Abstract

Event shape variables in $e^+e^-$ annihilation serve as an instrument to explore the interface between perturbative QCD and non-perturbative effects. Due to soft and collinear gluon emissions, perturbative series suffer from enhanced logarithmic coefficients. Another contribution to large coefficients are infrared renormalons that lead to non-perturbative power corrections. Dressed gluon exponentiation is applied to resum the large logarithms and the renormalons at the same time. Ambiguities in the result provide us with information on the shape of power corrections. The obtained descriptions of event shapes are confronted with the data.
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Chapter 1

Introduction

Event shape variables are very suitable observables to test and improve our understanding of QCD. For example, comparison of theoretical predictions of their distributions with the available data has revealed that gluons are vector particles. Furthermore, the value of the strong coupling can be deduced from the data by comparison with perturbative calculations. Recently however, the emphasis is on using the sensitivity of event shape distributions to long-distance effects to extend our knowledge of QCD somewhat into the non-perturbative regime. This is where the power corrections arise.

In this thesis we study event shapes of final states in $e^+e^-$ annihilation. In Chapter 2 the cross section for creation of a quark-antiquark pair is calculated up to first order in $\alpha_s$ to get acquainted with this process. Subsequently some event shape variables describing the geometrical aspects of the final state are defined in Chapter 3. We focus on event shapes that are small in the 2-jet limit, where soft and collinear gluons are emitted. As we will see in Chapter 4, the event shape distribution in this region is large due to logarithmic enhancements of the perturbative coefficients. Consequently, the series cannot be truncated at fixed order. In order to improve the validity of perturbation theory in this region, the large logarithms have to be resummed to all orders in the strong coupling. This is achieved with the help of the exponentiation property of event shape distributions, which is a consequence of the factorization of soft and collinear emissions from the hard process.

However, the results disagree with the measurements since final states in detectors consist of hadrons instead of the free quarks and gluons that are described by the perturbative theory. The hadronization effects behave as power corrections $(\Lambda/Q)^p$, where $Q$ is the center-of-momentum energy, and contribute significantly for most event shapes. Even though these effects cannot be described perturbatively, information on their contributions can be extracted from resummed perturbation theory.

A known source of disturbance of perturbative QCD are renormalons; diagrams containing chains of fermionic loops. Because of increasing sensitivity to ultraviolet as well as infrared momenta, the perturbative coefficients grow factorially and need to be resummed using the Borel transform. Borel summation of infrared renormalons leads to ambiguities in the result, which have to be cancelled on the non-perturbative level by power corrections.

The above-mentioned resummation of large logarithms is usually performed up to NLL (next-to-leading-logarithmic) order. However, due to renormalons, the coefficients of the subleading logarithms are factorially enhanced and need to be reorganized by Borel summation. Dressed gluon exponentiation (Chapter 5) is a method to resum both the large logarithms and the infrared renormalons. Useful information on the functional form of the non-perturbative effects is obtained this way.
Conventions

- Units in which $\hbar = c = 1$ are used
- The metric
  \[ g_{\mu\nu} = \text{Diag}\{1, -1, -1, -1\} \]
  is used throughout this thesis, such that the 4-vector dot product is
  \[ p^2 = g_{\mu\nu} p^\mu p^\nu = E^2 - \mathbf{p} \cdot \mathbf{p} = m^2 \]
Chapter 2

QCD in $e^+e^-$ Annihilation

Our process of interest is the annihilation of electrons ($e^-$) and positrons ($e^+$). Experimental data on electron-positron collisions are available in a wide range of energies from several colliders, the most recent one being the Large Electron Positron Collider (LEP) at CERN. Further below we want to consider the production of a quark ($q$) and an antiquark ($\bar{q}$) each producing jets of quarks and gluons in turn, but we start out by a detailed calculation of the cross section up to first order in $\alpha_s$.

2.1 $e^+e^- \rightarrow q\bar{q}$ Cross Section

The lowest order contribution comes from the production of $q\bar{q}$ without any additional gluons. The $q\bar{q}\gamma$ vertex factor is $(−ie_ee_\gamma\mu)$ where $e_q$ is the charge of the quark in units of the electric charge $e$ and $\gamma_\mu$ are the Dirac matrices in 4 dimensions, satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}_4.$$  \hfill (2.1)

The corresponding amplitude is

$$A_\mu(p_2, s_2; p_1, s_1) = −ie_ee_\gamma\mu u(p_2, s_2)\gamma_\mu v(p_1, s_1).$$  \hfill (2.2)

The $e^+e^-\gamma$ vertex factor is $(−ie\gamma_\mu)$ with an amplitude

$$A_\mu(p_b, s_b; p_a, s_a) = −ie\bar{u}(p_b, s_b)\gamma_\mu v(p_a, s_a).$$  \hfill (2.3)

The matrix element corresponding to the diagram in Figure 2.1 becomes (inserting the photon propagator with momentum $q$)

$$M(e^+e^- \rightarrow q\bar{q}) = −e^2e_ee_\gamma\mu u(p_2, s_2)\gamma_\mu v(p_1, s_1)−ig_{\mu\nu}\frac{q^2}{q^2}\bar{v}(p_a, s_a)\gamma_\nu u(p_b, s_b).$$  \hfill (2.4)
Using
\[ \sum_s u_\alpha(p,s) \bar{u}_\beta(p,s) = (\not{p} + m)_{\alpha\beta} \] \hspace{1cm} (2.5)
\[ \text{Tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] = 4(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}), \] \hspace{1cm} (2.6)
where the trace is taken over the spinor indices and neglecting the masses, we arrive at
\[ |\mathcal{M}|^2 = \sum_{s_1,s_2,s_a,s_b} |\mathcal{M}|^2 \]
\[ = \frac{32e^4e^2}{q^4} [p_1 \cdot p_a p_2 \cdot p_b + p_a \cdot p_2 p_b \cdot p_1]. \] \hspace{1cm} (2.7)
The dot products can be expressed in terms of the Mandelstam invariants (in the center of momentum frame \( \sqrt{q^2} = Q \)):
\[ s = (p_a + p_b)^2 = (p_1 + p_2)^2 = q^2 = Q^2 \] \hspace{1cm} (2.8)
\[ t = (p_1 - p_a)^2 = -2p_1 \cdot p_a \]
\[ = (p_b - p_2)^2 = -2p_2 \cdot p_b \]
\[ = -\frac{s}{2}(1 - \cos \theta_{\text{cm}}) \] \hspace{1cm} (2.9)
\[ u = (p_1 - p_b)^2 = -2p_1 \cdot p_b \]
\[ = (p_a - p_2)^2 = -2p_2 \cdot p_a \]
\[ = -\frac{s}{2}(1 + \cos \theta_{\text{cm}}), \] \hspace{1cm} (2.10)
where \( E_a = E_b = (p_a + p_b)/2 = \sqrt{s}/2 \) and \( E_1 = E_2 = (p_1 + p_2)/2 = \sqrt{s}/2 \) have been used to express the invariants in \( \theta_{\text{cm}} \). These follow from energy and momentum conservation and by neglecting the masses \( (p_i^2 = 0 \Rightarrow E_i = |p_i|) \). Now the spin averaged squared matrix element can be written as
\[ |\mathcal{M}|^2 = \frac{32e^4e^2}{q^4} \frac{1}{4} (t^2 + u^2) \]
\[ = \frac{32e^4e^2}{q^4} \frac{1}{8} s^2 (1 + \cos^2 \theta_{\text{cm}}) \] \hspace{1cm} (2.11)
and the differential cross section is
\[ \frac{d\sigma}{d\Omega_{\text{cm}}} = \frac{1}{64\pi^2 E^2_{\text{cm}}} \sum_{\text{colours}} |\mathcal{M}|^2 \]
\[ = \frac{3\alpha^2 e^2}{s} (1 + \cos^2 \theta_{\text{cm}}), \] \hspace{1cm} (2.12)
where \( \alpha = \frac{e^2}{4\pi} \) is the QED coupling.
Integration over \( \Omega_{\text{cm}} \) leads to the total cross section
\[ \sigma(e^+e^- \rightarrow q\bar{q}) = \frac{4\pi\alpha^2 e^2}{s}. \] \hspace{1cm} (2.13)

### 2.2 Born Term Cross Section \( \sigma_0 \)

The production of a final state from \( e^+e^- \) annihilation always happens via an off-shell photon. For completeness and later convenience we will also have a look at the decay of such a virtual photon \( \gamma^* \) into a \( q\bar{q} \) pair as shown in Figure 2.2. From (2.2) we see that
2.3. Gluon Emission

There are two possible diagrams in which a virtual photon decays into $q\bar{q}g$, for the gluon can be attached either to the quark or the antiquark (Figure 2.3). The quark-gluon vertex factor is

\[ A = \bar{u}(p_2, s_2)(-ig_s\gamma_\mu T^a_{ij})\left(\frac{i\gamma_\mu}{p^a_0}\right)(-iee_q\gamma_\mu)v(p_1, s_1)\epsilon_\mu(\lambda)\epsilon'_\nu(\lambda) \]

(2.20)

(−ig_sγμT^a_{ij}) where $T^a$ are the SU(3) colour matrices (Appendix A). After inserting the quark propagators with momenta $p_a$ and $p_b$ respectively, the corresponding amplitudes are

\[ [\bar{M}]^2 = \sum_\lambda \sum_{s_1, s_2} e^2\epsilon^2_q \bar{u}(p_2, s_2)\gamma_\mu v(p_1, s_1)\bar{v}(p_1, s_1)\gamma_\nu u(p_2, s_2)\epsilon_\mu(\lambda)\epsilon'_\nu(\lambda) \]

\[ = -e^2\epsilon^2_q 2\text{Tr} [p_2\gamma_\mu p_1\gamma_\mu] \]

\[ = 4e^2\epsilon^2_q Q^2, \quad (2.14) \]

by using the following:

\[ \sum_\lambda \epsilon_\mu(\lambda)\epsilon'_\nu(\lambda) = -\left(g_{\mu\nu} + \eta \frac{k_0^2}{k^2}\right) \]

(2.15)

with $\eta = 0$ in the Feynman gauge,

\[ \gamma_\mu\gamma_\alpha\gamma_\mu = (2 - N)\gamma_\alpha \]

(2.16)

\[ \text{Tr}[\not{a}\not{b}] = 4a\cdot b \]

(2.17)

and taking $N = 4$. The total two-body decay rate is

\[ W = \frac{1}{16\pi Q^2}[\bar{M}]^2 \]

(2.18)

and we end up with

\[ W(\gamma^* \rightarrow q\bar{q}) = \sigma_0 = 3\alpha e^2_q Q \]

(2.19)

with a factor of 3 from the sum over the quark colours.
\[ A' = \bar{u}(p_2, s_2)(-ie\epsilon_{\gamma\mu}) \left( \frac{i}{p_0} \right) (-ig_s\gamma_\nu T^j_{ij})\psi(p_1, s_1)\epsilon_\nu. \]  \hspace{1cm} (2.21)

Again we start by calculating the spin averaged squared matrix element,
\[ |\overline{M}(\gamma^* \rightarrow q\bar{q}g)|^2 = |A|^2 + |A'|^2 + A(A')^* + A'A^* \] \hspace{1cm} (2.22)

Notice that the 4-momentum of the photon satisfies \( q = p_1 + p_2 + p_3 \) and again in the center of momentum frame \( \sqrt{q^2} = Q \). At this point it is useful and customary to introduce the dimensionless quantities
\[ x_i = \frac{2E_i}{Q}. \] \hspace{1cm} (2.23)

Neglecting the masses of the decay products we get \( p_1 \cdot p_3 = \frac{1}{2}Q^2(1 - x_k) \). The Mandelstam variables in this case are
\[ s = (p_1 + p_3)^2 = 2p_1 \cdot p_3 = Q^2(1 - x_2) \] \hspace{1cm} (2.24)
\[ t = (p_2 + p_3)^2 = 2p_2 \cdot p_3 = Q^2(1 - x_1) \] \hspace{1cm} (2.25)
\[ u = (p_1 + p_2)^2 = 2p_1 \cdot p_2 = Q^2(1 - x_3). \] \hspace{1cm} (2.26)

Using (2.5) again, the first term in (2.22) is given by
\[ |A|^2 = \frac{g_s^2 e^2 e_q^2}{p_0^4} \text{Tr}[T_a T_a] \text{Tr}[\bar{p}_2 \gamma_\nu \bar{p}_a \gamma_\mu \bar{p}_1 \gamma_\mu \bar{p}_a \gamma_\nu]. \] \hspace{1cm} (2.27)

In \( N \) dimensions
\[ \gamma_\mu \gamma_\alpha \gamma_\mu = (2 - N)\gamma_\alpha \] \hspace{1cm} (2.28)
\[ \gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\mu = 4g_{\alpha\beta}I_N + (N - 4)\gamma_\alpha \gamma_\beta \] \hspace{1cm} (2.29)
\[ \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\mu = -2\gamma_\lambda \gamma_\nu \gamma_\alpha - (N - 4)\gamma_\alpha \gamma_\beta \gamma_\gamma \] \hspace{1cm} (2.30)
\[ \text{Tr}[T_a T_a] = \frac{1}{2}\delta_{\alpha\alpha} = 4 \] \hspace{1cm} (2.31)

and by using (2.6) again, the result in 4 dimensions is
\[ |A|^2 = \frac{g_s^2 e^2 e_q^2}{t^2} + 64 \left( \frac{1}{2}(u + s)t - \frac{1}{2}ut \right) = 32g_s^2 e^2 e_q^2 \frac{s}{t} \] \hspace{1cm} (2.32)
\[ = 32g_s^2 e^2 e_q^2 \frac{1 - x_2}{1 - x_1}. \] \hspace{1cm} (2.32)

Similarly
\[ |A'|^2 = 32g_s^2 e^2 e_q^2 \frac{1 - x_1}{1 - x_2}, \] \hspace{1cm} (2.33)
while
\[ A(A')^* = A^*A' \] \hspace{1cm} (2.34)
\[ = 32g_s^2 e^2 e_q^2 \frac{1 - x_3}{(1 - x_1)(1 - x_2)}. \] \hspace{1cm} (2.34)

So the spin averaged squared matrix element becomes
\[ |\overline{M}(\gamma^* \rightarrow q\bar{q}g)|^2 = 32g_s^2 e^2 e_q^2 \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}. \] \hspace{1cm} (2.35)
Now we can have a look at the differential decay rate
\[ dW = \frac{1}{2E_{\text{cm}}} |\mathcal{M}|^2 \left( \prod_{i=1}^{3} \frac{\text{d}^3 p_i}{(2\pi)^3(2E_i)} \right) (2\pi)^4 \delta^4(q - p_1 - p_2 - p_3) \] (2.36)

in which integrating over one of the 3-momenta leads to
\[ \int \text{d}^3p_3 \, \delta(q - p_1 - p_2 - p_3) = \delta(Q - E_1 - E_2 - E_3) \] (2.37)
or
\[ x_1 + x_2 + x_3 = 2 \] \[ s + t + u = Q^2. \] (2.38) (2.39)

For massless particles, \(|p_i| = E_i\), so
\[ \int \frac{\text{d}^3p_i}{2E_i} = \frac{1}{2} \int E_i \sin \theta_i \text{d} \theta_i \text{d} \phi_i \text{d} E_i \]
\[ = \frac{4\pi}{2} \int E_i \text{d} E_i. \] (2.40)

And
\[ \frac{1}{(2\pi)^5} \int \frac{\text{d}^3p_1 \, \text{d}^3p_2 \, \delta(Q - E_1 - E_2 - E_3)}{2E_1 2E_2 2E_3} \]
\[ = \frac{1}{(2\pi)^5} \frac{4\pi \cdot 2\pi}{4} \int_0^\pi \sin \theta_1 \text{d} \theta_1 \frac{\delta(Q - E_1 - E_2 - E_3)}{2E_3} E_1 \text{d} E_1 E_2 \text{d} E_2 \]
\[ = \frac{1}{2(2\pi)^3} \int_{-1}^1 \text{d} z \frac{\delta(Q - E_1 - E_2 - E_3)}{2E_0} E_1 \text{d} E_1 E_2 \text{d} E_2 \] (2.41)

where \( z = \cos \theta_{12} \) if the \( \hat{z} \)-axis is taken in the direction of particle 2.

Now we can use
\[ \delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \quad \text{if} \quad f(x_i) = 0 \] (2.42)

and from (2.24)-(2.26) we see that
\[ (1 - x_i) = \frac{1}{2} x_j x_k (1 - \cos \theta_{jk}), \] (2.43)

for \( i \neq j \neq k \), so the integrand in (2.41) can be rewritten as
\[ \frac{\delta(Q - E_1 - E_2 - E_3)}{2E_3} \]
\[ = \delta((Q - E_1 - E_2 - E_3)^2) \]
\[ = \delta(Q^2(2 - x_1 - x_2 - x_3)) \]
\[ = \delta(Q^2[1 - x_1 - x_2 + \frac{1}{2} x_1 x_2(1 - z)]). \] (2.44)

Integrating this expression over \( z \) just gives \( 2/(Q^2 x_1 x_2) \), such that (2.41) becomes
\[ \frac{Q^2}{16(2\pi)^3} \text{d} x_1 \text{d} x_2. \] (2.45)
Now (2.36) with $E_{cm} = Q$ leads to

$$dW = \frac{Q}{32(2\pi)^3} |\mathcal{M}|^2 dx_1 dx_2.$$  

(2.46)

Combining with (2.35) leads to the differential cross section

$$\frac{1}{\sigma_0} \frac{d^2 \sigma}{dx_1 dx_2} = \frac{2\alpha_s}{3\pi} \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)},$$  

(2.47)

in which $\alpha_s = \frac{g_s^2}{4\pi}$ and $\sigma_0$ as in (2.19).

The contribution to the total cross section is obtained by integrating over $x_1$ and $x_2$ while the boundaries are determined by the constraint $E_i \leq Q/2$, i.e. $0 \leq x_1, x_2 \leq 1$ and $x_1 + x_2 \geq 1$. This means the cross section is divergent at the boundaries $x_1 = 1$ and $x_2 = 1$. We see from (2.43) that these singularities occur when the gluon is \textit{collinear} with the quark or the antiquark ($\theta_{i3} \to 0$) or when the gluon is \textit{soft} ($E_3/Q \to 0$).

### 2.4 Virtual Gluon Corrections

Another order $\alpha_s$ contribution comes from the virtual gluon correction in Figure 2.4. The corresponding amplitude is

$$A_v = \bar{u}(p_2, s_2)(-ig_s\gamma_\beta T^a_{\mu\nu}) \frac{i}{p_2^\alpha}(-ie\gamma_\mu)(-i\frac{p_1}{p_a})(-ig_s\gamma_\alpha T^a_{\mu\nu})(-i\frac{k}{k^2}g_{\alpha\beta})v(p_1, s_1)\epsilon_\mu,$$  

(2.48)

where the gluon propagator $\frac{-i\not{k}}{k^2}(g_{\mu\nu} + \eta_{\mu\nu}k^2/k^2)$ has simplified in the Feynman gauge $\eta = 0$.

![Figure 2.4: virtual gluon correction](image)

We see that when this is added to the Born amplitude

$$A_0 = -ie\epsilon_\mu \bar{u}(p_2, s_2)\gamma_\mu v(p_1, s_1)\epsilon_\mu,$$  

(2.49)

and squared, the cross terms give an order $\alpha_s$ contribution:

$$2 \sum_{s_1, s_2} A_0 A_0^* = \frac{-2ie^2 g_s^2 g^2}{(p_1 - k)^2(p_2 + k)^2 k^2} \text{Tr}[T_a T_a] \text{Tr}[-\not{p}_2 \gamma_\mu \not{p}_1 \gamma_\beta \not{p}_2 \gamma_\mu \not{p}_1 \gamma_\beta]$$  

(2.50)

Using (2.29)-(2.31) with $N = 4$, the traces can be worked out, leading to

$$\frac{-2ie^2 g_s^2 g^2}{(p_1 - k)^2(p_2 + k)^2 k^2}(-128)(p_1 \cdot p_2 + p_1 \cdot k)(p_2 \cdot p_1 - p_2 \cdot k)$$  

(2.51)

$$= \frac{-32ie^2 g_s^2 g^2 \gamma^4 + 4(p_2 \cdot k - p_1 \cdot k)q^2 + 8p_1 \cdot k p_2 \cdot k}{(p_1 - k)^2(p_2 + k)^2 k^2}$$  

(2.52)
because \( p_a = p_1 - k, \ p_b = p_2 + k \) and \( q^2 = 2p_1 \cdot p_2 \).

Integrating over the internal momentum integration variable from for terms odd in \( k \) and using (2.18),

\[
\sigma_v = -2i g_s^4 \sum_0 \int \frac{d^4k}{(2\pi)^4} \frac{-2q^2 + 4(p_2 \cdot k - p_1 \cdot k) + \frac{8p_1 \cdot k p_2 \cdot k}{q^2}}{(p_1 - k)^2(p_2 + k)^2 k^2},
\]

with \( \sigma_0 \) as in (2.19). To evaluate the integral we use the Feynman parametrizations

\[
\frac{1}{ab} = \int_0^1 dy \frac{1}{[ay + b(1 - y)]^2}
\]

\[
\frac{1}{cd} = \int_0^1 dx \frac{2x}{[cx + d(1 - x)]^3}.
\]

If we take

\[
a = (p_1 - k)^2
\]

\[
b = (p_2 + k)^2
\]

the integral becomes

\[
\int \frac{d^4k}{(2\pi)^4} \int_0^1 dy \int_0^1 dx \frac{-2q^2 + 4(p_2 \cdot k - p_1 \cdot k) + \frac{8p_1 \cdot k p_2 \cdot k}{q^2}}{[k^2 - 2p_1 \cdot k y + 2p_2 \cdot k (1 - y)]^2 k^2}.
\]

Now we choose

\[
c = k^2 - 2p_1 \cdot k y + 2p_2 \cdot k (1 - y)
\]

\[
d = k^2,
\]

resulting in

\[
\int \frac{d^4k}{(2\pi)^4} \int_0^1 dy \int_0^1 dx \frac{2x[-2q^2 + 4(p_2 \cdot k - p_1 \cdot k) + \frac{8p_1 \cdot k p_2 \cdot k}{q^2}]}{[k^2 - 2xk \cdot (yp_1 - p_2(1 - y))]^3}
\]

\[
= \int \frac{d^4k}{(2\pi)^4} \int_0^1 dy \int_0^1 dx \frac{2x[-2q^2 + 4(p_2 \cdot k - p_1 \cdot k) + \frac{8p_1 \cdot k p_2 \cdot k}{q^2}]}{[K^2 - C]^3},
\]

with \( K = k - x(yp_1 - p_2(1 - y)) \) and \( C = x^2(yp_1 - p_2(1 - y))^2 = -y(1 - y)x^2q^2 \). Shifting the integration variable from \( k \) to \( K \),

\[
p_2 \cdot k - p_1 \cdot k \rightarrow p_2 \cdot x(yp_1 - p_2(1 - y)) - p_1 \cdot x(yp_1 - p_2(1 - y))
\]

\[
= (p_2 \cdot p_1)xy + (1 - y)x p_1 \cdot p_2
\]

\[
= \frac{1}{2}xy^2
\]

\[
p_1 \cdot k p_2 \cdot k \rightarrow p_1 \cdot K p_2 \cdot K + p_1 \cdot x(yp_1 - p_2(1 - y)) p_2 \cdot x(yp_1 - p_2(1 - y))
\]

\[
= \frac{1}{2}(q \cdot K)^2 - \frac{1}{4}x^2y(1 - y)q^4
\]

\[
= \frac{1}{8}q^2K^2 - \frac{1}{4}x^2y(1 - y)q^4,
\]

for terms odd in \( K \) do not contribute to the integral, the quark masses are being neglected and symmetric integration requires

\[
K_{\mu}K_{\nu} = \frac{1}{4} K^2 g_{\mu\nu}.
\]
This yields
\[ \int_0^1 \frac{d^4K}{(2\pi)^4} \int_0^1 dy \int_0^1 dx \frac{2x(-2y^2 + 2xy^2 + K^2 - 2x^2y(1-y)y^2)}{[K^2 - C]^3}. \] (2.66)

The integral over \( K \) is evaluated using [1]
\[ \int \frac{d^N k}{(2\pi)^N} \frac{(K^2)^R}{[K^2 - C]^M} = \frac{i(-1)^{R-M}}{(16\pi^2)^{N/4}} C^{R-M+N/2} \frac{\Gamma(R + N/2)\Gamma(M - R - N/2)}{\Gamma(N/2)\Gamma(M)}, \] (2.67)
such that in 4 dimensions
\[
\sigma_v = -2i g_s^2 \frac{4}{3} \sigma_0 \left[ -\frac{i}{32\pi^2} \left( \int_0^1 dy \int_0^1 dx \frac{-4 + 4x}{-xy(1-y)} + \int_0^1 dx \right) 
+ \int_0^1 dy \int_0^1 dx \int \frac{d^4K}{(2\pi)^4} \frac{2xK^2}{[K^2 - C]^3} \right] 
= \frac{2\alpha_s \sigma_0}{3\pi} \int_0^1 dy \int_0^1 dx \left( \frac{2x - 2}{xy(1-y)} - 1 \right) 
- 2i g_s^2 \frac{4}{3} \sigma_0 \int_0^1 dy \int_0^1 dx \int \frac{d^4K}{(2\pi)^4} \frac{2xK^2}{[K^2 - C]^3}. \] (2.68)

### 2.5 Dimensional Regularization

The infrared and ultraviolet divergences in (2.47) and (2.68) can be regularized simultaneously by dimensional regularization. In this regularization scheme calculations are done in \( N = 4 + \epsilon \) dimensions such that all divergences become explicit as \( 1/\epsilon \) poles.

**Born term**

First we have to reconsider \( \gamma^* \rightarrow g\bar{q} \) as in Figure 2.2. Instead of (2.18), we get
\[
\sigma_0^{\text{DR}} = \int \frac{d^{N-1}p_1}{(2\pi)^{N-1}(2E_1)} \int \frac{d^{N-1}p_2}{(2\pi)^{N-1}(2E_2)} (2\pi)^N \delta^N(q - p_1 - p_2) \frac{N!}{2Q}. \]

In spherical coordinates, using \( |p_1| = E_1 \) for massless particles,
\[
\frac{d^{N-1}p_1}{2E_1} = \frac{1}{2} E_1^{N-3} dE_1 \sin^{N-3} \theta_1 \sin^{N-4} \theta_2 \ldots \sin \theta_{N-3} d\theta_1 d\theta_2 \ldots d\theta_{N-2}. \] (2.70)
There is no other dependence on the angles, so they can be integrated out, where \( \theta_i \) runs from 0 to \( \pi \), except for one that runs from 0 to \( 2\pi \), leading to
\[
\frac{d^{N-1}p_1}{2E_1} = \pi \prod_{n=1}^{N-3} \int_0^\pi \sin^n \theta d\theta E_1^{N-3} dE_1 \] (2.71)

which can be evaluated using
\[
\int_0^\pi \sin^n \theta d\theta = \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}, \] (2.72)
\[
\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma(x + \frac{1}{2}), \] (2.73)
\[
\Gamma(x) = (x-1)\Gamma(x-1). \] (2.74)
After rearranging some terms, (2.71) reduces to
\[
\pi^{(N-2)/2} 2^{N-3} E_1^{N-3} dE_1.
\] (2.75)
The final integration over \( E_1 \) is performed using
\[
\frac{\delta(Q - E_1 - E_2)}{2E_2} = \delta[(q - p_1 - p_2)^2] = \delta(Q^2 - 2E_1Q)
\] (2.76)
such that
\[
\sigma_0^{DR} = \frac{1}{2^{N-1} \pi^{N/2-1}} \frac{\Gamma\left(\frac{N}{2} - 1\right)}{\Gamma(N-2)} Q^{N-4} |\mathcal{M}|^2 2Q.
\] (2.77)
In N dimensions, using (2.16),
\[
|\mathcal{M}(\gamma^* \rightarrow q\bar{q})|^2 = 2(N-2)e_q^2 e_N^2 Q^2.
\] (2.78)
Here the coupling \( e_N \) differs from the dimensionless quantity \( e \) in (2.14) as follows
\[
e_N = e m_D^{2-\frac{4}{N}}.
\] (2.79)
with \( m_D \) the so called dimensional regularization mass. This results in the Born term in \( N \) dimensions
\[
\sigma_0^{DR} = 3\alpha e_q^2 Q \frac{\Gamma\left(\frac{N}{2} - 1\right)}{\Gamma(N-2)} \left(\frac{Q^2}{4\pi m_D^2}\right)^{\frac{4}{N}-2}.
\] (2.80)
Taking \( N = 4 + \epsilon \), we see this becomes
\[
\sigma_0^{DR} = 3\alpha e_q^2 Q \frac{\Gamma(2 + \frac{3}{2})}{\Gamma(2 + \epsilon)} \left(\frac{Q^2}{4\pi m_D^2}\right)^{z/2}.
\] (2.81)
Obviously, this equation reduces to (2.19) when \( \epsilon = 0 \).

**Real gluon emission**

Now consider the three-body decay of a virtual photon into a quark, antiquark and a gluon (Figure 2.3). In \( N \) dimensions, the corresponding differential decay rate is
\[
dW = \frac{1}{2Q} |\mathcal{M}|^2 \left(\frac{\Gamma\left(\frac{N}{2} - 1\right)}{\Gamma(N-2)}\right) \left(\frac{Q^2}{4\pi m_D^2}\right)^{\frac{4}{N}-2}.
\] (2.82)
Similar to (2.37), the integral over \( p_3 \) leads to an energy conservation \( \delta \)-function, whereas (2.70) and (2.73)-(2.75) can be used to simplify the remaining momentum integrals:
\[
\int \int dE_1^{N-1} p_1 dE_2^{N-1} p_2 = \frac{\pi^{N-2} 2^{N-3}}{\Gamma(N-2)} E_1^{N-3} dE_1 E_2^{N-3} dE_2 \int_0^1 dz (1 - z^2)^{\frac{N}{2}-2}
\] (2.83)
Here \( z = \cos \theta_{12} \), with \( \theta_{12} \) the relative angle between the particles, when the axis of the one remaining angle is chosen along one of the particles.

Thus the differential decay rate is
\[
dW = \frac{1}{2Q} |\mathcal{M}|^2 \frac{\pi^{N-2} 2^{N-3}}{\Gamma(N-2)} E_1^{N-3} dE_1 E_2^{N-3} dE_2 \int_0^1 dz (1 - z^2)^{\frac{N}{2}-2} \frac{\delta(Q - E_1 - E_2 - E_3)}{2\pi(2N-4)(2E_3)}
\]
with \( z = 1 + \frac{2(1-x_1-x_2)}{x_1 x_2} \) from (2.44) and the \( x_i \) from (2.23). In \( 4 + \epsilon \) dimensions, this becomes

\[
\frac{1}{2\hat{Q}^2} |\vec{M}|^2 \left( \frac{Q^2}{16\pi^2} \right)^\epsilon \frac{1}{\Gamma(2+\epsilon)} (1-z^2)^{\epsilon/2} x_1 d x_1 x_2 d x_2.
\] (2.84)

Using (2.28)-(2.31) with \( N = 4 + \epsilon \), (2.22) is replaced by

\[
|\vec{M}|^2 = 32 g_N^2 \epsilon^2 \left( \frac{Q^2}{4\pi m_D^2} \right)^\epsilon \int^1_0 \left( 1 + \frac{\epsilon}{2} \right) \frac{1 - x_2}{1 - x_1} + \epsilon \left( \frac{1 - x_3}{1 - x_1} \right) x_1 x_2 + 2 - 2 x_1 - 2 x_2 \right) (1 - x_1) (1 - x_2)
\] (2.86)

where the dimensional coupling \( g_N \) is defined as

\[
g_N = \frac{g s}{m_D^{\epsilon/2}} \] (2.88)

Combining (2.84) and (2.86) leads to the differential cross section

\[
\frac{1}{\sigma_{DR}^{\epsilon}} \frac{d \sigma_{DR}}{d x_1 d x_2} = \frac{2\alpha_s}{3\pi} \left( \frac{Q^2}{4\pi m_D^2} \right)^\epsilon \left( \frac{x_1 x_2}{x_1^2 + x_2^2} F(x_1, x_2) \right)^\epsilon \left( \frac{1 - z^2}{4} \right)^\epsilon/2 \] (2.89)

Similar to section 2.3, \( \sigma \) is obtained by integrating over the triangular region

\[
0 \leq x_1 \leq 1,
\] (2.90)

\[
1 - x_1 \leq x_2 \leq 1.
\] (2.91)

These integrals are somewhat more doable if we define

\[
x_2 = 1 - v x_1,
\] (2.92)

with \( 0 \leq v \leq 1 \). We see that this leads to

\[
\left( \frac{1 - z^2}{4} \right)^{\epsilon/2} = v^{\epsilon/2} (1 - v)^{\epsilon/2} x_2^{-\epsilon} (1 - x_1)^{\epsilon/2}
\] (2.93)

and

\[
F(x_1, v) = \frac{1}{(1 - x_1) v x_1} \left[ \left( \frac{1 + \epsilon}{2} \right) x_1^2 + (1 - v x_1)^2 \right] + \epsilon \left( \frac{1 + \epsilon}{2} \right) (2 v x_1 - x_1 - v x_1^2) \] (2.94)

such that

\[
\frac{\sigma_{DR}^{\epsilon}}{\sigma_{DR}} = \frac{2\alpha_s}{3\pi} \left( \frac{Q^2}{4\pi m_D^2} \right)^\epsilon \int^1_0 dx_1 x_1^{1+\epsilon} (1 - x_1)^{\epsilon/2} \int^1_0 dv v^{\epsilon/2} (1 - v)^{\epsilon/2} F(x_1, v) \frac{1}{\Gamma(2 + \frac{\epsilon}{2})}.
\] (2.95)
2.5. Dimensional Regularization

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The integrals can be performed using
\[ \int_0^1 dx \, x^{R-1}(1-x)^{M-1} = \frac{\Gamma(R)\Gamma(M)}{\Gamma(R+M)}, \]  
and after some algebra, (2.95) reduces to
\[ \frac{\sigma^\text{DR}}{\sigma_0^\text{DR}} = \frac{2\alpha_s}{3\pi} \left( \frac{Q^2}{4\pi m_D^2} \right)^{\epsilon/2} \frac{\Gamma^2(1+\frac{\epsilon}{2})}{\Gamma(1+\frac{\epsilon}{2})} \left( \frac{8}{\epsilon^2} - \frac{6}{\epsilon} + \frac{19}{2} + \mathcal{O}(\epsilon) \right). \]  
(2.97)

To isolate the divergent terms, we use the following Taylor expansions
\[ \Gamma(1+\epsilon) = 1 - \epsilon \gamma_E + \frac{\epsilon^2}{2} \left( \gamma_E^2 + \frac{\pi^2}{6} \right) + \ldots \]  
(2.98)

\[ \frac{\Gamma^2(1+\frac{\epsilon}{2})}{\Gamma(1+\frac{\epsilon}{2})} = 1 + \frac{\epsilon}{2} \gamma_E + \frac{\epsilon^2}{48} (6\gamma_E^2 - 7\pi^2) + \ldots \]  
(2.99)

\[ \left( \frac{Q^2}{4\pi m_D^2} \right)^{\epsilon/2} = \exp \left( \frac{\epsilon}{2} \ln \left( \frac{Q^2}{4\pi m_D^2} \right) \right) \]  
(2.100)

\[ = 1 + \frac{\epsilon}{2} \ln \left( \frac{Q^2}{4\pi m_D^2} \right) + \frac{\epsilon^2}{8} \ln^2 \left( \frac{Q^2}{4\pi m_D^2} \right) + \ldots \]  
(2.101)

Here $\gamma_E = -\Gamma'(1)$ is Euler’s constant which is defined as
\[ \gamma_E = \lim_{n\to\infty} \sum_{i=1}^n \left[ \frac{1}{i} - \ln \left( 1 + \frac{1}{i} \right) \right] \]  
(2.102)

\[ \approx 0.5772 \]  
(2.103)

This way we can write the real gluon contribution as
\[ \frac{\sigma^\text{DR}}{\sigma_0^\text{DR}} = \frac{2\alpha_s}{3\pi} \left( \frac{8}{\epsilon^2} + \frac{1}{\epsilon} \left[ -6 + 4\ln \left( \frac{Q^2}{4\pi m_D^2} \right) + 4\gamma_E \right] + \frac{19}{2} - 3\gamma_E + \gamma_E^2 \right. \]  
\[ \left. + (2\gamma_E - 3) \ln \left( \frac{Q^2}{4\pi m_D^2} \right) + \ln^2 \left( \frac{Q^2}{4\pi m_D^2} \right) - \frac{7\pi^2}{6} + \mathcal{O}(\epsilon) \right) \]  
(2.104)

**Virtual gluon emission**

In $N$ dimensions, (2.50) becomes
\[ \frac{-32\pi^2}{(p_1 - k)^2(p_2 - k)^2k^2} \left[ (2-N)(q^4 + q^2(p_1 \cdot k - p_2 \cdot k) - 4p_1 \cdot k p_2 \cdot k) + (N-2)(N-4) \right] \]  
(2.105)

The two body phase space factor has been worked out for the Born term in (2.80) already, so we can express $\sigma^\text{DR}$ in $\sigma_0^\text{DR}$ in the following way
\[ \sigma_{v}^\text{DR} = \int \frac{d^N k}{(2\pi)^N} \frac{1}{2^{N-1}\pi^{N/2-1}} \frac{\Gamma \left( \frac{N}{2} - 1 \right)}{\Gamma(N-2)} \left| \frac{Q^{N-4} |M|^2}{2^Q} \right| \]  
(2.106)

\[ = -2g_N^2 \frac{4}{3} \sigma_0^\text{DR} \int \frac{d^N k}{(2\pi)^N} \int_0^1 dy \left| \frac{2x[-2q^2 + 4(p_2 \cdot k - p_1 \cdot k) + \frac{8p_1 \cdot k p_2 \cdot k}{N} + (N-4)k^2]}{|K^2 - C|^{3/2}} \right|, \]  
(2.107)

Proceeding in the same way as in section 2.4,
\[ \sigma_{\nu}^\text{DR} = -2g_N^2 \frac{4}{3} \sigma_0^\text{DR} \int \frac{d^N k}{(2\pi)^N} \int_0^1 dy \int_0^1 dx \frac{x[-2q^2 + 4(p_2 \cdot k - p_1 \cdot k) + \frac{8p_1 \cdot k p_2 \cdot k}{N} + (N-4)k^2]}{|K^2 - C|^{3/2}}, \]  
(2.108)
with \( k = K + x(y p_1 + p_2 (1 - y)) \) and \( C = -x^2 q^2 y (1 - y) \). In the present case, shifting the integration variable to \( K \) yields

\[
K_\mu K_\nu \rightarrow \frac{1}{N} K^2 g_{\mu \nu}
\]

(2.109)

\[
p_2 \cdot k - p_1 \cdot k \rightarrow \frac{1}{2} x q^2
\]

(2.110)

\[
p_1 \cdot k p_2 \cdot k \rightarrow \frac{q^2 K^2}{2 N} - \frac{1}{4} x^2 y (1 - y) q^4
\]

(2.111)

\[
k^2 \rightarrow K^2 - x^2 y (1 - y) q^2
\]

(2.112)

where terms odd in \( K \) have been neglected, such that the numerator in (2.108) becomes

\[
2 x \left[ -2 q^2 + 2 x q^2 + \frac{4}{N} K^2 - 2 x^2 y (1 - y) q^2 + (N - 4) K^2 - (N - 4) x^2 y (1 - y) q^2 \right]
\]

\[
= 2 x \left[ -2 q^2 + 2 x q^2 - (2 + \epsilon) x^2 y (1 - y) q^2 + \frac{4 (1 + \epsilon)}{N} K^2 \right].
\]

(2.113)

Now we can use (2.67) to perform the integral over \( K \) and (2.96) to integrate out \( x \) and \( y \). After using (2.74) repeatedly and rearranging some terms, we arrive at

\[
\frac{\sigma_{\nu}^{DR}}{\sigma_{0}^{DR}} = \frac{2 \alpha_s}{3 \pi} \frac{\alpha_{0}^{DR}}{\alpha_0^{DR}} \left( -\frac{q^2}{4 \pi m_D^2} \right)^{\epsilon / 2} \frac{\Gamma(1 - \frac{\epsilon}{2}) \Gamma^2(1 + \frac{\epsilon}{2})}{\Gamma(1 + \epsilon)} \left( \frac{-8}{\epsilon^2 (1 + \epsilon)} - \frac{1}{1 + \epsilon} - \frac{4}{\epsilon (2 + \epsilon)} \right) + \mathcal{O}(\epsilon)
\]

(2.114)

with \( m_D \) the dimensional regularization mass from (2.88). From (2.98) we get

\[
\frac{\Gamma(1 - \frac{\epsilon}{2}) \Gamma^2(1 + \frac{\epsilon}{2})}{\Gamma(1 + \epsilon)} = 1 + \frac{\epsilon}{2} \gamma_E + \frac{\epsilon^2}{8} \left( \gamma_E^2 - \frac{\pi^2}{6} \right) + \ldots.
\]

(2.115)

and with the use of (2.101) we can expand in \( \epsilon \):

\[
\frac{\sigma_{\nu}^{DR}}{\sigma_0^{DR}} = \frac{2 \alpha_s}{3 \pi} \left( -\frac{8}{\epsilon^2} + \frac{1}{\epsilon} \left[ 6 - 4 \gamma_E - 4 \ln \left( \frac{-q^2}{4 \pi m_D^2} \right) \right] - 2 \gamma_E + \frac{\pi^2}{6} \right) \left( 1 + \frac{\epsilon}{2} \gamma_E + \frac{\epsilon^2}{8} \left( \gamma_E^2 - \frac{\pi^2}{6} \right) \right) + \mathcal{O}(\epsilon).
\]

(2.116)

For timelike particles \( q^2 = Q^2 > 0 \), so we have to continue analytically from \( -q^2 \) to \( q^2 \) by using

\[
\ln(-q^2) = \ln(q^2) \pm i \pi.
\]

(2.117)

Keeping only the real terms,

\[
\frac{\sigma_{\nu}^{DR}}{\sigma_0^{DR}} = \frac{2 \alpha_s}{3 \pi} \left( -\frac{8}{\epsilon^2} + \frac{1}{\epsilon} \left[ 6 - 4 \gamma_E - 4 \ln \left( \frac{Q^2}{4 \pi m_D^2} \right) \right] - 8 \ln \left( \frac{Q^2}{4 \pi m_D^2} \right) \right) - \gamma_E^2 + \frac{\pi^2}{6} - (2 \gamma_E - 3) \ln \left( \frac{Q^2}{4 \pi m_D^2} \right) + 3 \gamma_E + \pi^2 + \mathcal{O}(\epsilon).
\]

(2.118)
2.5. Dimensional Regularization

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Resulting order $\alpha_s$ corrections

Combining (2.104) and (2.118), we see that the $1/\epsilon$ terms cancel. Therefore the total order $\alpha_s$ contribution to the cross section in $4 + \epsilon$ dimensions is

$$\sigma^{DR}_v + \sigma^{DR}_v = \frac{2\alpha_s}{3\pi} \sigma^{DR}_0 \left(\frac{19}{2} - \frac{7\pi^2}{6} - 8 + \frac{\pi^2}{6} + \pi^2\right)$$

$$= \frac{\alpha_s}{\pi} \sigma^{DR}_0.$$

(2.119)

This is well-behaved for $\epsilon \to 0$ and the total decay rate for the process $\gamma^* \to q\bar{q}$ in perturbation theory is

$$\sigma(\gamma^* \to q\bar{q}) = \sigma_0 \left(1 + \frac{1}{\pi} \alpha_s + \ldots\right),$$

(2.120)

which is finite. In fact, according to the KLN theorem [17, 19] infrared divergences cancel at all orders in perturbation theory when summing over degenerate initial and final states. This cancellation does not only apply to the total cross section, but to "weighted" cross sections as well. In the next Chapter these kind of cross sections, in which final states are weighted according to their geometrical shape, will be introduced.
Chapter 3

Event Shape Variables

In order to characterize the “shape” of an event, one can define certain quantities, called *event shapes*. These dimensionless observables each describe certain geometrical properties of the momentum distribution of the outgoing particles. Some examples of event shapes that are often used are

\[
\text{thrust } T = \max_n \frac{\sum_i |p_i \cdot n|}{\sum_i |p_i|} \quad (3.1)
\]

\[
\text{spherocity } S = \left( \frac{4\pi}{\pi} \right)^2 \min_n \left( \frac{1}{\sum_i |p_i|} \right) \left( \sum_i |p_i \times n| \right)^2 \quad (3.2)
\]

\[
\text{C-parameter } C = \frac{3 \sum_{i,j} |p_i||p_j| - (p_i \cdot p_j)^2}{(\sum_i |p_i|)^2} \quad (3.3)
\]

\[
\text{heavy jet mass } \rho_H = \max(\rho_1, \rho_2) \quad (3.4)
\]

\[
\text{single jet mass } \rho_m = \frac{(\sum_{i \epsilon H_m} p_i)^2}{(\sum_i p_i)^2} \quad (3.5)
\]

where the indices \( i \) and \( j \) run over all final-state particles, \( n \) runs over all unit vectors and \( m = 1, 2 \) refers to the two hemispheres \( H_m \) that are separated by a plane normal to the thrust axis \( n_T \).

One can easily show that in case of two opposite-moving thin pencil-like jets (Figure 3.1a), \( T = 1, S = 0, C = 0 \) and \( \rho_1 = \rho_2 = 0 \) for massless particles. In the other limiting case, a perfectly spherical event (in which the energy is distributed uniformly over a spherical surface), it is quite straightforward to show \( T = \frac{1}{2}, S = 1, C = 1 \) and \( \rho_1 = \rho_2 = \frac{1}{4} \). For example, since the thrust axis is arbitrary in that case, the thrust is just

\[
T = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi |\cos \theta| \sin \theta d\theta d\phi
= \frac{1}{2} \cdot 2 \int_0^{\pi/2} \cos \theta \sin \theta d\theta = \frac{1}{2} \quad (3.7)
\]
Figure 3.1: a) In the perfect 2-jet case, $n_T$ lies along the momenta of the particles, such that 
$$T = \frac{1}{2P}(|P n_T \cdot n_T| + |-P n_T \cdot n_T|) = 1.$$ b) In case of a perfect 3-jet structure, $n_T$ can be chosen along any of the 3 momenta. For massless particles $|p_i| = E_i = \frac{Q}{2}$, such that 
$$T = \frac{1}{Q} \left( \frac{Q}{3} + 2 \frac{Q}{2} |\cos(2\pi/3)| \right) = 2/3.
$$

All event shape variables are infrared and collinear safe, i.e. insensitive to the inclusion of a very low energetic particle,

$$\lim_{\lambda \rightarrow 0} X_n(p_1, \ldots, \lambda p_i, \ldots, p_n) = X_{n-1}(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n),$$

or replacement of one particle by two collinear particles with the same total momentum

$$X_n(p_1, \ldots, \lambda p_i, \ldots, p_j = (1 - \lambda)p_i, \ldots, p_n) = X_{n-1}(p_1, \ldots, p_i, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n).$$

This is achieved by making these quantities out of linear combinations of momenta. These properties ensure that the distribution of an event shape $X (d\sigma/dX)$ can be calculated in perturbation theory.

### 3.1 Thrust Distribution

At order $\alpha_s$ the event shape distribution $d\sigma/dX$ is obtained by integrating the differential cross section (2.47) over $x_1$ and $x_2$ with the constraint

$$\delta(X - f_X(x_1, x_2, x_3 = 2 - x_1 - x_2)).$$

(3.10)

Here $f_T(x_1, x_2, x_3 = 2 - x_1 - x_2)$ is determined in the following way

$$\max_i \frac{\sum_i |p_i \cdot n|}{\sum_i |p_i|} = \frac{1}{Q} \sum_i E_i |\vec{p}_i \cdot n_T| = \frac{1}{Q} \left( E_k + |\sum_{i\neq k} E_i \cos \theta_{ik}| \right)$$

$$= \frac{2}{Q} \max_i E_i = \max_i x_i,$$

(3.11)

by using momentum conservation in the second line:

$$\max_i E_i + \sum_{j\neq k} E_j \cos \theta_{jk} = 0,$$

(3.12)

with $E_k = \max_i E_i$.

Hence the thrust distribution equals

$$\frac{d\sigma}{dT} = \frac{2\alpha_s}{3\pi} \sigma_0 \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \delta(T - \max_i x_i).$$

(3.13)
3.1. Thrust Distribution

Chapter 3. Event Shape Variables

The regions of integration are determined by $\max_i x_i$ as is illustrated in Figure 3.2. The thrust is constant on contours like the dashed red one. Because of symmetry, the cases $T = x_1$ and $T = x_2$

are equivalent, so we can insert a factor of 2. In the third region, $T$ is constant on diagonal lines with end points $x_2 = 2 - 2x_1$ and $x_2 = 1 - \frac{x_1}{2}$. We know that in this region $x_1 + x_2 + T = 2$ and the integral in (3.13) becomes

\[
\begin{align*}
2 & \int_{2/3}^1 dx_1 \int_{2-2x_1}^{x_1} dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \delta(T-x_1) \\
+ & \int_T^{T(1-T)} dx_2 \frac{(2-T-x_2)^2 + x_2^2}{(T+x_2-1)(1-x_2)} \\
= & \frac{2}{1-T} \left( T^2 \ln \left( \frac{2T-1}{1-T} \right) + \int_{2T-1}^{T} dx \left[ \frac{(x-1)(x+1)}{1-x} + \frac{1}{1-x} \right] \right) \\
+ & \int_{1-T}^{2T-1} dy \left[ \frac{1}{T} \left( (1-T^2)^2 + 1 \right) \left( \frac{1}{T-y} + \frac{1}{y} \right) - 2 \right] \\
= & \frac{2}{1-T} \left[ (3T^2 - 3T + 2) \ln \left( \frac{2T-1}{1-T} \right) \right] + 3 \left( 3T - 2 \right) \left( T - 2 \right) \frac{1}{1-T}.
\end{align*}
\]

Figure 3.2: The regions in which $T = x_1$, provided that $x_1 + x_2 \geq 1$.

This leads to the following expression for the thrust distribution in (3.13)

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dT} = \frac{2\alpha_s}{3\pi} \left[ \frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \left( \frac{2T-1}{1-T} \right) + \frac{3(3T-2)(T-2)}{1-T} \right].
\]

(3.15)
Similar to section 2.3 the singularities for soft and collinear gluons are present at $T = 1$. From the constraint in the first line of (3.14) we see that in first order QCD, $T \geq \frac{2}{3}$. This bound is obtained by considering the maximally symmetric configuration for three outgoing particles (Figure 3.1b), where $x_1 = x_2 = x_3 = \frac{2}{3}$. The behaviour of the thrust distribution as a function of $\tau = 1 - T$ is illustrated by Figure 3.3.

### 3.2 C-parameter Distribution

In determining the C-parameter distribution, we need $f_C(x_1, x_2, x_3 = 2 - x_1 - x_2)$ in

$$
\frac{1}{\sigma_0} \frac{d\sigma}{dC} = \frac{2\alpha_s}{3\pi} \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \delta(C - f_C(x_1, x_2, x_3 = 2 - x_1 - x_2)). \tag{3.16}
$$

From (3.3) we see that at $O(\alpha_s)$ and in the center of momentum frame,

$$
C = \frac{3}{2} \frac{1}{Q^2} \sum_{i \neq j} E_i E_j (1 - \cos^2 \theta_{ij})
$$

$$
= \frac{3}{4} \sum_{i \neq j} x_i x_j (1 - \cos \theta_{ij})(1 + \cos \theta_{ij})
$$

$$
= \frac{3}{2} \sum_{i \neq j \neq k} (1 - x_k)(1 + \cos \theta_{ij})
$$

$$
= \frac{3}{2} \sum_{i \neq j \neq k} (1 - x_k)(2 - \frac{2(1 - x_k)}{x_i x_j})
$$

$$
= 6 \prod_{i}(1 - x_i) x_1 x_2 x_3,
$$

by using (2.43) in the third and the fourth line. Notice that in case of a perfect 3-jet structure $C = 3/4$, which is the upper bound for the C-parameter at this order. Evaluating the expression (3.16) is quite labourious and will not be done here. The result can be found in Refs. [8, 13].
3.3 Heavy Jet Mass Distribution

In case of three decay particles, the heavy jet mass

\[ \rho_H = \max_m \frac{\left( \sum_{i \in H_m} p_i \right)^2}{\left( \sum_i p_i \right)^2} \]  

(3.18)
can be expressed in \( x_1 \) and \( x_2 \) by using (2.43) in the following way

\[
\rho_H = \frac{1}{Q^2 \max_m} \left( \sum_{i \in H_m} p_i \right)^2
\]

\[
= \frac{1}{Q^2} \left( \sum_{i \neq k} p_i \right) \quad \text{for } i \neq j \neq k
\]

\[
= \frac{2}{Q^2} E_k E_j (1 - \cos \theta_{ij}) = 1 - \max_i x_i.
\]

Apparently at order \( \alpha_s \), the heavy jet mass \( \rho_H \) is exactly equal to \( 1 - T = \tau \), which means that its distribution is easily obtained from (3.15) to be

\[
\frac{1}{\sigma_0} \frac{d\sigma}{d\rho_H} = \frac{2\alpha_s}{3\pi} \left[ \frac{2(3\rho_H^2 - 3\rho_H + 2)}{\rho_H(1 - \rho_H)} \ln \left( \frac{1 - \rho_H}{\rho_H} \right) - \frac{3(1 - \rho_H)^2}{\rho_H^2} \right].
\]  

(3.19)

In what follows we focus on the two-jet limit (Figure 3.1a) where soft and collinear gluons are emitted. We saw that the distributions of event shapes that vanish in this limit, like \( \tau \) and \( \rho_H \), receive large contributions from this region of phase space.
Chapter 4

Sudakov Resummation

The property of asymptotic freedom (Appendix A.1) makes QCD analyzable by perturbation theory. However, quantities like event shape distributions receive large logarithmic contributions to their perturbative coefficients in the infrared (IR) region, as we will see in section 4.1. Instead of truncating the perturbation series at fixed order, it has to be resummed. Thanks to the factorization property of the cross section, which will be explained in section 4.2, one finds an exponentiated expression in which the large logarithms are reorganized. This result can be obtained for instance by using a branching algorithm to describe the partonic jets [7] or by studying the functional dependence of the factorized expression [21]. Both approaches will be discussed in this Chapter.

4.1 Double Logarithms

In general the leading behaviour of the differential cross section of an event shape variable $X$ that vanishes in the two-jet limit (e.g. $\tau$, $\rho_{H}$) is

$$\frac{1}{\sigma} \frac{d\sigma}{dX} \propto \alpha_s^n \frac{\ln^{2n-1} (1/X)}{X}$$

(4.1)

for small $X$ and at $n$th order. As an explicit example we look back at the first order thrust distribution in (3.15) which reduces to

$$\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} = \frac{2\alpha_s}{3\pi} \left[ \frac{4}{7} \ln \left( \frac{1}{\tau} \right) - \frac{3}{\tau} \right]$$

(4.2)

for small $\tau$ (corresponding to $T \simeq 1$). We define the radiation function $R(X)$ by

$$R(X) \equiv \int_0^X dX' \frac{1}{\sigma} \frac{d\sigma}{dX'},$$

(4.3)

which is unity at $X = X_{\text{max}} = 1$ and vanishes as $X \to 0$. From (4.1) a leading behaviour at $n$th order of

$$R_n(X) \propto \alpha_s^n \ln^{2n} \left( \frac{1}{X} \right) \equiv \alpha_s^n L^{2n}$$

(4.4)

is obtained. These double logarithms occur when gluons are emitted simultaneously soft and collinear to the emitting parton.

At small values of $X$, all orders in the perturbative expansion become important because of the large logarithms in the coefficients. Hence it is not justified to truncate the perturbation series at
fixed order. However, event shape cross sections have been determined explicitly up to next-to-leading order (NLO) and some up to NNLO only. They can be written generically as

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dX} = \frac{\alpha_s(\mu^2)}{2\pi} A(X) + \left( \frac{\alpha_s(\mu^2)}{2\pi} \right)^2 \left( 2\pi \beta_0 A(X) \ln \frac{\mu^2}{Q^2} + B(X) \right) + \mathcal{O}(\alpha_s^3),
\]

where \( \mu \) is the renormalization scale (cf. Appendix B). The coefficient functions \( A(X) \) and \( B(X) \) can be found for instance in [2] for many event shape variables. At higher orders the final state configurations of the partons induce a far more complex phase space. Moreover, perturbative predictions are unreliable due to the logarithmically enhanced coefficients. We see from (4.4) that \( \alpha_s L^2 \) needs to be small for a perturbative treatment of \( R(X) \), but even for small \( \alpha_s \) the region near the two-jet limit (\( X \to 0 \)) would violate this condition because of the large double logarithms. Nevertheless, perturbative predictions can be improved by resumming the large logarithmic terms to all orders. To this end we use the factorization property to find exponentiation for event shape distributions.

4.2 Factorization

Factorization is the property that long-distance effects behave independently from short-distance effects up to inverse powers of the energy scale. The fundamental idea is that if the scales at which the processes take place are sufficiently different, they behave incoherently. According to the factorization theorem [10], the cross section for hadronic collisions is the convolution over functions describing the long distance dynamics, like parton distribution functions (PDF’s) and fragmentation functions, and functions describing the short distance physics, which are the partonic cross sections. In the present case of \( e^+ e^- \) annihilation, PDF’s describing the probability to find a parton in a hadron do not appear, but the cross section does adopt a similar convoluted form. The hadronization process is separated from the short-distance physics (Figure 4.1) by a factorization scale \( \mu \).

![Figure 4.1: Factorization of long- and short-distance physics](image-url)
4.3 Exponentiation

The radiation function \( R(X) \) from (4.3) is said to \textit{exponentiate} if it can be written in the following way

\[
R(X) = C(\alpha_s) \exp \{ G(\alpha_s, L) \} + D(X, \alpha_s),
\]

where the remainder function \( D(X, \alpha_s) \) vanishes for \( X \to 0 \) and

\[
C(\alpha_s) = 1 + \sum_{n=1}^{\infty} C_n \left( \frac{\alpha_s}{2\pi} \right)^n,
\]

\[
G(\alpha_s, L) = \sum_{n=1}^{\infty} \sum_{m=1}^{n+1} G_{nm} \left( \frac{\alpha_s}{2\pi} \right)^n L^n = L g_1(\alpha_s, L) + g_2(\alpha_s, L) + \alpha_s g_3(\alpha_s, L) + \ldots
\]

(4.6)

(4.7)

(4.8)

(4.9)

\( \equiv \)

Evidently, the function \( g_1 \) resums all the leading logarithmic contributions (LL) \( \alpha_s^n L^n \), while all next to leading logarithms (NLL) \( \alpha_s^n L^n \) are contained in \( g_2 \), etcetera. The coefficient function \( C(\alpha_s) \) contains none of the large logarithms. By expanding in \( \alpha_s L \), the region of \( X \) where \( \alpha_s L \ll 1 \) can be included if the functions \( g_i \) are known, which is a strong improvement from the condition \( \alpha_s L^2 \ll 1 \).

Before combining the resummed and the full NLO expressions, it is necessary to check the renormalization scale dependence of the resummed expression (4.9). We know from Appendix B that modifying the renormalization scale from \( Q_2^2 \) to \( \mu^2 \) leads to

\[
\alpha_s(\mu^2) = \alpha_s(Q_2^2) - \beta_0 \alpha_s^2 \ln \left( \frac{\mu^2}{Q_2^2} \right) + \ldots
\]

(4.10)

Yet the physical quantity \( R(X) \) cannot possibly depend on the renormalization scale, which implies constraints on the \( g_i \). Since the LL term \( g_1 \) is renormalization group invariant, all terms appearing from the scale change that have explicit scale dependence must be cancelled by the non-leading functions. For instance by expanding \( L g_1(\alpha_s L, \mu^2) \) around \( \alpha_s(Q_2^2) \) we see that

\[
L g_1(\alpha_s L, \mu^2) = L \left[ g_1(\alpha_s L) - L \beta_0 \alpha_s^2 \ln \left( \frac{\mu^2}{Q_2^2} \right) g_1'(\alpha_s L) + \ldots \right]
\]

(4.11)

Since the second term on the right hand side is of order \( L^n \alpha_s^n \), it has to be cancelled by \( g_2(\alpha_s L, \mu^2) \). Consequently, in order to do any sensible determination of \( \alpha_s \), both \( g_1 \) and \( g_2 \) have to be established.

Exponentiation is a consequence of the factorization property of QCD matrix elements. The exponentiating property of the thrust and heavy jet mass distributions can be demonstrated in several ways as we will see in the following two sections.

4.4 Branching Algorithm

In Ref. [7] a branching algorithm for \( e^+ e^- \) annihilation was developed in order to describe the evolution of the jets. It takes into account the angular ordering of the emitted gluons \(^1\). Obviously, the branching process starts with the creation of a quark-antiquark pair at center of momentum

\(^1\)Angular ordering corresponds to decreasing branching angles. It is a way of including \textit{colour coherence}, the interference among the amplitudes of soft gluons radiated from colour-connected partons, which is quantum mechanically expected in QCD.
energy $Q$ (Figure 4.2). The algorithm is based on the fact that all momenta $k_i$ of the branches consist of fractions of two lightlike back-to-back momenta, $p = \frac{Q}{2}(1, n)$ and $\bar{p} = \frac{Q}{2}(1, -n)$, and a transverse component $k_{\perp,i}$. By examining the four-momentum squared, these fractions are found to be related in the following way

$$
\begin{align*}
\mu_i p^\mu &= x_i p^\mu + k_{\perp,i}^\mu + x_i \bar{p}^\mu \\
&= x_i p^\mu + k_{\perp,i}^\mu + \frac{k_{\perp,i}^2 + k_i^2}{x_i Q^2} \bar{p}^\mu \\
&= x_i p^\mu + k_{\perp,i}^\mu + \frac{w_i}{x_i} \bar{p}^\mu 
\end{align*}
(4.12)
$$

Recall that in case of the thrust and the heavy jet mass, two hemispheres $H_m$ are defined by the thrust axis $n_T$ (cf page 23). The total momentum $k_m$ in each of the hemispheres can be written as in (4.12). It follows from momentum conservation ($k_1 + k_2 = p + \bar{p}$) that the components must satisfy

$$
\begin{align*}
x_1 &= \frac{1}{2} (1 + w_1 - w_2 + t) \\
x_2 &= \frac{1}{2} (1 - w_1 + w_2 + t) \\
k_{\perp,1} &= -k_{\perp,2}
\end{align*}
(4.13)
$$

where $t$ is defined in the following way

$$
t = \sqrt{(w_2 - w_2 - 1)^2 - 4w_1}.
(4.15)
$$

Now for some axis $n$ we have

$$
\sum_i |p_i \cdot n| = \frac{Q}{2} \left( x_1 + x_2 - \frac{w_1}{x_1} - \frac{w_2}{x_2} \right) = Q t
(4.16)
$$

It can be shown [7] that if $p$ and $\bar{p}$ are chosen along the thrust axis, the transverse three-momentum components disappear (no momentum in the plane orthogonal to the thrust axis) such that $w_i$ simplifies in the following way

$$
w_i = \frac{k_i^2}{Q^2}.
(4.17)
$$

At this point we can use (4.16) to express the thrust (3.1) in terms of the simplified jet masses $w_1$ and $w_2$

$$
T = \frac{Q t}{\sum_i |p_i|}.
(4.18)
$$
where $t$ is defined by (4.15). Since we consider the final-state particles to be massless, this reduces to $T = t$. Hence in the 2-jet region (expanding $T$ around 1)

$$1 - T = \tau = \frac{k^2_1}{Q^2} + \frac{k^2_2}{Q^2} + 2\frac{k^2_1 k^2_2}{Q^4} + \ldots$$

(4.19)

Recalling that

$$\rho_H = \max_m \frac{\left(\sum_{m} k_i \right)^2}{Q^2},$$

(4.20)

and neglecting correlations between the hemispheres, it should not come as a surprise that to NLL accuracy the cross sections of the thrust and the heavy jet mass factorize in the following way

$$R_T(\tau) = \int_0^\infty dk_1^2 dk_2^2 J^0(Q^2, k_1^2) J^0(Q^2, k_2^2) \theta(\tau Q^2 - k_1^2 - k_2^2)$$

(4.21)

$$R_{\rho_H}(\rho_H) = \int_0^\infty dk_1^2 dk_2^2 J^0(Q^2, k_1^2) J^0(Q^2, k_2^2) \theta(\rho_H Q^2 - k_1^2 - k_2^2).$$

(4.22)

Here the single jet mass distribution $J^0(Q^2, k^2)$ represents the probability of a final state jet with total momentum $k$ coming from a parton $p$ that was produced at scale $Q$. Obviously, such a probability has to be normalized according to

$$\int_0^\infty dk^2 J^0(Q^2, k^2) = 1.$$  

(4.23)

The single jet mass distribution satisfies an evolution equation that follows from the branching algorithm. In Laplace space to NLL accuracy its solution was found to be [7]

$$\ln \mathcal{J}^0_n(Q^2) = \int_0^1 \frac{du}{u} \left(e^{-uQ^2} - 1\right) \left[\int_{u^2 Q^2}^{\infty} \frac{dq^2}{q^2} A(\alpha_s(q^2)) + \frac{1}{2} B(\alpha_s(u Q^2))\right] + \mathcal{O}(\alpha_s(\alpha_s \ln(\nu Q^2)^n)),$$

(4.24)

with

$$A(\alpha_s) = \frac{\alpha_s C_F}{\pi} + \frac{1}{2} C_F \left(\frac{\alpha_s}{\pi}\right)^2 K, \quad B(\alpha_s) = -\frac{3\alpha_s C_F}{2\pi},$$

(4.25)

$$K = C_A \left(\frac{67}{18} - \frac{\pi^2}{6}\right) = \frac{5}{9} N_f.$$  

To NLL accuracy we can insert a Heaviside step function instead of the exponent,

$$e^{-u Q^2} - 1 \simeq -\theta(u - \frac{1}{N}),$$

(4.26)

where $\bar{N} = e^{r_N N} = e^{r_N} \nu Q^2$. When we write $e^{-(1-x)^N - 1} \simeq x^N - 1$ for large $N$, (4.26) follows from the general distribution

$$\int_0^1 dx x^{N-1} \left(\frac{\ln(1-x)}{1-x}\right)_+ = \frac{1}{2} (\gamma_E + \ln N)^2 + \frac{1}{2} \zeta(2) + \mathcal{O}(1/N),$$

(4.27)

of which the right hand side can be written as

$$-\int_0^{1-1/N} \frac{dx}{1-x} \ln(1-x) = \int_0^1 \frac{dx}{1-x} \ln(1-x) \theta(1-x - 1/\bar{N}).$$

(4.28)
If we take into account LL contributions only, we have to work out the following integral

\[
\int_{1/N}^{1} \frac{du}{u} \int_{a(Q^2)}^{uQ^2} \frac{dq^2}{q^2} C_F \frac{\alpha_s(q^2)}{\pi},
\]

(4.29)

which can be simplified by converting to \( \alpha_s \) as an integration variable according to (cf. Appendix B)

\[
\frac{dq^2}{q^2} = -\frac{1}{\beta_0} \frac{d\alpha_s}{\alpha_s^2}
\]

(4.30)

and we arrive at

\[
\frac{C_F}{\pi \beta_0} \int_{1/N}^{1} \frac{du}{u} \int_{\alpha_s(u^2 Q^2)}^{\alpha_s(u Q^2)} \frac{d\alpha_s}{\alpha_s} = \frac{C_F}{\pi \beta_0} \int_{1/N}^{1} \frac{du}{u} \left[ \ln \left( \frac{\alpha_s}{1 + \beta_0 \alpha_s \ln u} \right) - \ln \left( \frac{\alpha_s}{1 + 2\beta_0 \alpha_s \ln u} \right) \right]
\]

\[
= \frac{C_F}{\pi \beta_0^{3/2} \alpha_s} \int_{-\lambda - \beta_0 \alpha_s \gamma_c}^{0} dt \left[ \ln \left( \frac{\alpha_s}{1 + t} \right) - \ln \left( \frac{\alpha_s}{1 + 2t} \right) \right]
\]

\[
= \frac{C_F}{\pi \beta_0^{3/2} \alpha_s} \left[ (1 - \lambda) \ln(1 - \lambda) - \frac{1}{2} (1 - 2\lambda) \ln(1 - 2\lambda) \right]
\]

(4.31)

where \( \lambda \) is defined as \( \beta_0 \alpha_s \ln(\nu Q^2) \) and the integration in the second line was performed from \(-\lambda\) to 0 because the \( \gamma_c \) term contributes only to the NLL part.

It turns out that when taking into account NLL contributions as well, the complete expression is [7]

\[
\ln \tilde{J}_p^\beta(Q^2) = \ln(\nu Q^2) f_1 (\beta_0 \alpha_s \ln(\nu Q^2)) + f_2 (\beta_0 \alpha_s \ln(\nu Q^2)) + \mathcal{O} \left( \alpha_s^n \ln^{n-1}(\nu Q^2) \right)
\]

(4.32)

with

\[
f_1(\lambda) = -\frac{C_F}{2\pi \beta_0 \lambda} \left[ (1 - 2\lambda) \ln(1 - 2\lambda) - 2(1 - \lambda) \ln(1 - \lambda) \right],
\]

(4.33)

\[
f_2(\lambda) = -\frac{2 C_F K}{2\pi \beta_0} \left[ 2 \ln(1 - \lambda) - \ln(1 - 2\lambda) \right]
\]

\[
- \frac{3 C_F}{2\pi \beta_0} \ln(1 - \lambda) - \frac{C_F \gamma_E}{\pi \beta_0} \left[ \ln(1 - \lambda) - \ln(1 - 2\lambda) \right]
\]

\[
- \frac{C_F \beta_0}{2\pi \beta_0^3} \left[ \ln(1 - 2\lambda) - 2 \ln(1 - \lambda) + \frac{1}{2} \ln^2(1 - 2\lambda) - \ln(1 - \lambda) \right].
\]

(4.34)

Using the Laplace representation of the \( \theta \)-function in (4.21) and then applying the inverse Laplace transformation on the entire expression, we arrive at

\[
R_T(\tau) = \frac{1}{2\pi i} \int_{C} \frac{d\nu}{\nu} e^{\nu \tau} Q^2 \left[ \tilde{J}_p^\beta(Q^2) \right]^2
\]

\[
= \frac{\exp[2L f_1(\lambda) + 2 \beta_0 \alpha_s \ln(\mu^2 / Q^2)]}{\Gamma[1 - 2f_1(\lambda) - 2\lambda f_1(\lambda)]},
\]

(4.35)

where \( \lambda = \beta_0 \alpha_s \ln(1/\tau) \) and \( L = \ln(1/\tau) \). Now the explicit form of the corresponding functions \( g_1 \) and \( g_2 \) as in (4.9) can be easily seen to be

\[
g_1(\alpha_s L) = 2 f_1(\beta_0 \alpha_s L),
\]

(4.37)

\[
g_2(\alpha_s L) = 2 f_2(\beta_0 \alpha_s L) - \ln \Gamma \left[ 1 - 2 f_1(2 \beta_0 \alpha_s L) - 2 \beta_0 \alpha_s L f_1(\beta_0 \alpha_s L) \right].
\]

(4.38)
Similarly, from (4.22) we obtain
\[ R_H(\rho_H) = \frac{\exp[2Lf_1(\lambda) + 2f_2(\lambda) + 2\lambda^2 f'_1(\lambda) \ln(\mu^2/Q^2)]}{(\Gamma[1 - f_1(\lambda) - \lambda f'_1(\lambda)])^2}, \] (4.39)
where of course \( \lambda \) is now given by \( \beta_0 \alpha_s \ln(1/\rho_H) \) and
\[
\begin{align*}
g_1(\alpha_s, L) &= 2f_1(\beta_0 \alpha_s L) \\
g_2(\alpha_s, L) &= 2f_2(\beta_0 \alpha_s L) - \ln \Gamma [1 - f_1(\beta_0 \alpha_s L) - \beta_0 \alpha_s L f'_1(\beta_0 \alpha_s L)]
\end{align*}
\] (4.40) (4.41)
To NNL the heavy jet mass distribution differs from \( R_T \) only in the argument of the \( \Gamma \) since the only difference is in the \( \theta \) constraints.

Finally we can determine the thrust distribution from the normalized cross section \( R_T(\tau) \) by taking the derivative,
\[
\frac{1}{\sigma_0} \frac{d\sigma}{dT} = -\frac{1}{1-T} \left. \frac{d}{d\ln \tau} R_T(\tau) \right|_{\tau=1-T}. \] (4.42)
When considering only the leading terms \( \alpha_s^n \ln^{2n}(\tau) \) in \( R_T \), the so-called double logarithmic (DL) approximation, this was found [7] to be
\[
\frac{1}{\sigma_0} \frac{d\sigma}{dT} = -2\frac{C_F}{\pi} \frac{\alpha_s}{1-T} \exp \left\{ -C_F \frac{\alpha_s}{\pi} \ln^2(1-T) \right\}. \] (4.43)
This coincides with the leading behaviour of the thrust distribution to first order in \( \alpha_s \) that we came across in (4.2). Clearly, \( \sigma/dT \) vanishes when \( T \to 1 \) (the limit in which no gluons are emitted) which is the so called Sudakov suppression. This can be traced back to the property of an accelerated (colour) charged particle to be accompanied by its bremsstrahlung. The probability for a perfect 2-jet configuration with no emission at all is zero. The LO and NLO calculations where the large logarithms were not resummed do not satisfy this physically expected property at all. In fact they get huge (negative) contributions in the 2-jet limit (Figure 4.3).

![Figure 4.3: Comparision of LO, NLO and NNL calculations](image)

Indeed all the leading and next-to-leading logarithms in the thrust and heavy jet mass radiation functions are resummed by the functions \( g_1 \) and \( g_2 \) as described in section 4.3. The agreement with experiment is much better than for the NLO calculation. As we will see in section 5.3 the shape of the resummed distribution is equivalent to that of the data (Figure 5.2). Nevertheless, in the region near the peak non-perturbative corrections, originating from hadronization effects, are of the same order of magnitude as the perturbative contributions and have to be taken into account. This can be achieved by introducing a shape function, as will be described in section 4.6.
4.5 Functional Properties

The exponentiating behaviour of the thrust (or heavy jet mass) distribution can also be derived in a more elegant way. Instead of the detailed analysis of the kinematics we make use of functional properties and separation of variables. In the two-jet limit the cross section factorizes into a hard part \( H \) containing the hard scattering, a soft part \( S \) and the two jets \( J_1 \) and \( J_2 \) containing soft and collinear gluons (Figure 4.4),

\[
\frac{d\sigma}{dk_1^2 dk_2^2} = J_1(k_1, \mu, \xi) J_2(k_2, \mu, \xi) H(k_1, k_2, \mu, \xi),
\]

where \( \mu \) is the factorization scale which can be set equal to the renormalization scale and \( \xi^\mu \) is the gauge vector.

![Figure 4.4: Diagram representing the factorization of the cross section in the Sudakov limit](image)

Contributions to the soft part, in which soft gluons connect the two jets, are of the single logarithmic type and may be neglected in the double logarithmic approximation. If we use the axial gauge \( \xi \cdot A = 0 \) instead of the Feynman gauge of Chapter 2, the gluon propagator becomes

\[
G_{\mu\nu} = \left( -g_{\mu\nu} + \frac{k_\mu \xi_\nu + \xi_\mu k_\nu}{\xi \cdot k} - \frac{k_\mu k_\nu \xi^2}{(\xi \cdot k)^2} \right) \frac{1}{k^2 + i\epsilon}.
\]

Since this is invariant under rescalings of the gauge vector \( \xi^\mu \), the \( \xi \) dependence of the functions \( H \) and \( J_i \) has to be of the form \( k_i \cdot \xi / \sqrt{\xi^2} \), so we normalize \( \xi \) by the condition \( \xi^2 = 1 \). Recall from Chapter 3 that the thrust distribution is obtained by integrating the differential cross section in (4.44) with the constraint (4.19). Hence, in the region \( T \sim 1 \), we arrive at the convoluted expression

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dT} \simeq \int_0^{Q^2} dk_1^2 dk_2^2 \delta \left( 1 - T - \frac{k_1^2}{Q^2} - \frac{k_2^2}{Q^2} \right) H \left( \frac{k_1 \cdot \xi}{\mu}, \frac{k_2 \cdot \xi}{\mu} \right. \left. \alpha_s(\mu) \right) J_1 \left( \frac{k_1^2}{\mu^2}, \frac{k_1 \cdot \xi}{\mu}, \alpha_s(\mu) \right) J_2 \left( \frac{k_2^2}{\mu^2}, \frac{k_2 \cdot \xi}{\mu}, \alpha_s(\mu) \right).
\]

In the \( T \to 1 \) limit, we can expand the \( k_i \) about two opposite lightlike momenta \( \frac{Q}{2}(1, n) \) and \( \frac{Q}{2}(1, -n) \) with \( k_1^\mu \) in the plus and \( k_2^\mu \) in the minus direction. This leads to

\[
k_1^+ \simeq \frac{1}{\sqrt{2}} \left( \frac{Q}{2} + \frac{Q}{2} \right) = \frac{Q}{\sqrt{2}} \simeq k_2^-
\]

\[
k_1^- \simeq 2k_1^+ k_1^- \simeq \sqrt{2} Q k_1^-
\]

\[
k_2^- \simeq 2k_2^+ k_1^- \simeq \sqrt{2} Q k_2^+.
\]
To leading power in $Q$ we may take $k_1 \cdot \xi \approx k_1^\perp \xi^-$ and $k_2 \cdot \xi \approx k_2^\perp \xi^+$ and it follows that the $k_i^2$ are independent of $k_i \cdot \xi$ at this order. Hence the hard part $H$ may be pulled out of the integral in (4.46).

The convolution can be disentangled by taking the Mellin transform

$$\tilde{\sigma}(n) = \frac{1}{\sigma_0} \int_0^1 dT \, T^{n-1} \frac{d\sigma}{d(1-T)},$$

(4.49)

which is equivalent to the Laplace transform

$$\frac{1}{\sigma_0} \int_0^\infty d\tau \, e^{-\tau} \frac{d\sigma}{d\tau}$$

(4.50)

for large $n$ since in that case $e^{-\tau} \approx (1 - \tau)^n$. A function that is finite at $T = 1$ will fall off as $1/n$ for $n \to \infty$ in Mellin space, while the large logarithms of $1 - T$ transform into logarithms of $n$ in the following way

$$\int_0^1 dT \frac{T^n - 1}{1 - T} \ln^m(1 - T) \equiv \frac{(-1)^m}{m+1} \ln^{m+1} n + O(1/n).$$

(4.51)

Integrating out the delta distribution in (4.46), it follows that the convolution becomes a regular product in Mellin space,

$$\tilde{\sigma}(n) = H \left( \frac{k_1 \cdot \xi}{\mu}, \frac{k_2 \cdot \xi}{\mu}, \alpha_s(\mu) \right) \tilde{J}_1 \left( \frac{Q^2}{n \mu^2}, \frac{k_1 \cdot \xi}{\mu}, \alpha_s(\mu) \right) \tilde{J}_2 \left( \frac{Q^2}{n \mu^2}, \frac{k_2 \cdot \xi}{\mu}, \alpha_s(\mu) \right),$$

(4.52)

with

$$\tilde{J}_i \left( \frac{Q^2}{n \mu^2}, \alpha_s(\mu) \right) = \int_0^\infty d\tau J_i e^{-\tau n} \ln H \left( \tau, \frac{Q^2}{n \mu^2}, \frac{k_i \cdot \xi}{\mu}, \alpha_s(\mu) \right)$$

(4.53)

and

$$\tau_J = \frac{k_i^2}{Q^2}.$$  

(4.54)

Because the physical cross section must be independent of the renormalization scale $\mu$, $\ln \tilde{\sigma}$ must be as well. The renormalization group equation for the jet functions (which have two external lines) becomes

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right] \ln \tilde{J}_i = -2\gamma_0(\alpha_s(\mu))$$

(4.55)

where $\gamma_0$ is the corresponding quark anomalous dimension and $g$ is the coupling constant. Consequently $H$ must satisfy

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right] \ln H = 4\gamma_0(\alpha_s(\mu)).$$

(4.56)

All together $\tilde{\sigma}$ has to be gauge-independent as well, which means that when boosting $\xi^+$ and $\xi^-$ while keeping $\xi^2 = 1$, the variation of $\tilde{\sigma}$ needs to satisfy

$$\frac{\partial \tilde{\sigma}}{\partial \ln \xi^-} = -\frac{\partial \tilde{\sigma}}{\partial \ln \xi^+}.$$  

(4.57)

Recalling that $k_1 \cdot \xi \approx k_1^\perp \xi^-$ and $k_2 \cdot \xi \approx k_2^\perp \xi^+$ to leading power in $Q$, this can be written explicitly as

$$\frac{\partial \ln \tilde{J}_1 \left( \frac{Q^2}{n \mu^2}, \frac{k_1^\perp}{\mu}, \alpha_s \right)}{\partial \ln k_1^\perp} + \frac{\partial \ln H \left( \frac{k_1^\perp}{\mu}, \frac{k_2^\perp}{\mu}, \alpha_s \right)}{\partial \ln k_1^\perp} = -\frac{\partial \ln \tilde{J}_2 \left( \frac{Q^2}{n \mu^2}, \frac{k_2^\perp}{\mu}, \alpha_s \right)}{\partial \ln k_2^\perp} - \frac{\partial \ln H \left( \frac{k_1^\perp}{\mu}, \frac{k_2^\perp}{\mu}, \alpha_s \right)}{\partial \ln k_2^\perp}.$$  

(4.58)
Some important properties of the $J_i$ and $H$ follow from this equation because of the difference in their arguments. For instance, the first term on the left hand side may obviously depend on $k_1^+ / \mu$ as well as on $Q^2 / n \mu^2$. However, its $k_1^+$ dependent part has to be cancelled by a derivative of $H$, while the remaining must be equal to the first term on the right hand side. In other words, its dependance on $k_1^+$ has to be independent of the $Q^2 / n \mu^2$ dependence and the two need to behave additively,

$$
\frac{\partial}{\partial \ln k_1^+} \ln \tilde{J}_1 \left( \frac{Q^2}{n \mu^2}, \frac{k_1^+}{\mu}, \alpha_s(\mu) \right) = K \left( \frac{Q^2}{n \mu^2}, \alpha_s(\mu) \right) + G \left( \frac{k_1^+}{\mu}, \alpha_s(\mu) \right)
$$

(4.59)

where $K$ contains the infrared behaviour and $G$ the ultra-violet contributions. Similarly, the $k_2^-$ dependence of the first term on the right hand side has to be cancelled by a contribution from $H$ and the following must hold

$$
\frac{\partial \ln H}{\partial \ln k_1^+} + \frac{\partial \ln H}{\partial \ln k_2^-} = -G \left( \frac{k_1^+}{\mu} \right) - G \left( \frac{k_2^-}{\mu} \right).
$$

(4.60)

The renormalization group equations can be combined with this gauge-independence as follows. From (4.55) it can be seen that

$$
\frac{d^2}{d\mu d\tilde{\mu}} \ln \tilde{J}_1 = 0
$$

(4.61)

When applying this to (4.59) we obtain a renormalization group equation for $K + G$,

$$
\mu \frac{d}{d\mu} (K + G) = 0,
$$

(4.62)

such that the Sudakov anomalous dimension $\gamma_K$ arises, which satisfies

$$
\mu \frac{d}{d\mu} K = -\mu \frac{d}{d\mu} G = -\gamma_K(\alpha_s(\mu)).
$$

(4.63)

This anomalous dimension $\gamma_K$ can depend on $\alpha_s$ only, as this is the one variable that $K$ and $G$ have in common. Integrating both sides with the convenient lower bounds $Q^2 / n$ and $Q$ we can make the $n$ dependence explicit,

$$
K \left( \frac{Q^2}{n \mu^2}, \alpha_s(\mu) \right) - K(1, \alpha_s(Q^2 / n)) = -\frac{1}{2} \int_{Q^2 / n}^{\mu^2} \frac{d\mu^2}{\mu^2} \gamma_K(\alpha_s(\mu^2))
$$

(4.64)

$$
G \left( \frac{Q}{\mu}, \alpha_s(\mu^2) \right) - G(1, \alpha_s(Q^2)) = \frac{1}{2} \int_{Q}^{\mu^2} \frac{d\mu^2}{\mu^2} \gamma_K(\alpha_s(\mu^2))
$$

(4.65)

such that

$$
K \left( \frac{Q^2}{n \mu^2}, \alpha_s(\mu) \right) + G \left( \frac{Q}{\mu}, \alpha_s(\mu^2) \right) = K(1, \alpha_s(Q^2 / n)) + G(1, \alpha_s(Q^2))
$$

$$
-\frac{1}{2} \int_{Q^2 / n}^{\mu^2} \frac{d\mu^2}{\mu^2} \gamma_K(\alpha_s(\mu^2)).
$$

(4.66)

Furthermore, in order to get the same running coupling for $K$ and $G$, we can rewrite the first term on the right hand side as

$$
K(1, \alpha_s(Q^2)) + \int_{Q^2}^{\mu^2} \frac{d\mu^2}{\mu^2} \beta(\gamma) \frac{\partial}{\partial g} K(1, \alpha_s(\mu^2))
$$

(4.67)

$$
= \frac{1}{2} \int_{Q^2 / n}^{\mu^2} \frac{d\mu^2}{\mu^2} \beta(\gamma) \frac{\partial}{\partial g} K(1, \alpha_s(\mu^2)),
$$

(4.68)
since $\beta(g) = \mu \frac{\partial g}{\partial \ln k_1}$. So all together we arrive at

$$\frac{\partial}{\partial \ln k_1} \ln \tilde{J}_1 = K(1, \alpha_s(Q)) + G(1, \alpha_s(Q)) - \frac{1}{2} \int_{Q^2/n}^\infty \frac{d\lambda^2}{\lambda^2} \Gamma_J(\alpha_s(\lambda^2))$$

(4.69)

$$\Gamma_J(\alpha_s) = \gamma_k(\alpha_s) + \beta(g) \frac{\partial}{\partial g} K(1, \alpha_s)$$

(4.70)

while of course the same equation holds for $\partial_{k_\perp} (\ln \tilde{J}_2)$. The differential equations for the $\tilde{J}_i$ follow by studying their arguments (as in (4.58)). Differentiating with respect to $\ln n$ is equivalent to differentiating with respect to $\ln \mu^2$ provided that we compensate for the $k_i^{+/−}$ derivative that follows from the second argument, so

$$\frac{\partial}{\partial \ln n} \ln \tilde{J}_i \left( \frac{Q^2}{n \mu^2}, \frac{k_i^{+/−}}{\mu}, \alpha_s(\mu^2) \right) = \left[ \frac{\partial}{2 \partial \ln \mu} - \frac{1}{2} \frac{\partial \ln(1/k_i^{+/−})}{\partial g} \right] \ln \tilde{J}_i \left( \frac{Q^2}{n \mu^2}, \frac{k_i^{+/−}}{\mu}, \alpha_s(\mu^2) \right)$$

$$= \frac{1}{2} \left( -2 \gamma_q - \beta(g) \frac{\partial}{\partial g} \ln \tilde{J} + K(1, \alpha_s(Q)) + G(1, \alpha_s(Q)) \right) - \frac{1}{2} \int_{Q^2/n}^\infty \frac{d\lambda^2}{\lambda^2} \Gamma_J(\alpha_s(\lambda^2)),$$

(4.71)

where (4.55) and (4.69) have been inserted in the second equality. At this point we fix $k_\perp, \xi$ at the center of momentum energy $Q$. The above can be interpreted as a renormalization group equation with anomalous dimensions $\Gamma$ and $\Gamma'$ by writing (4.71) as

$$\left[ \frac{\partial}{\partial \ln n} + \frac{\beta(\alpha_s)}{2 \partial g} \right] \ln \tilde{J}_i \left( \frac{Q^2}{n \mu^2}, \frac{k_i^{+/−}}{\mu}, \alpha_s(\mu^2) \right) = \frac{1}{2} \Gamma'_J(\alpha_s(\mu^2)) - \frac{1}{4} \int_{Q^2/n}^\infty \frac{d\lambda^2}{\lambda^2} \Gamma_J(\alpha_s(\lambda^2)).$$

(4.72)

The solution of this differential equation is the sum of the homogeneous solution and a particular solution,

$$\ln \tilde{J}_i \left( \frac{Q^2}{n \mu^2}, \frac{k_i^{+/−}}{\mu}, \alpha_s(\mu^2) \right) = \ln \tilde{J}_i \left( \frac{Q^2}{\mu^2}, \frac{k_i^{+/−}}{\mu}, \alpha_s(\mu^2) \right) + \frac{1}{2} \int_{Q^2/n}^\infty \frac{d\lambda^2}{\lambda^2} \left[ \Gamma_J(\alpha_s(\lambda^2)) \right],$$

(4.73)

(4.74)

which can be verified by using

$$\beta(g) \frac{\partial \Gamma(\alpha_s(\lambda^2))}{\partial g} = \lambda \frac{\partial \Gamma(\alpha_s(\lambda^2))}{\partial \lambda}$$

(4.75)

Apparently in Mellin space the behaviour of the jet functions is determined by the exponentiation of logarithms of $n$ which are generated by the expansion of $\alpha_s(Q^2/n)$, $\alpha_s(Q^2/n)$,

$$\tilde{J}_i \left( \frac{Q^2}{n \mu^2}, \frac{k_i^{+/−}}{\mu}, \alpha_s(\mu^2) \right) = \tilde{J}_i \left( \frac{Q^2}{\mu^2}, \frac{k_i^{+/−}}{\mu}, \alpha_s(\mu^2) \right) e^{\frac{1}{2} \int_{Q^2/n}^\infty \frac{d\lambda^2}{\lambda^2} \left[ \Gamma_J(\alpha_s(\lambda^2)) - \ln(\frac{Q^2}{\lambda^2}) \Gamma_J(\alpha_s(\lambda^2)) \right]},$$

(4.75)
4.6 Shape Functions

In this section the so called shape function is introduced, pursuing Korchemsky and Sterman \[18\]. It parametrizes the non-perturbative contributions to event shape distributions. The exponentiating property of the radiation functions described above is based on the factorization property of the cross section. Yet this holds up to inverse powers of $Q$ only and non-perturbative corrections are expected to be of the form $1/Q^p$.

Recall that the distribution for an event shape $X$ is obtained by integrating the differential cross section after insertion of a delta function $\delta(X - f_X)$. Following the notation of Ref. \[18\], in which such a distribution is denoted as $\langle \delta(X - f_X) \rangle$, it follows that the radiation function as defined in (4.3) satisfies

$$R(X) = \int_0^X dX' \frac{1}{\sigma} \frac{d\sigma}{dX'} = \langle \theta(X - f_X) \rangle. \quad (4.76)$$

We came across the weight functions $f_X$ for the thrust and the heavy jet mass already in (4.19) and (4.20),

$$f_\tau = \frac{k_1^2 + k_2^2}{Q^2} \quad \text{for small } \tau \quad (4.77)$$

$$f_{\rho_H} = \frac{k_1^2}{Q^2} \theta(k_1^2 - k_2^2) + \frac{k_2^2}{Q^2} \theta(k_2^2 - k_1^2), \quad (4.78)$$

where $k_m$ is the total momentum in hemisphere $H_m$. The corresponding radiation functions become

$$R_{\rho_H}(\rho_H) = \left\langle \theta \left( \rho_H - \frac{k_1^2}{Q^2} \theta(k_1^2 - k_2^2) - \frac{k_2^2}{Q^2} \theta(k_2^2 - k_1^2) \right) \right\rangle$$

$$= \left\langle \theta \left( \rho - \frac{k_1^2}{Q^2} \right) \theta \left( \rho_H - \frac{k_2^2}{Q^2} \right) \right\rangle. \quad (4.79)$$

$$R_T(\tau) = \left\langle \theta \left( \tau - \frac{k_1^2 + k_2^2}{Q^2} \right) \right\rangle. \quad (4.80)$$

In the limit $X \to 0$ the squared mass in each hemisphere can be approximated by

$$k_m^2 = Q \sum_{i=1}^{N_{soft}} E_i (1 - |\cos \theta_i|), \quad (4.81)$$

where $\theta_i$ describes the angle between the thrust axis and the 3-momentum of the $i$th soft particle in the corresponding hemisphere. Furthermore, since the soft gluons contribute additively, small non-perturbative corrections to the hemisphere masses can be separated by writing

$$k_m^2 \to k_m^2 + \varepsilon_m Q, \quad \varepsilon_m = \sum_{i=1}^{N_{soft}} (1 - |\cos \theta_i|). \quad (4.82)$$

Due to the difference in scales, the emission of soft gluons factorizes from the jet physics as described in Section 4.2. Correspondingly, the radiation functions become convolutions,

$$R_{\rho_H}(\rho_H) = \int_0^\mu d\varepsilon_1 d\varepsilon_2 f(\varepsilon_1, \varepsilon_2) \left\langle \theta \left( \rho_H - \frac{k_1^2}{Q^2} - \varepsilon_1 \right) \theta \left( \rho_H - \frac{k_2^2}{Q^2} - \varepsilon_2 \right) \right\rangle \quad (4.83)$$

$$R_T(\tau) = \int_0^\mu d\varepsilon_1 d\varepsilon_2 f(\varepsilon_1, \varepsilon_2) \left\langle \theta \left( \tau - \frac{k_1^2 + k_2^2}{Q^2} - \varepsilon_1 + \varepsilon_2 \right) \right\rangle. \quad (4.84)$$

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Here the shape function $f(\varepsilon_1, \varepsilon_2)$ is introduced, describing the probability density for total soft momentum $\varepsilon_1$ in one hemisphere and $\varepsilon_2$ in the other. Such a shape function does not depend on the hard scale $Q$. Note from (4.84) that taking into account non-perturbative effects amounts to a shift in the argument of the thrust radiation function. A similar property would arise in $R_H(\rho)$ if there would be no correlation $\Delta f(\varepsilon_1, \varepsilon_2)$ between the hemispheres in

$$f(\varepsilon_1, \varepsilon_2) = f(\varepsilon_1) f(\varepsilon_2) + \Delta f(\varepsilon_1, \varepsilon_2). \quad (4.85)$$

However, this condition is not satisfied in general.

Infrared shape functions, or at least similar functions, can be identified in resummed perturbation theory as well. As we saw in the previous two sections, in the 2-jet limit the single jet mass distribution exponentiates in Laplace space. The corresponding (Sudakov) exponent was found to be

$$\tilde{S}(\nu, Q^2) = \int_0^1 \frac{du}{u} \left( e^{-u \nu} - 1 \right) \left[ \int_{u^2 Q^2}^0 \frac{dq^2}{q^2} A(\alpha_s(q^2)) + \frac{1}{2} B(\alpha_s(u Q^2)) \right]. \quad (4.86)$$

By introducing a factorization scale $\mu$, one can separate the non-perturbative contributions from the perturbative ones such that the exponent becomes

$$\tilde{S}(\nu, Q^2) = \tilde{S}_{PT}(\nu, Q^2, \mu) + \tilde{S}_{NP}(\nu, Q^2, \mu). \quad (4.87)$$

The non-perturbative behaviour is exclusively related to terms in $A((\alpha_s(q^2))$ at momentum scales $< \mu$. Consequently, it can be rewritten as an expansion in the following way,

$$\tilde{S}_{NP}(\nu, Q^2, \mu) = \int_0^{\mu^2} \frac{dq^2}{q^2} \int_0^{\nu/Q} \frac{du}{u} \sum_{n>0} \frac{1}{n!} \left( -\frac{\nu}{Q} \right)^n u^{n-1} q^{n-2} A(\alpha_s(q^2))$$

$$= \sum_{n>0} \frac{\lambda_n(\mu)}{n!} \left( -\frac{\nu}{Q} \right)^n \int_0^{\mu^2} q^{n-2} A(\alpha_s(q^2))$$

$$= \sum_{n>0} \lambda_n(\mu) \left( -\frac{\nu}{Q} \right)^n. \quad (4.88)$$

Because the parameters $\lambda_n(\mu)$ are independent of $Q$, exponentiating both sides in (4.88) leads to

$$e^{\tilde{S}_{NP}(\nu/Q, \mu)} = \int_0^\infty d\zeta e^{-\nu \zeta/Q} f(\zeta, \mu). \quad (4.89)$$

The function $f(\zeta, \mu)$, which is independent of $Q$, can indeed be interpreted as an infrared shape function.

In the next Chapter the non-perturbative corrections to event shape distributions will be calculated more explicitly, albeit in the context of a model.
Chapter 5

Dressed Gluon Exponentiation

The discrepancy between experiment and the NLL distributions may be attributed (at least partly) to infrared renormalons. These are diagrams containing chains of fermionic loops that give rise to factorially increasing perturbative coefficients, as described in Appendix C. As a consequence, subleading logarithms have to be taken into account as well. This is achieved by dressed gluon exponentiation [12], in which both Sudakov logarithms and renormalons are resummed. First the cross section for single gluon emission is calculated in the 2-jet region to get the logarithmically enhanced terms while a small mass is assigned to the gluon. Subsequently the gluon is dressed with fermionic loops in the dispersive approach (as described in Appendix D). This way higher-order perturbative terms are included and power corrections are identified by Borel summation of the renormalons.

Under the assumption that gluons are emitted independently, the single dressed gluon (SDG) cross section serves as an exponentiation kernel in Laplace space. In the Sudakov limit, the total cross section for multiple gluon emission can be approximated by exponentiating the SDG cross section.

5.1 Single Dressed Gluon

We start by calculating the cross section of the renormalon diagram that corresponds to the emission of a gluon with a chain of fermion loop insertions, a dressed gluon. This corresponds to considering an off-shell gluon with virtuality \( k^2 = m^2 = \epsilon Q^2 \) and inserting the one-loop running coupling at the vertices (Appendix C). The final state quarks however are once again considered to be massless. It is sufficient to consider the case in which the gluon propagates in the hemisphere of the quark and account for the other hemisphere by inserting a factor of 2. If we define light-cone coordinates such that the momentum of the quark and the antiquark can be written as \( p = (p_+, 0, 0) \) and \( \bar{p} = (0, p_-, 0) \) respectively, the situation simplifies considerably in the light-cone axial gauge \( A_+ = 0 \) because of suppression of the coupling to the antiquark. The matrix element corresponding to the hadronic part of the remaining diagram is

\[
\bar{u}(\bar{p}, s)(-ig_s\gamma^\mu T^a)(\frac{\bar{p}^+ + \bar{k}^+}{(\bar{p} + \bar{k})^2})(-ie\epsilon_{\mu\lambda}(\lambda)\epsilon_{\rho}(\lambda')).
\]

\[5.1\]

The leptonic contribution is just

\[
e^2 [\gamma_\mu \gamma^\rho \gamma_\sigma],
\]

\[5.2\]

Figure 5.1: Single dressed gluon emission
where \( q_1 \) and \( q_2 \) are the electron and positron momenta. Inserting the photon propagator, we arrive at

\[
|\mathcal{M}|^2 = \frac{N_C \alpha^4 C_F g_s^2}{Q^4(p + k)^4} \sum q_m^2 (\sqrt{-g_{\mu
u}}) D_{\mu\nu} Tr [\not{p} \gamma^\mu (\not{q} + \not{k}) \gamma^\rho \not{p} \gamma^\sigma (\not{q} + \not{k}) \gamma^\nu]
\]  

(5.3)

by using (2.5) and replacing \( \sum_\lambda \epsilon_\mu(\lambda) \epsilon_\nu(\lambda) \) by

\[
D_{\mu\nu}(k) = \left[ -g_{\mu\nu} + \frac{k_\mu p_\nu + p_\mu k_\nu}{k \cdot \bar{p}} \right],
\]

(5.4)

where \( D_{\mu\nu}/k^2 \) is the gluon propagator (4.45) in the present gauge.

All together the trace becomes

\[
2 \left( 2 Tr [\not{p} (\not{q} + \not{k}) \not{p} (\not{q} + \not{k})] + \frac{1}{k \cdot \bar{p}} Tr [\not{q} \not{p} \not{k} (\not{p} + \not{k}) \not{p} (\not{q} + \not{k})] + \frac{1}{k \cdot \bar{p}} Tr [\not{k} \not{q} \not{p} (\not{q} + \not{k}) \not{p} (\not{q} + \not{k})] \right).
\]

(5.5)

At this point the so called Sudakov decomposition

\[
k = (k_+, k_-, k_\perp) = \alpha \bar{p} + \beta p + k_\perp
\]

(5.6)

is introduced. In these variables the propagator squared in (5.3) is proportional to \((\alpha + \lambda)^{-2}\).

Since we are interested in singular terms in the cross section only \cite{12}, we consider the regions where both \( \alpha \) and \( \lambda \) are small. Then it follows from \( \alpha \beta = \gamma + \lambda \) that \( \gamma \) has to be small compared to \( \beta \) as well, although \( \beta \) can be either large or small. This way we are clearly focussing on soft and collinear gluons.

Substituting the Sudakov parameters, (5.3) becomes

\[
|\mathcal{M}|^2 = \frac{8 N_C \alpha^4 C_F g_s^2}{Q^4(\alpha + \lambda)^2} \sum_q e_q^2 \left[ \frac{\alpha \beta + 2 \alpha + \lambda + \alpha}{\beta} + 2 \frac{\lambda}{\beta} \right]
\]

\[
= \frac{8 N_C \alpha^4 C_F g_s^2}{Q^4(\alpha + \lambda)^2} \sum_q e_q^2 \left[ -\frac{\lambda}{(\alpha + \lambda)^2} (1 + \beta) + \frac{1}{\alpha + \lambda} \frac{\beta^2 + 2 \beta + 2}{\beta} \right].
\]

(5.8)

Recalling (2.36) and inserting a factor of 2 to account for the other half of phase space, the cross section in the center of mass frame is

\[
\sigma = \frac{1}{Q^2} \int \frac{d^4 p d^4 \bar{p}}{(2\pi)^6} \delta(p^2) \delta(\bar{p}^2) \int \frac{d^4 k}{(2\pi)^4} \delta(k^2 - m^2) (2\pi)^4 \delta^4(p + \bar{p} + k - q) |\mathcal{M}|^2
\]

\[
= \frac{C_F \alpha_s \sigma_0}{\pi} \int \frac{d\alpha d\beta d\lambda}{(1 + \alpha + \beta + \lambda)^3} \left[ -\frac{\lambda}{(\alpha + \lambda)^2} (1 + \beta) + \frac{1}{\alpha + \lambda} \frac{\beta^2 + 2 \beta + 2}{\beta} \right]
\]

\[
\approx \frac{C_F \alpha_s \sigma_0}{\pi} \int \frac{d\alpha d\beta d\lambda (1 + \beta)}{(1 + \beta)^3} \left[ -\frac{\lambda}{(\alpha + \lambda)^2} (1 + \beta) + \frac{1}{\alpha + \lambda} \frac{\beta^2 + 2 \beta + 2}{\beta} \right]
\]

(5.9)

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with $\sigma_0$ the bare cross section from (2.13). The phase space measure was rewritten in terms of the Sudakov parameters by looking back at (2.45). The energy fractions $x_1$ and $x_2$ are related to the Sudakov parameters as follows,

$$
x_1 = \frac{1 + \alpha}{1 + \alpha + \beta + \lambda},
$$
$$
x_2 = \frac{1 + \beta}{1 + \alpha + \beta + \lambda}.
$$

The Jacobian corresponding to the transformation is

$$
(1 - \lambda)(1 + \alpha + \beta + \lambda),
$$

which leads to (5.9) considering $\alpha$ and $\lambda$ are small in our case of interest. The delta function confines $\epsilon \equiv m^2/Q^2$ to a fixed value in terms of the Sudakov parameters

$$
\epsilon = \frac{m^2}{(k+p+p')^2} = \frac{\lambda}{1 + \alpha + \beta + \lambda} \approx \frac{\lambda}{1 + \beta}.
$$

In order to determine the thrust and heavy jet mass distributions in the single dressed gluon case, first we rewrite the definition of the heavy jet mass in case of single gluon emission in the current variables

$$
\rho_H = \max\left(m\left(\sum_i \epsilon_i m_i p_i\right)^2\frac{Q^2}{(k+p+p')^2}\right) = \frac{\alpha + \lambda}{1 + \alpha + \beta + \lambda} \approx \frac{\alpha + \lambda}{1 + \beta}.
$$

Recall that our interest lies with the limit of vanishing transverse momentum ($\gamma \rightarrow 0$), which corresponds to $\alpha\beta = \lambda$ and it follows that in performing the integral, we should substitute the lower limit $\beta = \frac{\epsilon}{\rho_H}$ and neglect other contributions. This results in

$$
\frac{1}{\sigma} \frac{d\sigma}{d\rho_H} = \frac{C_F \alpha_s}{\pi} \int \frac{d\beta}{(1 + \beta)^3} \left( -\frac{\epsilon}{\rho_H} + \frac{1}{\rho_H} \frac{\beta^2 + 2\beta + 2}{\beta(1 + \beta)} \right).
$$

When using the gluon Bremsstrahlung definition of the running coupling [15, 9],

$$
\bar{\alpha_s} = \alpha_s \left( 1 + \frac{\alpha_s}{2} \left( C_A \left( \frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{5}{9} \frac{\alpha_s}{N_f} \right) \right)
$$

and using the two-loop renormalization group equation, the thrust and heavy jet mass distributions are exact to NLL accuracy. We define $\bar{A}(k^2) \equiv \beta_0 \bar{\alpha}_s(k^2)/\pi$ and use the renormalization scheme invariant Borel representation [6, 16, 3] in which

$$
\bar{A}(k^2) = \int_0^\infty du \exp(-u \ln \frac{k^2}{\Lambda^2}) \bar{A}_B(u).
$$

In the dispersive approach [11] an effective coupling $\alpha_{\text{eff}}$ is introduced to get a finite description at large distances (see Appendix D). Using (D.12), the corresponding $\bar{A}_{\text{eff}}$ is

$$
\bar{A}_{\text{eff}}(Q^2) = \int_0^\infty du \exp \left( -u \ln \frac{Q^2}{\Lambda^2} \right) \frac{\sin \pi u}{\pi u} \bar{A}_B(u).
$$
Now according to (D.24) the SDG heavy jet mass distribution is obtained from (5.14) by performing the following integral,

\[
\left. \frac{1}{\sigma} \frac{d\sigma}{d\rho_H}(Q^2, \rho_H) \right|_{SDG} = \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}(\mu^2) \left( -\mu^2 \frac{d}{d\mu^2} \right) \frac{C_F \alpha_s}{\pi} \left( \frac{2}{\rho_H} \ln \frac{\rho_H}{\epsilon} - 3 \frac{1}{2 \rho_H} + \frac{\epsilon}{\rho_H^2} + \frac{1}{2 \rho_H^2} \right) = \frac{C_F}{\beta_0} \int_{\rho_H}^{\rho_H^\text{max}} \frac{d\epsilon}{\epsilon} \tilde{A}_{\text{eff}}(\epsilon Q^2) \left( \frac{2}{\rho_H} - \frac{\epsilon}{\rho_H^2} - \frac{\epsilon^2}{\rho_H^2} \right),
\]

(5.18)

where the integration limits are determined by \( \alpha > 0 \) and \( \alpha < \beta \) (to ensure gluon is in the hemisphere of the quark). The boundaries are associated with collinear gluon emission (\( \alpha = 0 \) means both \( k_2^\perp \) and \( k_- \) vanish) and soft emission (when \( \alpha = \beta \)). Substituting the expression (5.17) for \( \tilde{A}_{\text{eff}}(\epsilon Q^2) \) and performing the integral over \( \epsilon \) we obtain

\[
\frac{1}{\sigma} \frac{d\sigma}{d\rho_H} \bigg|_{SDG} = \frac{C_F}{\beta_0} \int_0^{\rho_H^\text{max}} \frac{d\epsilon}{\epsilon} \tilde{A}_{\text{eff}}(\epsilon Q^2) \left( \frac{2}{\rho_H} - \frac{\epsilon}{\rho_H^2} - \frac{\epsilon^2}{\rho_H^2} \right) B_{SDG}(\rho_H, u),
\]

(5.19)

where terms non-singular in \( \rho_H \) have been left out and the Borel function is given by

\[
B_{SDG}(\rho_H, u) = \left[ \frac{2}{u} e^{2u \ln \frac{1}{\rho_H}} - \left( \frac{2}{u} + 1 - \frac{1}{2} \right) e^{u \ln \frac{1}{\rho_H}} \right].
\]

(5.20)

This is well-defined since the pole at \( u = 0 \) is cancelled within \( B_{SDG} \) and the poles at \( u = 1 \) and \( u = 2 \) are regulated by the sine factor.

The single jet mass distribution \( \frac{d\sigma}{d\rho} \) is defined as the average of the distributions of the two single jet masses in (3.5). It is the common building block of the thrust and heavy jet mass distributions. To leading order in \( \alpha_s \) one of the single jet masses vanishes, since we assume the quarks to be on-shell. Consequently, at order \( \alpha_s \),

\[
\frac{1}{\sigma} \frac{d\sigma}{d\rho_H} = 2 \frac{1}{\sigma} \frac{d\sigma}{d\rho},
\]

(5.21)

and the SDG single jet mass distribution is obtained simply by dividing (5.19) by 2.

### 5.2 Multiple Gluons

The squared matrix element for the single dressed gluon emission can be easily shown to satisfy the factorization property [12]. As a consequence, as we saw in sections 4.4 and 4.5, the single jet mass distribution exponentiates in Laplace space,

\[
\ln \tilde{J}(\nu, Q^2) = \int_0^1 \frac{1}{\sigma} \frac{d\sigma}{d\rho}(Q^2, \rho) \bigg|_{SDG} (e^{-\nu \rho} - 1) d\rho,
\]

(5.22)

under the assumption that gluon emissions contribute independently. Inserting the Borel representation of the SDG distribution (5.19), this becomes

\[
\ln \tilde{J}(\nu, Q^2) = \frac{C_F}{2 \beta_0} \int_0^\infty d\nu B(\nu, u) e^{-u \ln \frac{Q^2}{\rho_H^2}} \frac{\sin \pi u}{\pi u} \tilde{A}_{\text{eff}}(\epsilon Q^2),
\]

(5.23)
with

\[ B(\nu, u) = \int_0^1 \frac{d\rho}{\rho} \left( e^{-\nu \rho} - 1 \right) \left[ \frac{2}{u} e^{2u \ln \frac{1}{2}} - \left( \frac{2}{u} + \frac{1}{1-u} + \frac{1}{2-u} \right) e^{u \ln \frac{1}{2}} \right] \]

\[ \simeq \frac{2}{u} \left( \nu^{2u} \Gamma(-2u) + \frac{1}{2u} \right) - \left( \frac{2}{u} + \frac{1}{1-u} + \frac{1}{2-u} \right) \left( \nu^{\mu} \Gamma(-u) + \frac{1}{u} \right) \]  

(5.24)

where the Gamma function is given by its integral representation

\[ \Gamma(z) \equiv \int_0^\infty dt \ t^{z-1} e^{-t}. \]  

(5.25)

Singularities of \( B(\nu, u) \) occur at all positive half integer values of \( u \) due as well as at all positive integer values of \( u \). Double poles arise at \( u = 1 \) and \( u = 2 \). The poles at half integers originate from the soft boundary (\( \alpha = \beta \) in (5.18)) and the integer ones are associated with collinear divergences (\( \alpha = 0 \)). However, the sine factor in (5.23) regulates the simple poles at integer values and reduces the double poles to simple ones.

Applying the inverse Laplace transformation on (5.23) we obtain the single jet mass distribution, which can be expressed in terms of the single jet mass radiation function \( R_\rho(\nu, Q^2, \rho) \) in the following way

\[ \frac{1}{\sigma} \frac{d\sigma}{d\tau}(\rho, Q^2) = \int \frac{d\nu}{2\pi i} e^{\nu \rho} J(\nu, Q^2) \delta(\rho_1 + \rho_2 - \tau) \]

\[ = - \int d\rho_1 d\rho_2 \int_C \frac{d\nu}{2\pi i} e^{\nu \rho_1 + \ln \tilde{J}(\nu, Q^2)} \int_{C'} \frac{d\nu'}{2\pi i} e^{\nu' \rho_2 + \ln \tilde{J}(\nu')} \frac{d}{d\rho_1} \delta(\rho_1 + \rho_2 - \tau) \]

\[ = \frac{d}{d\tau} \int_C \frac{d\nu}{2\pi i} e^{\nu \rho + \ln \tilde{J}(\nu, Q^2)} \]

(5.27)

Similarly, since \( \rho_H = \max\{\rho_1, \rho_2\}, \)

\[ \frac{1}{\sigma} \frac{d\sigma}{d\rho_H}(\rho_H, Q^2) = \int d\rho_1 d\rho_2 J(\rho_1, Q^2) J(\rho_2, Q^2) [\delta(\rho_1 - \rho_H) \theta(\rho_1 - \rho_2) + \delta(\rho_2 - \rho_H) \theta(\rho_2 - \rho_1)] \]

\[ = \int d\rho_1 d\rho_2 J(\rho_1, Q^2) J(\rho_2, Q^2) \frac{d}{d\rho_H} \left[ \theta(\rho_H - \rho_1) \theta(\rho_H - \rho_2) \right] \]

\[ = \frac{d}{d\rho_H} \int d\rho_1 \int_{C'} \frac{d\nu}{2\pi i} e^{\nu \rho_1 + \ln \tilde{J}(\nu, Q^2)} \int_{C'} \frac{d\nu'}{2\pi i} e^{\nu' \rho_2 + \ln \tilde{J}(\nu')} \delta(\rho_H - \rho_1) \delta(\rho_H - \rho_2) \]

\[ = \frac{d}{d\rho_H} \left[ \int \frac{d\nu}{2\pi i} e^{\nu \rho_H + \ln \tilde{J}(\nu, Q^2)} \right]^2. \]

(5.28)

### 5.3 Power Corrections

Since physical quantities cannot depend on the choice of an integration contour, the renormalon ambiguities in the thrust and heavy jet mass distributions have to be cancelled by non-perturbative
corrections. As a consequence, some information on the functional dependence of these corrections can be obtained from the ambiguities.

Denoting by $J^{PT}$ the contribution that follows from perturbation theory (satisfying (5.23)) and introducing a non-perturbative contribution $J^{NP}$, the radiation function can be expressed as

$$R_\rho(\rho, Q^2) = \int_C \frac{d\nu}{2\pi i} e^{\nu\rho + \ln J^{PT}(\nu, Q^2) + \ln J^{NP}(\nu, Q^2)}. \quad (5.29)$$

Here we used the property that, due to factorization, the perturbative piece is simply multiplied by the non-perturbative piece in Laplace space. This corresponds to the convoluted expression

$$R_\rho(\rho, Q^2) = R^{PT}_\rho(\rho, Q^2) \ast f_{NP}(\rho, Q^2) \quad (5.30)$$

where

$$f_{NP}(\rho, Q^2) = \int_C \frac{d\nu}{2\pi i} e^{\nu\rho + \ln J_{NP}(\nu, Q^2)}. \quad (5.31)$$

The renormalon ambiguities arising from (5.23) provide us with information on the functional form of $f_{NP}$ in the following way. The poles at half integer values of $u$ give rise to ambiguities in $\ln J(\nu, Q^2)$ proportional to odd powers of $-\nu \Lambda / Q$. This follows immediately from the residue in $u = \frac{2n-1}{2}$, with $n$ integer, using

$$\text{Res}_{u=-n} \Gamma(u) = \frac{(-1)^n}{n!}. \quad (5.32)$$

The collinear singularities transform into power corrections of the form $\sim \left(\frac{\nu \Lambda}{Q^2}\right)^p$ in the same way. Since they are suppressed compared to the soft contributions, we focus on the latter. The form of the ambiguities in $\ln J(\nu, Q^2)$ implies a functional dependence on a single argument $\nu \Lambda / Q$ only. Inspired by section 4.6 and Ref. [18], we write

$$\ln J_{NP} \left(\frac{\nu \Lambda}{Q}\right) = \sum_{n=1, \text{odd}}^{\infty} \frac{\lambda_n}{n!} \left(\frac{-\nu \Lambda}{Q}\right)^n. \quad (5.33)$$

By exponentiating both sides,

$$J_{NP} \left(\frac{\nu \Lambda}{Q}\right) = \int_0^\infty d\zeta f(\zeta, \mu) e^{-\zeta \nu \Lambda / Q}, \quad (5.34)$$

we recover the shape function $f(\zeta, \mu)$: a single-argument function, independent of $Q$, describing the leading non-perturbative contributions. Expanding both sides of (5.34) in powers of $\frac{\nu \Lambda}{Q}$ provides us with relations like

$$\int_0^\infty d\zeta f(\zeta, \mu) = 1$$

$$\int_0^\infty d\zeta \zeta f(\zeta, \mu) = \lambda_1$$

$$\int_0^\infty d\zeta \zeta^2 f(\zeta, \mu) = \lambda_1^2 - \lambda_2 \quad (5.35)$$
and so on. It follows that $\lambda_n$ are the central moments $^1$ of the shape function, that is

$$\lambda_1 = \int_0^\infty d\zeta \, \zeta f(\zeta, \mu)$$

$$\lambda_2 = \int_0^\infty d\zeta \, \zeta(\zeta - \lambda_1)^2 f(\zeta, \mu),$$  \hspace{1cm} (5.36)

et cetera. If $\nu$ is small enough, it is sufficient to take into account the contribution of order $\frac{\nu \Lambda}{Q}$ in (5.33) only. Since this corresponds to the shape function

$$f(\zeta) = \delta(\zeta - \lambda_1),$$  \hspace{1cm} (5.37)

it follows from (5.34) that the main non-perturbative effects can be taken into account by simply shifting the argument of the resummed perturbative expression. Yet small $\nu$ corresponds to large conjugate variables $\tau$ and $\rho_H$ such that the above approximation is no longer valid in the 2-jet region when $\tau, \rho_H \simeq \Lambda/Q$. A more accurate shape function is required there.

---

$^1$For a probability density function $f(x)$ the moment about the mean $\mu$ is

$$\mu_k = \int_{-\infty}^{\infty} dx (x - \mu)^k f(x)$$
Looking back at (5.30), we observe that the thrust and heavy jet mass radiation functions can be expressed as convolutions as well. From (5.27) we arrive at

\[ R_{\tau}(\tau, Q^2) = \int_0^\tau d\tilde{\tau} \int_0^{\pi/2} d\nu \exp(\nu \tau + 2 \ln \tilde{J}(\nu, Q^2)) \]

\[ = \int_0^\tau d\tilde{\tau} \int_0^{\pi/2} d\nu \exp(\nu \tilde{\tau} + \ln \tilde{J}(\nu)) \int_0^\infty d\nu' \exp(\nu' \tau - \tilde{\tau}) \ln \tilde{J}(\nu') \]

\[ = R_{\rho}(\tau, Q^2) \otimes R_{\rho}(\tau, Q^2) \]

whereas (5.28) leads to

\[ R_{\rho H}(\rho_H, Q^2) = \left\{ R_{\rho}(\rho_H, Q^2) \right\}^2 \]

\[ = \left\{ \left( R_{\rho H}^{PT}(\rho_H, Q^2) \right)^{1/2} \otimes f_{NP}(\rho_H, Q^2) \right\}^2. \]

Applying the non-perturbative contributions to the thrust and the heavy jet mass radiation functions are described by the same shape function. Therefore it should be possible to fit one of them to the data to determine the unknown parameters in \( f_{NP} \) and use this to predict the other radiation function. This was done in Ref. [14]. The thrust distribution was fitted to the data (Figure 5.2 a) and the corresponding parameters lead to the heavy jet mass distribution described by the solid line in Figure 5.2 b. The predicted curve for the distribution is in good agreement with experiment at all values of the heavy jet mass, from the completely perturbative regime to the peak region which is dominated by non-perturbative effects.

Note that as claimed in section 4.4 the calculated perturbative distribution without any power corrections, as described by the dotted line in Figure 5.2, is not satisfactory.

Figure 5.2: a) The calculated thrust distribution as a function of 1-thrust without power corrections (dotted), with just the shift (dashed) and the best fit with a shape function (solid) to the data. b) The calculated heavy jet mass distribution without power corrections (dotted) and with the shape function using the parameters of the fitted thrust (solid) compared to the data.
Chapter 6

Conclusion

The thrust and heavy jet mass distributions for $e^+e^-$ annihilation were determined in the straightforward perturbative fashion, subsequently with resummed large logarithms and finally from the DGE model. Although a strong improvement on calculations at fixed order in the coupling, resummation up to fixed logarithmic accuracy (e.g., NLL) is not satisfactory when compared to the data. Non-perturbative corrections become important at small values of the event shape and power corrections need to be included. From renormalon theory we know that the shape of non-perturbative corrections can be extracted from perturbation theory itself. Via dressed gluon exponentiation these power corrections were derived using the exponentiation property that follows from factorization and applying it to a dressed gluon. The power corrections are described by a shape function that by convolution with the perturbative contribution leads to the event shape distribution. The obtained distributions are in good agreement with the available data. Moreover, the parameters in the shape functions that describe the contribution of the power corrections seem to be universal. This should be investigated more thoroughly by studying more event shape variables. Dressed gluon exponentiation is succesful in revealing certain aspects of what happens at the interface between perturbative and non-perturbative QCD, but we cannot speak of a full description yet.
Appendix A

The QCD Lagrangian and Feynman Rules

Quantum Chromodynamics (QCD) is a non-abelian gauge theory based on the $SU(3)$ group. It is described by the following Lagrangian

$$\mathcal{L}_{QCD} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} + \bar{\psi}_i (i \slashed{D} - m_i) \psi_i$$  \hspace{1cm} (A.1)

$$F_{\mu\nu}^a = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} + gf^{abc} A^b_{\mu} A^c_{\nu}$$

$$D_\mu = \partial_\mu - ig A^a_\mu T^a (R)$$

where $A_\mu$ is the gluon field, $\psi_i$ is the quark field with flavour $i(= 1, \ldots, n_f)$, $g$ is the colour charge, $f^{abc}$ are the $SU(3)$ structure constants (so $a$ runs from 1 to 8) and $T_a(R)$ are the generators in the representation $R$, meaning

$$[T^a, T^b] = i f^{abc} T^c. \hspace{1cm} (A.2)$$

The usual choice for the generators in the fundamental representation is

$$T^a = \frac{1}{2} \lambda^a \hspace{1cm} (A.3)$$

where $\lambda^a$ are the Gell-Mann matrices and they are normalized according to

$$\text{Tr}[T^a T^b] = T_F \delta^{ab} = \frac{1}{2} \delta^{ab}. \hspace{1cm} (A.4)$$

The Casimirs $C_F$ and $C_A$, corresponding to the fundamental and the adjoint representation respectively, are defined by

$$(T^a)_{ij} (T^a)_{jk} = C_F \delta_{ik} \hspace{1cm} (A.5)$$

$$f^{abc} f^{abd} = C_A \delta^{cd} \hspace{1cm} (A.6)$$

and in a $SU(N)$ theory they satisfy

$$C_F = \frac{N^2 - 1}{2N} \hspace{1cm} (A.7)$$

$$C_A = N. \hspace{1cm} (A.8)$$

The Feynman rules for vertices and propagators in momentum space follow from the QCD Lagrangian (A.1) and are displayed in Figure A.1. Quarks are considered as massless particles in this thesis. The expressions for the gluon 3- and 4- selfinteraction vertices are not needed in this thesis.
<table>
<thead>
<tr>
<th>Quark propagator</th>
<th>( \frac{i\delta_{ij} \not{k}}{k^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gluon propagator</td>
<td>( \frac{-i\delta_{ab}(g_{\mu\nu} + \eta_{\mu\nu} k_{\mu} k_{\nu})}{k^2} )</td>
</tr>
<tr>
<td>Quark-gluon vertex</td>
<td>( -ig_s \gamma^\mu T^a_{ij} )</td>
</tr>
<tr>
<td>Gluon 3-vertex</td>
<td></td>
</tr>
<tr>
<td>Gluon 4-vertex</td>
<td></td>
</tr>
</tbody>
</table>

Figure A.1: QCD Feynman rules
Appendix B

Renormalization and the Running of the Coupling

The constant $g$ in (A.1) determines the strength of the colour interaction. It is related to the strong coupling constant $\alpha_s$ by $\alpha_s = \frac{g^2}{4\pi}$ and thus interpreted as the colour charge. However, the renormalized coupling will turn out not to be a true constant.

In quantum field theories we encounter divergences when calculating loop diagrams. The antidote is to absorb the infinities into physical quantities such as charge or mass by introducing counterterms. In this renormalization procedure, first the infinities have to be separated from the finite contributions by means of a regularization scheme. There are various schemes available of which dimensional regularization is probably the most convenient in QCD. It may not be the most intuitive one, but it preserves gauge invariance as well as the symmetries of the theory. All calculations are done in $N = 4 + \epsilon \in \mathbb{C}$ dimensions such that the original singularities appear explicitly as $1/\epsilon$ poles. The physical limit is obtained by removing the regulator, which corresponds to $\epsilon \to 0$ in this case. The counterterms can be chosen such that they cancel the $1/\epsilon$ poles exactly, which is called the minimal subtraction ($\overline{MS}$) renormalization scheme, but often one includes $\ln \frac{4\pi}{\gamma_E}$ in the subtraction, which is called modified minimal subtraction ($\overline{MS}$). Now, because of dimensional regularization, the bare coupling constant $g$ has obtained a dimension. It is factored out by introducing a parameter $\mu$ with the dimension of mass, an arbitrary mass scale, such that the renormalized coupling $g_R$ is dimensionless

$$g = g_R(\mu)Z_g\mu^{-\epsilon/2}. \quad (B.1)$$

Similarly the masses and the fields in (A.1) become

$$m = m_R(\mu)Z_m \quad (B.2)$$

$$\psi = \psi_R(\mu)Z_\psi^{1/2}. \quad (B.3)$$

Since the renormalization scale $\mu$ is entirely arbitrary, no physical observable $O = O_R(\mu)Z_O$ should depend on it, so

$$\mu \frac{\partial}{\partial \mu} O = 0 = Z_O \left( \mu \frac{\partial}{\partial \mu} + \beta(g, \mu) \frac{\partial}{\partial g} \right) O_R(\mu) + O_R(\mu) \mu \frac{d}{d\mu} Z_O(\mu) \quad (B.3)$$
and we arrive at the *renormalization group equation* (RGE)

\[
\left[ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \gamma(g) \right] O_R(\mu) = 0,
\]

(B.4)

where the renormalization group coefficients defined by

\[
\beta(g) = \mu \frac{dg}{d\mu},
\]

(B.5)

\[
\gamma(g) = \frac{\mu}{2} \frac{d}{d\mu} \ln Z_O
\]

(B.6)

are referred to as the \( \beta \) function and the anomalous dimension respectively. In case of a \( n \)-point Green’s function \( \Gamma^{(n)} \), the renormalized function picks up a factor \( Z^{1/2} \) for each external line and the RGE becomes

\[
\left[ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + n\gamma(g) \right] \Gamma^{(n)} = 0.
\]

(B.7)

Substituting \( \alpha_s = \frac{g^2}{4\pi} \) into (B.5) we obtain a differential equation that provides the renormalization scale dependence of the coupling constant,

\[
\mu \frac{d\alpha_s}{d\mu} = \frac{g}{2\pi} \beta(g).
\]

(B.8)

As it is no longer a constant, \( \alpha_s(\mu) \) is referred to as the *running coupling*. The \( \beta \) function can be calculated order by order in perturbation theory,

\[
\beta(\alpha_s) = -\sum_{i=0}^{\infty} \beta_i \left( \frac{\alpha_s}{4\pi} \right)^{i+2},
\]

(B.9)

where the first 4 coefficients (to four loops) are actually known these days. In this thesis we only need the first one that is given by [5]

\[
\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_f n_f.
\]

(B.10)

The solution of the differential equation in (B.8) to one loop accuracy is

\[
\alpha_s(\mu) = \frac{1}{\alpha_s(\mu_0)} + \frac{\beta_0}{2\pi} \ln \frac{\mu}{\mu_0},
\]

(B.11)

for some reference scale \( \mu_0 \). We can introduce an energy scale \( \Lambda_{QCD} \) satisfying

\[
\frac{1}{\alpha_s(\mu_0)} = \frac{\beta_0}{2\pi} \ln \frac{\mu_0}{\Lambda_{QCD}}
\]

(B.12)

such that the running coupling to one loop accuracy becomes

\[
\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln \frac{\mu^2}{\Lambda_{QCD}^2}}.
\]

(B.13)

Apparently in QCD the coupling decreases with increasing energy scale \( \mu \)! That is, \( \beta_0 \) is positive as long as \( n_f < 17 \), but there have been discovered no more than 6 quark flavours so far. This is the property of *asymptotic freedom*; at very short distances, quarks and gluons behave as almost free particles. This behaviour is opposite to the running of the electromagnetic coupling, where
Appendix B. Renormalization and the Running of the Coupling

creation of electron-positron pairs in the vacuum induces a screening effect such that the effective electric charge decreases with distance. In non-abelian theories an additional "antiscreening" effect takes place, which exceeds the screening effect. It originates in the parallel alignment of the magnetic moments of virtual gluons surrounding the quark generating the colour field. The net effect is an enhancement of the coupling constant at large distances. An important consequence is that perturbation theory can be performed at sufficiently high energies since $\alpha_s$ is small there. At low energy scales the partons are confined in colourless compositions. In fact, it was not until 1971 that QCD was shown to be a renormalizable theory \[23, 22, 24\].
Appendix C

Renormalons and Borel Summation

In field theories it is possible to express an observable $R$ as a perturbative expansion in the coupling $\alpha$,

$$R(\alpha) = \sum_r r_n \alpha^n.$$  \hfill (C.1)

However, such a series is usually divergent and at best asymptotic $^1$. In QCD, a large contribution to the divergence comes from renormalons. These are diagrams that are associated with factorially increasing coefficients in perturbation theory, typically containing bubble chains (Figure C.1).

![Figure C.1: Renormalon diagram](image)

The most convenient way of handling such factorially divergent series is Borel summation. For a series $R(\alpha) = \sum r_n \alpha^{n+1}$, we define the Borel transform as

$$B[R](u) = \sum_{n=0}^{\infty} \frac{r_n}{n!} u^n,$$  \hfill (C.2)

which has an infinite radius of convergence in the Borel plane. Provided that the original series has a nonzero radius of convergence and $B[R](u)$ has no poles on the positive real axis and does $^1$

A series is asymptotic to a function $R(\alpha)$ if

$$\left| R(\alpha) - \sum_{n=0}^{N} r_n \alpha^n \right| = O(\alpha^{N+1})$$.
not increase more than exponentially along this axis, the Borel sum of $R$ is given by

$$\tilde{R} = \int_0^\infty du e^{-u/\alpha} B[R](u). \quad (C.3)$$

The divergent behaviour of $R(\alpha)$ is translated into singularities in the Borel plane. The faster the original divergence, the closer these poles are to the origin $u = 0$. However, if there is a singularity $u_0$ on the positive real axis, the Borel integral can only be performed by deforming the contour above or below the singularity (Figure C.2). This results in an ambiguity in the outcome. Its strength follows from Cauchy’s theorem to be $2\pi i \text{Res}_{u=u_0} \{e^{-u/\alpha} B[R](u)\}$.

The contribution of a fermionic loop to a gluon with momentum $k$ is $\beta_{0,f} \alpha_s \left( \ln \frac{k^2}{\mu^2} + C \right)$ [20], with $C = 5/3$ in the $\overline{\text{MS}}$ scheme and where $\beta_{0,f}$ is the fermionic part of the one-loop $\beta$ function ($\frac{4}{3} T_f n_f$). The contributions from ghost and gluon bubbles to the selfenergy is accounted for by substituting $\beta_{0,f} \rightarrow -\beta_0$. This procedure is called naive nonabelization for obvious reasons. The extension to a chain of $n$ bubbles gives [4]

$$\sum_{n=0}^\infty \alpha_s(\mu^2) \int_0^\infty \frac{dk^2}{k^2} F(k^2) \left[ -\beta_0 \alpha_s(\mu^2) \left( \ln \frac{k^2}{\mu^2} + C \right) \right]^n, \quad (C.4)$$

where $F(k^2)$ is the integrand of the Feynman diagram. This can be rewritten by interchanging the integral and the sum to give

$$\int_0^\infty \frac{dk^2}{k^2} F(k^2) \frac{\alpha_s(\mu^2)}{1 + \beta_0 \alpha_s(\mu^2) \ln(e^C k^2/\mu^2)} = \int_0^\infty \frac{dk^2}{k^2} F(k^2) \alpha_s(e^C k^2). \quad (C.5)$$

It follows that the set of diagrams containing a chain of bubbles is obtained from the skeleton diagram by inserting the one-loop running coupling at the vertices and integrating over the gluon momentum. At large $n$, the integral (C.4) is dominated by logarithmic enhancements of both large and small gluon momenta. In the infrared (IR) regime it turns out to be proportional to $\sum_n \beta_0^n n! \alpha_s^{n+1}$ and in the ultraviolet (UV) regime to $\sum_n (-\beta_0)^n n! \alpha_s^{n+1}$ [4]. Consequently, in non-Abelian theories such as QCD, in which $\beta_0$ is positive, the IR integral corresponds to a fixed-sign factorial divergence while the UV leads to a sign-alternating divergence. The procedure of naive nonabelization interchanged the role of IR and UV renormalons. Since the IR renormalons are associated with poles on the positive real axis in the Borel plane (which follows immediately from the geometric series that

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Figure C.2: A singularity on the positive real axis in the Borel plane
Appendix C. Renormalons and Borel Summation

is obtained from (C.2), ambiguities will arise on their behalf. Therefore we focus on infrared renormalons in this thesis in view of the power corrections. For example, according to (C.3), the IR pole at $u = \frac{1}{\beta_0}$ in the Borel plane leads to an ambiguity $2\pi i e^{-\frac{1}{\beta_0 \alpha_s}}$. Substituting the one-loop running coupling, this is proportional to $\frac{\Lambda^2 Q^2}{\beta_0}$. Since ambiguities should be absent from the result, power corrections at the non-perturbative level are suggested this way.
Appendix D

Dispersive Approach

In [11] a dispersive representation of the running coupling $\alpha_s(k^2)$ was introduced in order to take into account non-perturbative effects at low scales. The Ansatz is the extension from the known Abelian theory to QCD.

In QED, because of the Ward identities, the running of the coupling is determined by the photon propagator corrections only. Considering a photon exchange with negative virtuality $-k^2$ (we define $k^2 > 0$ as from now) we can write

$$\alpha(k^2) = \alpha(0) Z_A(-k^2). \quad \text{(D.1)}$$

The full propagator in turn can be written in the Källén-Lehmann representation as

$$\Delta(k^2) = \int_0^\infty \frac{d\mu^2}{k^2 - \mu^2 - i\epsilon} \rho(\mu^2) \quad \text{(D.2)}$$

where $\rho(\mu^2)$ is the spectral density function which is positive definite in Abelian theories. All together we can write

$$\alpha(k^2) = \Delta(-k^2) = \int_0^\infty d\mu^2 \rho(\mu^2) = \alpha(0) + k^2 \int_0^\infty \frac{d\mu^2}{\mu^2} \rho(\mu^2). \quad \text{(D.3)}$$

This corresponds to a branch cut on the negative real axis. According to Cauchy’s theorem the coupling at a fixed point $k^2_0$ is given by an integral in the complex plane over a contour $C$ enclosing the fixed point but no singularities of $\alpha(\mu^2)$

$$\alpha(k^2_0) = \frac{1}{2\pi i} \int_C d\mu^2 \frac{\alpha(\mu^2)}{k^2_0 - \mu^2}. \quad \text{(D.5)}$$

The contour can be extended to infinity except for a curve around the branch cut (Figure D.1). Since there is no contribution from the part of the contour where $|\mu| \to \infty$, the coupling is just an integral over the lines parallel to the branch cut at a distance $\epsilon$ from the axis

$$\alpha(k^2) = \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{d\mu^2}{k^2 - \mu^2} [\alpha(\mu^2 + i\epsilon) - \alpha(\mu^2 - i\epsilon)] = \frac{1}{2\pi i} \text{Disc}\{\alpha(-\mu^2)\} \quad \text{(D.6)}$$

$$= \frac{1}{2\pi i} \int_0^\infty \frac{d\mu^2}{k^2 + \mu^2} \alpha(0) \left[ Z_A(\mu^2 - i\epsilon) - Z_A(\mu^2 + i\epsilon) \right] = -\frac{\alpha(0)}{\pi} \text{Im}\{Z_A(k^2)\}. \quad \text{(D.7)}$$
This provides us with an expression for the spectral density function by comparing with the dispersion relation (D.3),

$$\rho(\mu^2) = -\frac{1}{2\pi i} \text{Disc}\{\alpha(-\mu^2)\}$$
$$= -\frac{\alpha(0)}{\pi} \text{Im}\{Z_A(\mu^2)\}$$

\hspace{\textwidth}

(D.8)

Why not try and extend this procedure to non-Abelian theories? Following the Abelian approach, we consider the single gluon corrections responsible for the running of the strong coupling. In case of a gluon with virtuality $-k^2$, we assume that a dispersion relation similar to (D.3) will hold,

$$\alpha_s(k^2) = \alpha_s(0) Z(-k^2) = -\int_0^{\infty} \frac{d\mu^2}{\mu^2 + k^2} \rho_s(\mu^2)$$

(D.9)

$$\rho_s(\mu^2) = -\frac{1}{2\pi i} \text{Disc}\{\alpha_s(-\mu^2)\} = -\frac{\alpha_s(0)}{\pi} \text{Im}\{Z(\mu^2)\}$$

(D.10)

Notice that in QCD $\alpha_s(k^2)$ decreases with increasing $k^2$ (asymptotic freedom) which implies that $\rho_s(\mu^2)$ is negative. Therefore it cannot be interpreted as a spectral density function in the same way as $\rho(\mu^2)$.

Contributions from real diagrams corresponding to gluon virtuality $k^2 \geq 0$ are described by an on-shell part and a branching gluon propagator,

$$\alpha_s(0) \delta(k^2) + \frac{\rho_s(k^2)}{k^2}.$$  

(D.11)

The dispersion relation can be rewritten as

$$\alpha_s(k^2) = -\int_0^{\infty} \frac{dv}{1 + v} \rho_s(vk^2)$$

$$= -\int_0^{\infty} \frac{dv}{1 + v} \exp\left\{\ln v \frac{d}{d\ln k^2}\right\} \rho_s(k^2)$$

$$= \frac{\pi}{\sin \left(\pi \frac{d}{d\ln k^2}\right)} \rho_s(k^2)$$

(D.12)

The second equality can be understood by expanding $\rho_s(vk^2) = \rho_s(e^t k^2)$ around $t = 0$ and the last follows from Euler’s reflection formula,

$$\Gamma(z)\Gamma(1-z) = \int_0^{\infty} dt \frac{e^{-t}}{1 + t} = \frac{\pi}{\sin(\pi z)}.$$  

(D.13)
If we define an effective coupling $\alpha_{\text{eff}}(\mu^2)$ by

$$\rho_s(k^2) = \frac{d}{d\ln \mu^2} \alpha_{\text{eff}}(\mu^2), \quad (D.14)$$

it follows from partial integration that

$$\alpha_s(k^2) = k^2 \int_0^\infty d\mu^2 \frac{\alpha_{\text{eff}}(\mu^2)}{(\mu^2 + k^2)^2}, \quad (D.15)$$

and the inverse relation follows from (D.12):

$$\alpha_{\text{eff}}(\mu^2) = \sin \left( \pi \frac{d}{d\ln \mu^2} \right) \alpha_s(\mu^2). \quad (D.16)$$

If we consider $\alpha_{\text{eff}}$ to hereby be an effective measure of the QCD interactions strength, the physical perturbative coupling is extended to the non-perturbative domain.

A collinear safe dimensionless quantity $F(Q^2, \{x\})$, where $\{x\}$ represents some dimensionless parameters, can be treated in the dispersive approach by using (D.9) for virtual and (D.11) for real contributions. If we denote by $F^{\text{real}}$ the squared amplitude for the emission of a gluon with $k^2 \geq 0$ where the coupling is left out and defined such that it vanishes outside $0 < k^2 < k_{\text{max}}^2$, the contribution to $F(Q^2, x)$ is just

$$F^{\text{real}}(Q^2, \{x\}) = \alpha_s(0) F(0) + \int_0^\infty \frac{dk^2}{k^2} \rho_s(k^2) F^{\text{real}}(k^2). \quad (D.17)$$

By means of (D.9) the contribution of a virtual gluon can be expressed in terms of the integrand $\mathcal{M}$ for the emission of a gluon with virtuality $-k^2 < 0$ where the coupling is left out:

$$F^{\text{virt}}(Q^2, \{x\}) = \int_0^\infty \frac{dk^2}{k^2} \alpha_s(k^2) \mathcal{M}(Q^2, \{x\}, -k^2). \quad (D.18)$$

Substituting the dispersive representation of $\alpha_s(k^2)$ and rewriting it in a way similar to (D.4) we obtain

$$F^{\text{virt}} = \alpha_s(0) F(0) + \int_0^\infty \frac{d\mu^2}{\mu^2} \rho_s(\mu^2) F^{\text{virt}}(\mu^2). \quad (D.19)$$

This can be written in the same form as (D.17) by defining

$$F^{\text{virt}}(Q^2, \{x\}, \mu^2) = \int_0^\infty \frac{dk^2}{k^2 + \mu^2} \mathcal{M}(Q^2, \{x\}, -k^2) \quad (D.20)$$

such that

$$F^{\text{virt}} = \alpha_s(0) F(0) + \int_0^\infty \frac{d\mu^2}{\mu^2} \rho_s(\mu^2) F^{\text{virt}}(\mu^2). \quad (D.21)$$

Subsequently the contributions (D.17) and (D.21) can be combined into

$$F = \alpha_s(0) F(0) + \int_0^\infty \frac{d\mu^2}{\mu^2} \rho_s(\mu^2) F(\mu^2) \quad (D.22)$$

where

$$F(Q^2, \{x\}, \mu^2) \equiv F^{\text{real}}(Q^2, \{x\}, \mu^2) + F^{\text{virt}}(Q^2, \{x\}, \mu^2).$$
is known as the characteristic function. The two terms can be combined into one integral since \( \alpha_s(0) \) satisfies the following equation in the dispersive approach,

\[
\alpha_s(0) = - \int_0^\infty \frac{d\mu^2}{\mu^2} \rho_s(\mu^2).
\]  

(D.23)

Inserting the effective coupling (D.14) and applying partial integration, we arrive at

\[
F(Q^2, \{x\}) = \int_0^\infty \frac{d\mu^2}{\mu^2} \rho_s(\mu^2) \left( \mathcal{F}(\mu^2) - \mathcal{F}(0) \right)
\]

\[
= - \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}(\mu^2) \frac{d}{d\ln \mu^2} \left( \mathcal{F}(\mu^2) - \mathcal{F}(0) \right)
\]

\[
= \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}(\mu^2) \hat{F}(Q^2, \{x\}, \mu^2),
\]  

(D.24)

where

\[
\hat{F} \equiv - \frac{\partial \mathcal{F}}{\partial \ln \mu^2}.
\]  

(D.25)

The idea is that \( \alpha_{\text{eff}} \) coincides with \( \alpha_s \) in the perturbative region, but takes into account the confinement physics at lower scales. Once the characteristic function for a specific process is known, the power corrections to a certain observable \( F(Q^2, \{x\}) \) are determined by the behaviour of \( \alpha_{\text{eff}} \) at low energy scales. It is a phenomenological function though, whose behaviour is to be established by experiment. Yet under the assumption that it is universal, power corrections in different processes are related by it. The characteristic function for a certain process is computed by summing the relevant one-loop graphs where a finite mass \( \mu \) is assigned to the gluon. Subsequently, the expression for \( F(Q^2, \{x\}) \) is obtained by inserting the effective coupling and integrating over \( \mu^2 \).
Bibliography


