange diagram is then given by

$$G(x_1, x_2, x_3, x_4) = \int \frac{d^{d+1}w}{w_0^{d+1}} A^\mu\nu(w, x_1, x_3) T^\mu\nu(w, x_2, x_4), \quad (B.18)$$

where $A^\mu\nu$ is the $z$-integral given by

$$A^\mu\nu(w, x_1, x_3) = \int \frac{d^{d+1}z}{z_0^{d+1}} G_{\mu\nu\rho\sigma} T^{\rho\sigma}(z, x_1, x_3). \quad (B.19)$$

In this equation $G_{\mu\nu\rho\sigma}$ is the bulk-to-bulk propagator for the symmetric tensor, and $T^{\rho\sigma}$ is given by

$$T^{\rho\sigma}(z, x_1, x_3) = \nabla^\rho K_{\Delta_1}(x_1) \nabla^\sigma K_{\Delta_3}(x_3) - \frac{1}{2} g^{\rho\sigma} \left[ \nabla^\rho K_{\Delta_1}(x_1) \nabla^\rho K_{\Delta_3}(x_3) 
+ \frac{1}{2} (m_1^2 + m_3^2 - f) K_{\Delta_1}(x_1) K_{\Delta_3}(x_3) \right]. \quad (B.20)$$
where \( m_1 \) and \( m_3 \) are the masses of the two scalar fields, and \( f \) is the mass of the symmetric tensor, and the coordinate \( z \) is suppressed in the scalar propagators \( K_.\)

As before, we don’t need the explicit form of the bulk-to-bulk propagator (see [39]), but only the defining wave equation (see Appendix E of [24]):

\[
W_{\mu\nu}^{\rho\lambda} G_{\rho\lambda'\nu'}(w, z) = \left( g_{\mu\nu'} g_{\nu\lambda'} + g_{\mu\lambda'} g_{\nu'\nu} + \frac{2}{1 - d} g_{\mu\nu} g_{\nu'\nu'} \right) \delta(z, w),
\]

where \( W_{\mu\nu}^{\rho\lambda} \) is the modified Ricci operator, defined by

\[
W_{\mu\nu}^{\rho\lambda} \phi_{\rho\lambda} = -\nabla_{\rho} \nabla_{\lambda} \phi_{\rho\mu} + \nabla_{\mu} \nabla_{\rho} \phi_{\rho\lambda} + \nabla_{\nu} \nabla_{\rho} \phi_{\rho\mu} - \nabla_{\mu} \nabla_{\nu} \phi_{\rho}^\rho - \left( 2 - f \right) \phi_{\mu\nu} + \frac{6 - f}{1 - d} g_{\mu\nu} \phi^\rho_{\rho}. \tag{B.21}
\]

As before, we translate the point \( x_1 \) the boundary and apply conformal inversion. This gives

\[
A_{\mu\nu}(w, x_1, x_3) = \frac{1}{|x_{13}|^{2\Delta_3} (w^2)^2} J_{\mu\lambda}(w) J_{\nu\rho}(w) I_{\lambda\rho}(w' - x_{13}),
\]

where \( I_{\lambda\rho} \) is the tensor integral

\[
I_{\mu\nu}(w) = \int \frac{d^{d+1}z}{z_0} G_{\mu\nu\mu'} \bar{T}_{\nu'\nu'}(z). \tag{B.22}
\]

In this equation \( \bar{T} \) is the tensor acquired by using the conformal symmetries in (B.20), and we symmetrized with respect to \( \mu \) and \( \nu \):

\[
\bar{T}_{\mu\nu} = \frac{1}{2} \nabla_{\mu} \frac{z_0}{z_2} - \frac{\Delta_3}{2} \nabla_{\nu} + \frac{1}{2} \nabla_{\mu} \nabla_{\nu} \left( \frac{z_0}{z_2} \right) \left( \frac{z_0}{z_2} \right) + \frac{1}{2} \left( m_1^2 + m_2^2 - f \right) z_0 (\frac{z_0}{z_2}) \Delta_3.
\]

\[
= \frac{1}{2} \Delta_3 \frac{z_0}{z_2} \left( P_{\mu\nu} t \Delta_3 - \frac{P_{\mu\nu}}{z_2} t \Delta_3 \right) - \frac{1}{2} \frac{z_0}{z_2} \left( t \Delta_3 - 2 t \Delta_3 + 1 + \frac{1}{2} \left( m_1^2 + m_2^2 - f \right) t \Delta_3 \right), \tag{B.23}
\]

where in the last line we acted with the derivatives. We now use the wave equation under the integral sign in equation (B.22). To this end, note the following formulas

\[
\left( g_{\mu\nu'} g_{\nu'\nu} + g_{\mu\nu'\nu'} - \frac{2 g_{\mu\nu} g_{\nu'\nu'}}{d - 1} \right) g_{\mu'\nu'} = -\frac{4}{d - 1} g_{\mu\nu},
\]

\[
\left( g_{\mu'\nu'} g_{\nu'\nu} + g_{\mu'\nu'\nu'} - \frac{2 g_{\mu'\nu} g_{\nu'\nu'}}{d - 1} \right) P_{\mu'\nu'} = 2 P_{\mu'\nu'} - \frac{2}{d - 1} g_{\mu'\nu'},
\]

\[
\left( g_{\mu'\nu'} g_{\nu'\nu} + g_{\mu'\nu'\nu'} - \frac{2 g_{\mu'\nu} g_{\nu'\nu'}}{d - 1} \right) P^\nu_{\nu'\mu'} = 2 P^\nu_{\nu'\mu'} - \frac{4}{d - 1} g_{\mu'\nu'} z_0^2.
\]

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If we now plug equation (B.23) into (B.22), use the wave equation and integrate out the $\delta$, we find using these relations

$$w^\Delta_{13} \left( 2\Delta_1 \Delta_3 P_\mu P_\nu t^{\Delta_3} - 2\Delta_1 \Delta_3 \frac{P_\mu w_\nu}{w^2} t^{\Delta_3} + g_{\mu\nu} \frac{m^2 + m_3^2 - f}{d-1} t^{\Delta_3} \right).$$ (B.24)

Note that in the case where $\Delta_1 = \Delta_3$, this result indeed coincides with that found in the case of coupling to conserved currents.

The next step is to propose a suitable ansatz for the tensor integral $I_{\mu\nu}$. Based on the success in the massive vector exchange case, we modify the ansatz in [41] by multiplying it with a factor $w_0^{\Delta_{13}}$. Hence our ansatz is

$$I_{\mu\nu}(w) = w_0^{\Delta_{13}} g_{\mu\nu} h(t) + w_0^{\Delta_{13}} P_\mu P_\nu \phi(t) + w_0^{\Delta_{13}} \nabla_\mu \nabla_\nu X(t) + w_0^{\Delta_{13}} \nabla_\mu (P_\nu Y(t),$$ (B.25)

where $\{\ldots\}$ denotes symmerization in the indices, and $P_\mu = \delta_{0\mu}/w_0$. Note the following useful identities

$$P_\mu P^\mu = 1, \quad \nabla_\mu P_\nu = -g_{\mu\nu} + P_\mu P_\nu, \quad \nabla^\sigma P_\sigma = -d.$$ (B.26)

The biggest task is now to calculate the action of the modified Ricci operator on the ansatz. To this end, it is useful to note the following formula, for $H$ a general scalar, vector or tensor function:

$$\nabla_\mu \nabla_\nu w_0^{\Delta_{13}} H = w_0^{\Delta_{13}} (\Delta_{13}^2 + \Delta_{13}) P_\mu P_\nu H - g_{\mu\nu} \Delta_{13} w_0^{\Delta_{13}} H$$

$$+ \Delta_{13} w_0^{\Delta_{13}} P_\nu \nabla_\mu H + w_0^{\Delta_{13}} \nabla_\mu \nabla_\nu H,$$ (B.27)

which can be easily obtained by using the Leibniz rule repeatedly, and applying (B.26). From this formula it follows that

$$W_{\mu\nu}^{\rho\lambda}[w_0^{\Delta_{13}} H_{\rho\lambda}] = w_0^{\Delta_{13}} \tilde{W}_{\mu\nu}^{\rho\lambda}[H_{\rho\lambda}] + w_0^{\Delta_{13}} W_{\mu\nu}^{\rho\lambda}[H_{\rho\lambda}],$$ (B.28)

where $\tilde{W}$ represent the extra contributions (in comparison with the case $\Delta_{13} = 0$), proportional to $\Delta_{13}$ and $\Delta_{13}^2$. The last term has already been calculated in [24], so we will concentrate on the extra contributions.

Calculating the extra contributions is straightforward, but very cumbersome. One can use (B.27) repeatedly for each derivative term in the modified Ricci operator. Another identity we frequently use is

$$P_\mu \nabla_\nu f(t) = P_\mu \left( 2 t P_\nu - \frac{2 w_0^2 w_\nu}{(w^2)^2} \right) f'(t) = 2 P_\mu P_\nu t f'(t) - 2 t \frac{P_\mu w_\nu}{(w^2)^2} f'(t).$$

From this equation it’s easy to find

$$P^\sigma \nabla_\sigma f(t) = P^\sigma \left( 2 t P_\sigma - \frac{2 w_0^2 w_\sigma}{(w^2)^2} \right) f'(t) = -2 t (t-1) f'(t).$$

We now calculate the extra contributions $W_{\mu\nu}^{\rho\lambda}[w_0^{\Delta_{13}} H_{\rho\lambda}]$. For $H_{\rho\lambda} = g_{\rho\lambda} h(t)$, calculations can be simplified a bit by noting that the metric commutes with
covariant derivatives. Omitting the tedious calculations, we find that the final result for the extra contribution is
\[
g_{\mu\nu} \left( -\Delta_{13}^2 + \Delta_{13}(2d - 1)h(t) + 4\Delta_{13}t(t - 1)h'(t) \right) \\
+ P_{\mu}P_{\nu} \left( (\Delta_{13}^2 + \Delta_{13})(1 - d)h(t) + 4t\Delta_{13}(1 - d)h'(t) \right) \\
- 2t\Delta_{13}(1 - d)P_{\mu}\frac{w_{\nu}}{w^2}h'(t).
\]

This is multiplied by the factor \( w^{\Delta_{13}}_0 \) in the rhs of equation (B.28).

The extra contribution for \( w^{\Delta_{13}}_0 P_{\mu}P_{\nu}\phi(t) \) can be found in the same manner. This is basically a matter of repeatedly applying equations (B.26). The result is
\[
-\Delta_{13}P_{\mu}P_{\nu}(d - 1)\phi(t) - \Delta_{13}g_{\mu\nu}\phi(t),
\]
multiplied by an overall factor of \( w^{\Delta_{13}}_0 \).

In calculating the extra contribution from \( w^{\Delta_{13}}_0 \nabla_\mu \nabla_\nu X \), one should be a bit careful, and take into account the Riemann tensor contribution when commuting variables (c.f. equation (4.8)). The result is
\[
\nabla_\mu \nabla_\nu X (-\Delta_{13}^2 + d\Delta_{13} - 2\Delta_{13}) + \Delta_{13}g_{\mu\nu}(4t^2(t - 1)X'' - 2dtX') \\
+ P_{\mu}P_{\nu}(-4(\Delta_{13}^2 + \Delta) t^2(t - 1)X'' + 2\Delta_{13}td(d(\Delta_{13} - 1) + 2(2 + \Delta_{13} - 2t(1 + \Delta_{13})))X') \\
+ \frac{w_{\mu}P_{\nu}}{w^2}(4t^2(t - 1)(\Delta_{13}^2 + \Delta_{13})X'' + (2\Delta_{13}^2(4t^2 - 3t) + 2\Delta_{13}(4t - 4 + d))X')
\]

again multiplied by an overall factor of \( w^{\Delta_{13}}_0 \).

For the extra contribution of the term \( w^{\Delta_{13}}_0 \nabla_{\{\mu}P_{\nu\}}Y(t) \), note that
\[
\nabla_{\{\mu}P_{\nu\}}Y(t) = -2g_{\mu\nu}Y(t) + 2P_{\mu}P_{\nu}Y(t) + P_{\{\mu}\nabla_{\nu\}}Y(t). \tag{B.29}
\]

If we now multiply this by \( w^{\Delta_{13}}_0 \), we see that we have already calculated the extra contribution from the first two terms. For the remaining term, first note that
\[
W_{\mu\nu}^{\lambda\sigma}[w^{\Delta_{13}}_0 (P_{\mu} \nabla_\nu Y + P_{\nu} \nabla_\mu Y)] = -\nabla_\sigma [w^{\Delta_{13}}_0 (P_{\mu} \nabla_\lambda Y + P_{\lambda} \nabla_\gamma Y)] \\
- 2\nabla_\mu \nabla_\nu [w^{\Delta_{13}}_0 P_\sigma \nabla_\sigma Y] \\
+ \nabla_\mu \nabla_\nu [w^{\Delta_{13}}_0 (P_{\mu} \nabla_\nu Y + P_{\nu} \nabla_\mu Y)] \\
+ \nabla_\nu \nabla_\sigma [w^{\Delta_{13}}_0 (P_{\sigma} \nabla_\mu Y + P_{\mu} \nabla_\sigma Y)] + (..), \tag{B.30}
\]
where (..) denotes the terms without derivatives. These terms can then be straightforwardly found by the same procedure as for the other terms. Adding everything together, we find that the final extra contribution for the \( w^{\Delta_{13}}_0 \nabla_{\{\mu}P_{\nu\}}Y(t) \) term is
\[
2\Delta_{13} \nabla_\mu \nabla_\nu Y + g_{\mu\nu} \left[ (2\Delta_{13}^2 - 4\Delta_{13}d)Y - 8\Delta_{13}t(t - 1)Y' \right] \\
+ P_{\mu}P_{\nu} \left[ 8\Delta_{13}t^2(t - 1)Y'' + 4\Delta_{13}td(d - 4 + 4t)Y' + 2\Delta_{13}^2(d - 1)Y \right] \\
\frac{P_{\{\mu}w_{\nu\}}}{w^2} \left( -4\Delta_{13}t(4t + d - 4)Y' - 8\Delta_{13}t^2(t - 1)Y'' \right),
\]

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We can simplify this by adding equation (B.31), to arrive at
\[\nabla_{\mu} \nabla_{\nu}(-3h - \phi + (f - \Delta_{13}^2 + d\Delta_{13} - 2\Delta_{13})X + 2\Delta_{13}Y) = 0\]

For the tensor \(g_{\mu\nu}\) we get
\[4t^2(t-1)h'' + 4t((t+1) + \Delta_{13}(t-1))h' + \left(\frac{8}{3}(f + 3) - \Delta_{13}(\Delta_{13} + 1 - 2d)\right)h\]
\[-\left(-\frac{f}{3} + \Delta_{13}\right)4t^2(t-1)X'' - \left(\frac{4f}{3}t + \frac{1}{2} + 2d\Delta_{13}t\right)X'\]
\[-\left(-\frac{f}{3} + 2\Delta_{13}\right)4t(t-1)Y' + \left(-\frac{14f}{3} + 2\Delta_{13}(\Delta_{13} - 2d)\right)Y\]
\[+4t(t-1)\phi' + \frac{f + 24 - 3\Delta_{13}}{3} = \frac{m_1^2 + m_2^2 - f_3\Delta_{13}}{d - 1}\]

For the tensor structure \(P_{\mu\nu\omega}\) this gives
\[2\Delta_{13}(d-1)h' + 4t(t-1)\phi'' + 8t\phi'\]
\[+(\Delta_{13}^2 + \Delta_{13})4t(t-1)X'' + (2\Delta_{13}^2(4t - 3) + 2\Delta_{13}(4t - 4 + d)\Delta_{13})Y'\]
\[-8\Delta_{13}t(t-1)Y'' - 2(f + 2\Delta_{13}(4t + d - 4))Y' = -2\Delta_{13}\Delta_{13}t\Delta_{13} - 1.\]

This equation is trivially integrated, and gives
\[2\Delta_{13}(d-1)h + 4(\Delta_{13}^2 + \Delta_{13})t(t-1)X' - 2\Delta_{13}(\Delta_{13} + 2 - d)X\]
\[+4t(t-1)\phi' + 4\phi - 8\Delta_{13}t(t-1)Y' + 2(-f + 4\Delta_{13} - 2d\Delta_{13})Y = -2\Delta_{13}\Delta_{13} + c_1,\]

where \(c_1\) is an arbitrary integration constant.

Finally, equating the \(P_{\mu}P_{\nu}\) tensor structures,
\[-(\Delta_{13}^2 + \Delta_{13})(d-1)h - 4t^2(t-1)\phi'' - 8t\phi' + (f - \Delta_{13}(d-1))\phi\]
\[-4(\Delta^2 + \Delta)t^2(t-1)X'' + 2\Delta_{13}t(d(\Delta_{13} - 1) + 2(2 + \Delta_{13} - 2t(\Delta_{13})))X'\]
\[8\Delta_{13}t^2(t-1)Y'' + 4t(f + \Delta_{13}(d - 4 + 4t))Y' + 2(f + \Delta_{13}^2(d - 1))Y\]
\[-4\Delta_{13}(d-1)t\phi' = 2\Delta_{13}\Delta_{13}t\Delta_{13}\]
(B.32)

We can simplify this by adding equation (B.31), to arrive at
\[-2t\Delta_{13}(d-1)h' - (\Delta_{13}^2 + \Delta_{13})(d-1)h + (f - \Delta_{13}(d-1))\phi\]
\[2\Delta_{13}t(d-1)X' + 2tfY' + 2(f + \Delta_{13}^2(d - 1))Y = 0.\]
These four differential equations of four unknown functions can be used to compute the most general exchange integrals of massive tensors.

Given the complicated expressions we found, we do not attempt to give a general solution, but we specialise to our case of interest, where $\Delta_1 = 3$, $\Delta_2 = 2$ and $f = 5$. In this case, $m_1^2 = -3$ and $m_2^2 = -4$, and $\Delta_{13} = 1$. The differential equations then reduce to

\[
\nabla_\mu \nabla_\nu (-3h - \phi + 6X + 2Y) = 0 \quad (B.33)
\]
\[
-6h - 6t h' + 16Y + 10t Y' + 2\phi + 6t X' = 0 \quad (B.34)
\]
\[
3h + X - 9Y + 2\phi + 4t(t - 1)X' - 4t(t - 1)Y' + 2t(t - 1)\phi' = -3t^2 + \frac{c_1}{2} \quad (B.35)
\]
\[
41h - 56Y + 13\phi + 12t^2 h' - 2t(5t + 11)X' - 22t(t - 1)Y' + 6t(t - 1)\phi' + 6t^2(t - 1)h'' - 16t^2(t - 1)X'' = -6t^2 \quad (B.36)
\]

For equation (B.33) we pick up the trivial solution

\[
h = -\frac{\phi}{3} + 2X + \frac{2}{3}Y, \quad (B.37)
\]

to avoid singularities appearing at $t = 0$ and $t = 1$. We now substitute this solution in (B.34). This results in

\[
4(\phi - 3X + 3Y) + 2t(\phi' - 3X' + 3Y') = 0,
\]

which reduces to $2g + tg' = 0$, if we define $g(t) = \phi(t) - 3X(t) + 3Y(t)$. The solution of this differential equation is $g(t) = c_2 t^{-2}$, where $c_2$ is an arbitrary constant. Hence, $\phi = 3(X - Y) + c_2 t^{-2}$. With these solutions for $h$ and $\phi$, equation (B.35) becomes

\[
10t^2 \theta + 10t^3(t - 1)\theta' = -3t^4 + \frac{c_1 t^2}{2} + c_2 (4t - 5),
\]

where we defined $\theta = X - Y$. By requiring that $\theta(t)$ is regular for $t \to 0$ and $t \to 1$, we can fix the two new constants appearing in the solution (these appear as the two homogeneous solutions of the second-order equation), fix $c_2 = 0$, and then find

\[
\theta(t) = \frac{c_1 - 6t}{20},
\]

and hence $Y = X + \frac{6t - c_1}{20}$. With the solutions we found so far, equation (B.36) reduces to

\[
X = -\frac{3t + c_1}{40}.
\]

The other functions are then given by

\[
Y = \frac{9t - 3c_1}{40}, \quad h = \frac{6t - 3c_1}{20}, \quad \phi = \frac{-18t + 3c_1}{20}.
\]

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If we substitute these results in the ansatz (B.25), the integration constant \( c_1 \) cancels as expected, and we get

\[
I_{\mu\nu}(t) = \frac{3w_0}{40} \left( 4tg_{\mu\nu} - 12tP_{\mu}P_{\nu} - \nabla_{\mu}\nabla_{\nu}t + 3\nabla_{\{\mu}P_{\nu\}}t \right)
\]

The next step is to calculate the action of the covariant derivatives. We get

\[
\nabla_{\mu}\nabla_{\nu}t = 6tP_{\mu}P_{\nu} - 6t\frac{P_{\mu w_{\nu}}}{w^2} - 2tg_{\mu\nu} + \frac{8tw_{\mu}w_{\nu}}{(w^2)^2},
\]

and for the remaining term

\[
\nabla_{\{\mu}P_{\nu\}}t = -2tg_{\mu\nu} + 6tP_{\mu}P_{\nu} - 2t\frac{P_{\mu w_{\nu} w_{\nu}}}{w^2}.
\]

Hence, the exchange integral is

\[
I_{\mu\nu}(t) = -\frac{3w_0t}{5} \frac{w_{\mu}w_{\nu}}{(w^2)^2}
\]

(\( B.38 \))

The exchange diagram can then be calculated by transforming back to the original coordinates, and do the \( w \)-integral to find the contribution in terms of \( D \)-functions.
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